Geometry, High School

11 September 2011

The work of teaching geometry

During middle school students develop an intuitive understanding of geometric definitions and theorems, through hands on experience with physical or virtual geometric objects. They begin to reason mathematically, seeing, for example, how the angle-sum of a triangle can be derived from facts about transversals an parallel lines. Students enter high school with varying conceptions of definition and proof, and varying levels of preparedness to understand proofs. Although reasoning and proof are important in all areas of mathematics, the study of geometry provides a particularly important opportunity for students to make the transition from empirical to deductive reasoning. It is important to focus on understanding the ideas and the logical relationships between them rather than the form or structure of the proof:

> ... it is not the deductive scheme that commands most attention. It is, in fact, the mathematical ideas, whose relationships are illuminated by the proof in a new way, which appeal for understanding, and it is the intuitive bridging of the gaps in logic that forms the essential component of that understanding. When a mathematician evaluates an idea, it is significance that is sought, the purpose of the idea and its implications, not the formal adequacy of the logic in which it is couched. ... what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinized and revised. While these skills may involve a degree of formalization, the emphasis must be clearly placed on the clarity of the ideas.
>
> —G Hanna, *More than formal proof*, For the Learning of Mathematics 9 (1989), no. 1, 20-23.

Clear ideas are based on precise definitions, and therefore another important part of the work of teaching geometry is helping students understand the role of definitions. This involves creating a classroom environment in which definitions are analyzed and discussed, not simply received on authority.

Dynamic geometry software environments provide ways to illustrate the meaning of interesting theorems, e.g. the concurrency theorems for triangles, and reveal the element of surprise in such theorems, with the potential for triggering in the student an appreciation of the need to arrive at conviction by means other than repeated empirical observations or acceptance of authority.

Key understandings to support this work

1. Understand that the angle- and distance- preserving properties of rotations, reflections, rotations and dilations may be taken as an intermediate platform of axioms on which to build the more advanced theorems of Euclidean geometry, without requiring a long march from more primitive axioms through seemingly obvious statements.

2. Understand that the intuitive notions of congruence and similarity in terms of size and shape may be given a precise form using rotations, reflections, translations, and dilations.

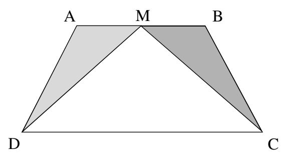
3. Understand that the standard triangle congruence and similarity criteria can be derived using the definition of congruence and similarity in terms of transformations. Familiarity with multiple ways of proving the basic theorems of Euclidean geometry.

4. Understand the way measures such as length, area, and volume scale under similarity transformations, and the application of this to modeling probems involving scale drawings.

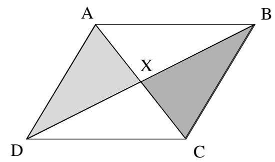
5. Understand that Cartesian coordinates are based on Euclidean geometry, particularly the theorems about transversals intersecting pairs of parallel lines, and that coordinates in turn provide an algebraic way of proving the basic theorems of Euclidean geometry.

Illustrative examples

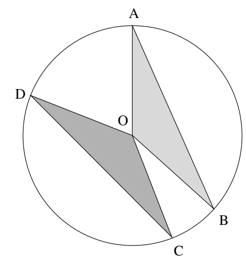
Experience with deductive reasoning Situations where students are asked to judge for themselves whether a given statement is true or not, and whether enough information has been given to judge, give them experience with the practices of mathematics rather than the form. For example, students could be asked to decide whether there is enough information to prove that two triangles are congruent, rather than to prove that they are congruent:



$ABCD$ is a trapezoid with $\overline{AB}||\overline{CD}$, and M is the midpoint of side $\overline{AB}$



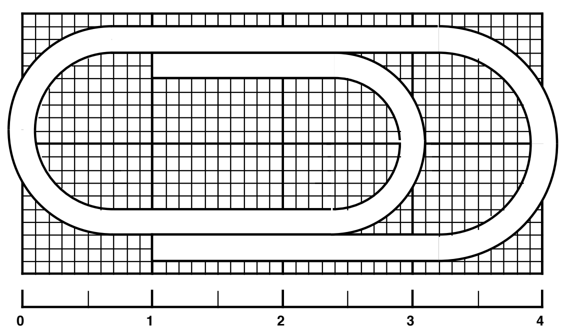
$ABCD$ is a parallelogram



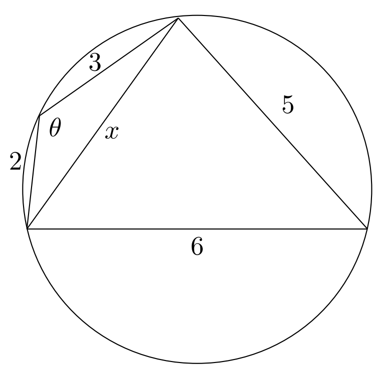
The circle is centered at point O, and the arc AB is congruent to the arc CD

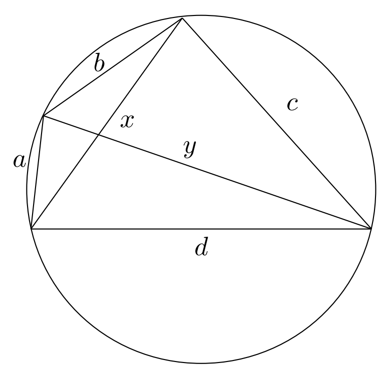
Modeling with geometry Geometry provides students with opportunities for modeling and looking for structure, as illustrated by the following example.

> This paper clip is just over 4 cm long. How many paper clips like this may be made from a straight piece of wire 10 metres long?



Connection between geometry and algebra The interplay between geometry and algebra provides students with opportunities to generalize repeated calculations and see structure in expressions and geometric figures.





For example, students who have studied trigonometry can use the law of cosines to express the square of the diagonal in the cyclic quadrilateral on the left in terms of the sides in two different ways:

$$\begin{aligned} x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\ &= 25 + 36 + 2 \cdot 30 \cos\theta. \end{aligned}$$

By eliminating $\cos\theta$ they can find the value of x.

Whereas students might stop there, teachers in professional development might conduct a further exploration with the figure on the right, generalizing the calculation to an arbitrary cyclic quadrilateral

$$\begin{aligned} x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\ &= \frac{(ac + bd)(ad + bc)}{ab + cd}. \end{aligned}$$

The surprising factorization of the numerator is a sophisticated illustration of looking for structure, where regrouping the squared factor in each term reveals an underlying structure of the expanded form of a product of two binomials. A similar formula for y^2 results in a remarkable simplification, namely Ptolemy's Theorem that the product of the two diagonals in a cyclic quadrilateral is the sum of the products of opposite sides:

$$xy = ac + bd.$$

Acknowledgments: the examples in this section were provided by Cody Patterson, The Shell Center, and Dick Askey.