Challenges, Issues, and Expectations of Preservice Teachers

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The MET recommendations present a vision of mathematics instruction for future teachers which requires a careful rethinking of what constitutes an appropriate and useful preparation. The first priority of preservice mathematics programs must be to help prospective elementary teachers rekindle their own powers of mathematical thought and help them develop a deep understanding of the mathematics they will teach. The goal is to have students develop mathematical habits of mind—developing careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems with the focus on a thorough development of basic mathematical ideas. In this paper, some of the practical issues of implementation raised by these recommendations are addressed—issues which need to be addressed in open forums by the various stakeholders in the mathematics preparation of teachers.

Introduction

The MET recommendations, the NCTM Standards, and the many recently mandated state tests set high expectations for students and for those who teach prospective teachers. Students are being challenged to develop a deeper, more coherent conceptual understanding of mathematics by becoming more flexible, relational thinkers, learning to see connections, accepting and using the language of mathematics, reflecting on and retaining what they have learned. They are supposed to be able to communicate their ideas about mathematics effectively and coherently, as well as attain computational proficiency. Acknowledgement of the need to address the special nature of mathematics knowledge needed by teachers has profound implications for those of us who teach mathematics courses at institutions of higher learning.

As educators of pre-service elementary teachers we face a constant challenge: our students' limited understanding of what constitutes mathematics and a mathematical approach to problems. A significant number of prospective K-8 teachers begin their college mathematics coursework at two-year colleges enrolled in developmental programs. They take one or more semesters of remedial algebra prior to taking the content course. As they begin the mathematics content course, pre-service elementary teachers' attitudes to mathematics are generally instrumental, focused on formulas and correct answers. Statements reflecting their beliefs about mathematics in their autobiographies are typical of many students' prior mathematical experiences:

> "I was used to having a formula and all I cared about was getting the right answer."
>
> "We were taught the rules and how to use them. There are many times I remember seeing the book show how they came about an equation and most teachers would ignore that part of it."
>
> "All throughout school, we have been taught that mathematics is simply plugging numbers into a learned equation. The teacher would just show us the equation dealing with what we were studying and we would complete the equation given different numbers because we were shown how to do it."
>
> "When I began learning mathematics everything was so simple. As I got older there were many more rules being taught to me. The more rules I learned, the easier it became to forget some of the older rules."

The Classroom Reality

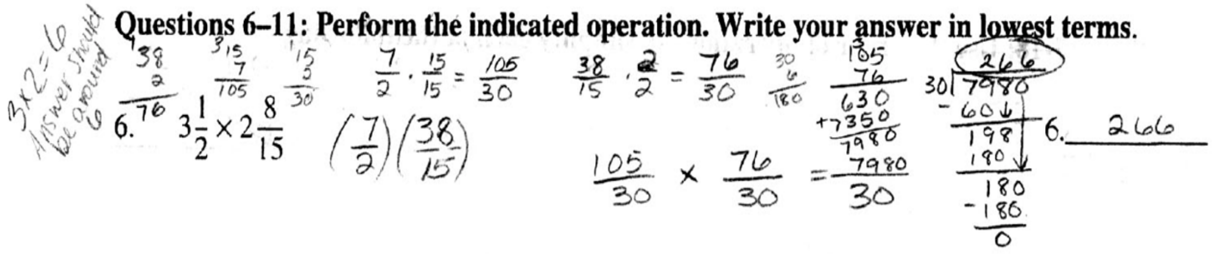
A careful rethinking of what constitutes an appropriate and useful preparation for prospective teachers is needed in light of these expectations and standards given the reality we face in our classrooms—a reality that clearly illustrates the magnitude of the challenge we face. The following problems were included on a pencil-and-paper only departmental competency exam given preservice teachers during the first week of the Fall 2001 semester:

$$\text{(a) } 3\frac{1}{2} \times 2\frac{8}{15} \qquad\qquad \text{(b) } 1\frac{3}{4} \div \frac{1}{2}$$

Fourteen different incorrect responses were given to the first problem including:

$$\frac{76}{105},\ 6\frac{2}{15},\ 10,\ 14,\ 48,\ 266, \text{ and } 450\frac{102}{30}.$$

The work of Student A illustrates how she arrived at the answer of 266.



Questions 6–11: Perform the indicated operation. Write your answer in lowest terms.

FIGURE 1. Work of Student A on Initial Competency Exam (week 1)

Fifteen different incorrect responses were given to the second problem including:

$$\frac{7}{8},\ 1\frac{1}{4},\ 2{,}125,\ \frac{19}{3},\ 4\frac{2}{3},\ 10,\ 14, \text{ and } 16\frac{5}{6}.$$

We are concerned when preservice teachers figure the answer to $9 + 3\frac{5}{8}$ by adding $\frac{72}{8} + \frac{29}{8}$. We are frustrated when, after building towers of height 4, using blocks of 2 different colors, they justify their result with the statement: “I know I have found all possible towers because other groups in the classroom got the exact same answer.” This strictly utilitarian perspective of mathematics often limits students’ mathematical vision. Were they to be anything other than teachers this restricted view of mathematics might not be so important. For the education of their students, however, overcoming these limitations is critical—particularly if the goals are to deepen students’ understanding of the mathematical concepts and content of the K-8 mathematics curriculum, to enhance their flexibility of thinking, and to improve their ability to see and value connections.

Changing students’ utilitarian view of mathematics and changing what they value in mathematics is much more difficult and time-consuming than teaching them algorithms. Effecting lasting changes in those beliefs and values present significant practical and theoretical challenges for mathematicians who teach the content courses to prospective teachers. How do we change this severely procedural orientation to mathematics focused on ‘correct answers’ and peer validation? In our experience, pre-service elementary teachers first need to be brought face-to-face with their unexamined belief that mathematics is only about applying formulas to find answers. A feature of the course is the intensive focus on building connections—constructing relationships between parts of mathematics that students see as different—explicitly emphasizing connections between different representations of a problem and focusing students’ attention on themselves as learners of mathematics and what it means to “do” mathematics. Problems as simple as enumerating all towers of height 4 and 5, built with one or two colors, are sufficient to set this process in motion. We initially set a number of connected combinatorial problems specifically designed to:

(a) be relatively straightforward to begin working on, given the students’ backgrounds;
(b) not be susceptible to easy solution by known or remembered formulas;
(c) set up strong episodic memories as a result of students discussing their solutions in class.

Students are asked to write reflectively after each of the combinatorial problem sessions, and re-writes are encouraged. The combinatorial problems are followed by problems which introduce students to various common arithmetic and geometric sequences, a few of which directly relate to the problems already investigated. After completing the problem investigations assigned during the first four weeks, students are asked to write a “Letter to Mork” describing the connections and the mathematics of the problems they have investigated. The structure and form of the early weeks of the course, focused on seeing connections and building strong episodic memories, assist students in changing their beliefs about the nature of mathematics and their attitudes about what it means to learn mathematics.

Opportunities for making connections with this early work are provided during the semester using questions students have not seen previously on three group and two individual exams. They submit a detailed written self-evaluation of their knowledge and skills twice during the semester. A midterm self-evaluation occurs during the 8th/9th week. The end-of-course self-evaluation takes place prior to the Final Exam. Students meet individually with the instructor to discuss these self-evaluations. The final course grade is based on the evidence contained in the student's portfolio—work which must clearly demonstrate the student's understanding, competence, and growth with respect to the course objectives.

Addressing the question: What constitutes *"profound understanding"*?

Questions of what constitutes evidence of deep understanding and what are reasonable expectations of students at various times in the semester were explored during break out sessions at the National Summit on the Mathematical Preparation of Teachers. Participants examined examples of students' work on a problem from the first group exam—The Tower of Hanoi. They discussed their observations and insights in small groups, then summarized the results of their small group discussions in an open forum. The exam was given a week after students' letters to Mork had been graded and returned to them. During the four weeks prior to the exam, as they investigated combinatorial problems and various sequences, students were introduced to constant first finite differences (linear functions), constant second finite differences (quadratic functions) and constant finite ratios (exponential functions—particularly the doubling sequence). Students were given a very brief history of the problem along with the rules for moving a disk. They were instructed to:

(a) Create an organized way of recording moves.

(b) Increase the number of disks with each investigation: 3 disks, 4 disks, 5 disks, etc.

(c) Record all moves for each investigation, as well as the fewest number of moves for each collection of disks.

(d) Analyze the results of your investigation. What generalized pattern describes the least number of moves for any number of disks?

One example of students' work discussed in the break out session is shown below, together with the students' explanation of how their answer was determined.

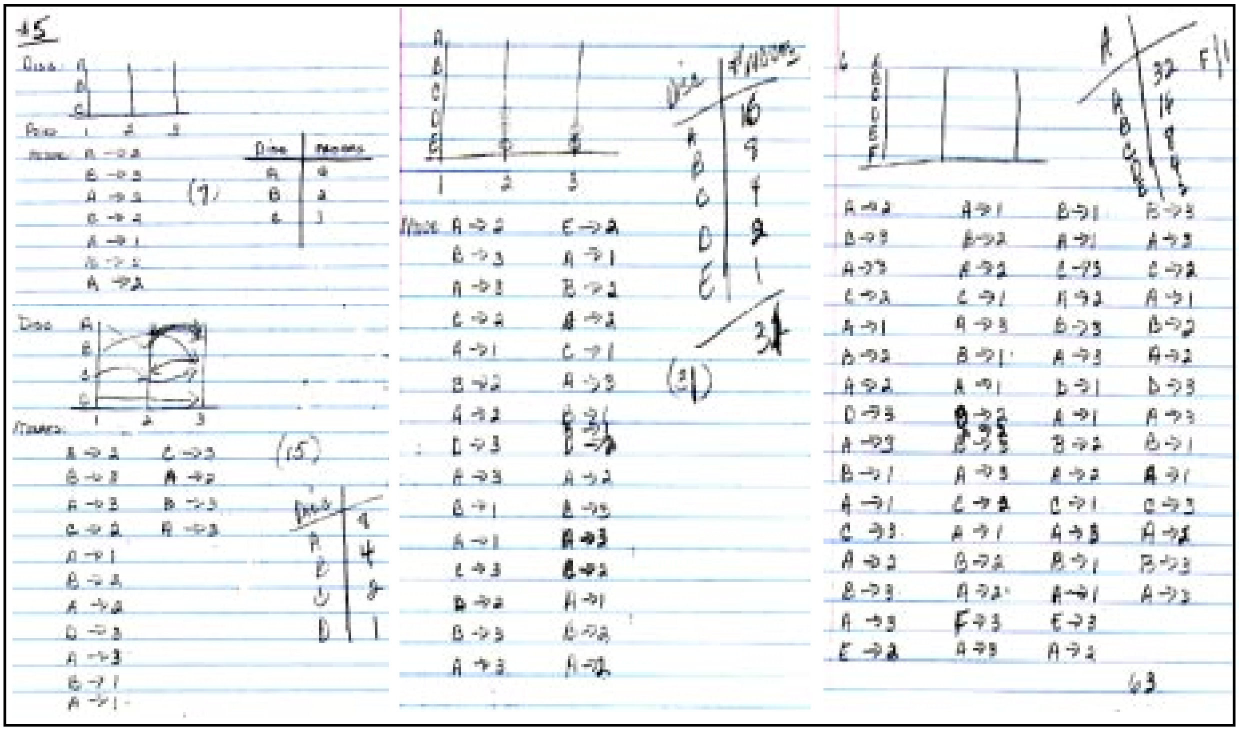


FIGURE 2. Group I: Tower of Hanoi

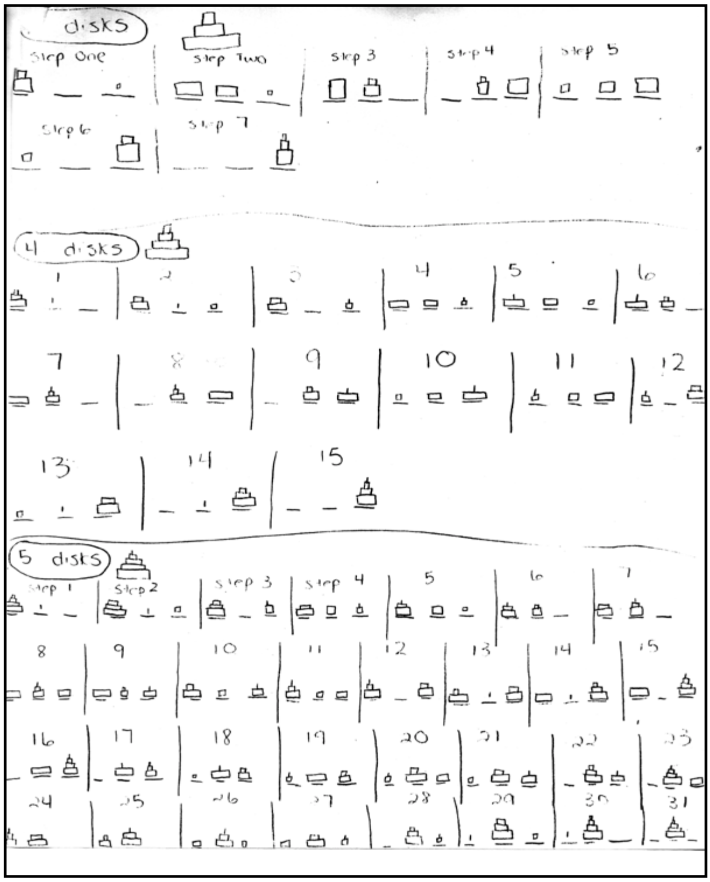
Group 1

We noticed the doubling sequence as we tracked our moves, so we listed the doubling sequence and compared our total moves for 3 disks, 4 disks, five disks, etc.

# disks/ towers	Total # moves	Doubling
1	1	$2^1 = 2$
2	3	$2^2 = 4$
3	7	$2^3 = 8$
4	15	$2^4 = 16$
5	31	$2^5 = 32$
6	63	$2^6 = 64$

The total number of moves in each case is one less than the total number of towers in the doubling sequence so the formula for number of moves for n disks is: *Hanoi* (n) = $2^n - 1$.

The second example of student work examined during the break out session is shown below, together with the students' explanation of how their answer was determined.



# disks	Total moves		
1	1	> 2	
2	3	> 4	4/2 = 2
3	7	> 8	8/4 = 2
4	15	> 16	16/8 = 2
5	31	> 32	32/16 = 2
6	63		

FIGURE 3. Group 2: Tower of Hanoi

Group 2:

We noticed that the first finite difference is the doubling sequence which has a constant finite ratio of 2: $Dbl(n) = 2^n$.

We also noticed that when you compare the total number of disk moves with the doubling sequence term by term, the moves for a given number of disks is one less than the corresponding term of the doubling sequence.

We concluded that the total number of moves for any number of disks, *n* could be written: $Hanoi\ (n) = 2^n - 1$.

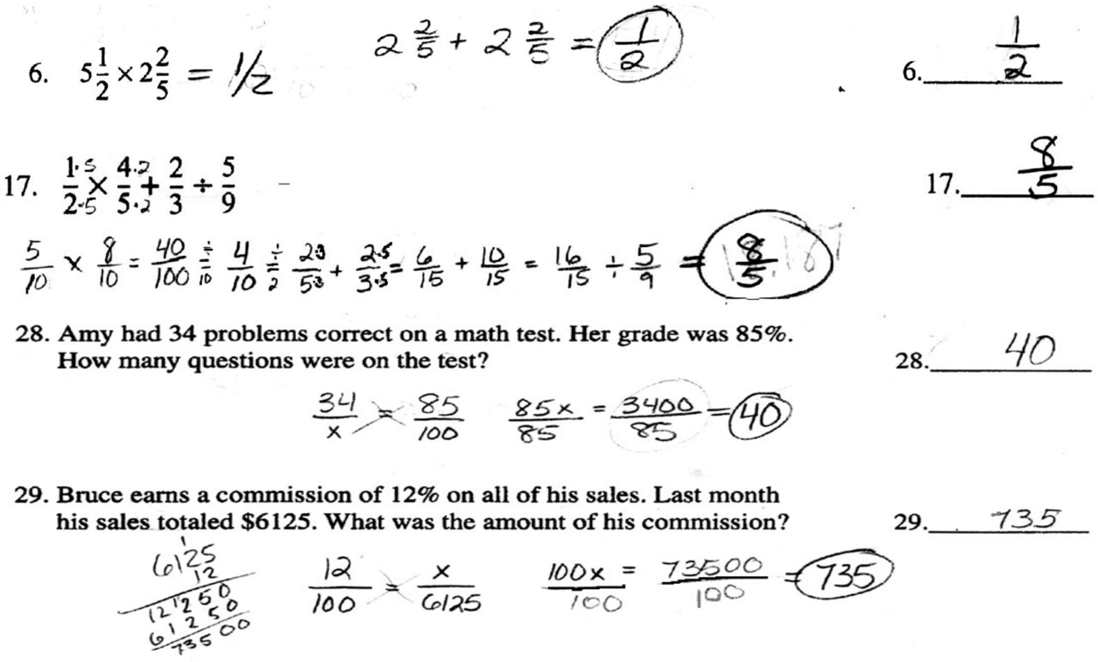
The Challenges of Interpretation and Implementation

Many of us teaching the mathematics content courses for preservice teachers are faced with the necessity of determining what constitutes evidence of “deep understanding” independently, in the absence of any consensus among the various constituencies involved in teacher preparation. We constantly wrestle with the question of whether there are various stages in the development of profound understanding of mathematics and we ask ourselves what degree/level of understanding is appropriate after sixteen weeks of instruction, after two semesters, upon graduation, etc.

The complexity of the task of assessment is illustrated by an examination of the work of Student A who answered only 6 of 30 questions correctly (20%) on her initial departmental competency exam at the beginning of the semester. Eight weeks later, on her second attempt, Student A, like many others in the course, more than doubled her initial score. However, for Student A and the other 28 of 47 students, doubling the initial score on the second attempt of the competency exam half-way through the semester does not yet meet the required passing score of 80%.

Her work on Question 6 on the 2nd attempt of the competency exam (multiplication of two mixed numbers) is unexplainable (Figure 4). The work shown for Question 17 does not justify her answer and, in fact, we are left wondering how it was actually determined. Though her work on Questions 27 and 28 is procedural and not as efficient as one might hope, nevertheless on these two problems she has represented the problems appropriately and has written correct answers—questions not even attempted on the first competency exam eight weeks earlier.

In an interview after taking the competency exam for the second time, Student A said that she had studied the textbook’s explanation of *theorem of addition of fractions with unlike denominators* and the *definition of multiplication of fractions* to figure out how to do problems like Question 17 (Figure 5). As she had not attended any of several review sessions due to scheduling conflicts, Student A attempted to learn mathematics she knew she did not know. Her initiative is to be applauded—she is highly motivated—she wants to be a teacher. However, her interpretation of the text is an example of “a little learning is a dangerous thing.” What is symbolic and meaningful for those who have the mathematical knowledge and understanding to interpret the symbols is simply a rule to follow in order to get the answer for students with limited mathematical understanding who attempt to interpret the marks—particularly when the text provides little guidance for students with a weak mathematics background. Compare the work of Student A on problem 17 (Figure 4) with the highlighted statements and examples of the text (Figure 5).



6. $5\frac{1}{2}\times 2\frac{2}{5} = 1/2$ $\quad 2\frac{2}{5} + 2\frac{2}{5} = \boxed{\frac{1}{2}}$ 6. $\frac{1}{2}$

17. $\frac{1\cdot5}{2\cdot5}\times\frac{4\cdot2}{5\cdot2}+\frac{2}{3}\div\frac{5}{9}$ 17. $\frac{8}{5}$

$\frac{5}{10}\times\frac{8}{10}=\frac{40}{100}\frac{\div}{\div 10}\frac{4}{10}\frac{\div}{\div 2}\frac{2\cdot3}{5\cdot3}+\frac{2\cdot5}{3\cdot5}=\frac{6}{15}+\frac{10}{15}=\frac{16}{15}\div\frac{5}{9}=\boxed{\frac{8}{5}}$

28. Amy had 34 problems correct on a math test. Her grade was 85%. How many questions were on the test? 28. 40

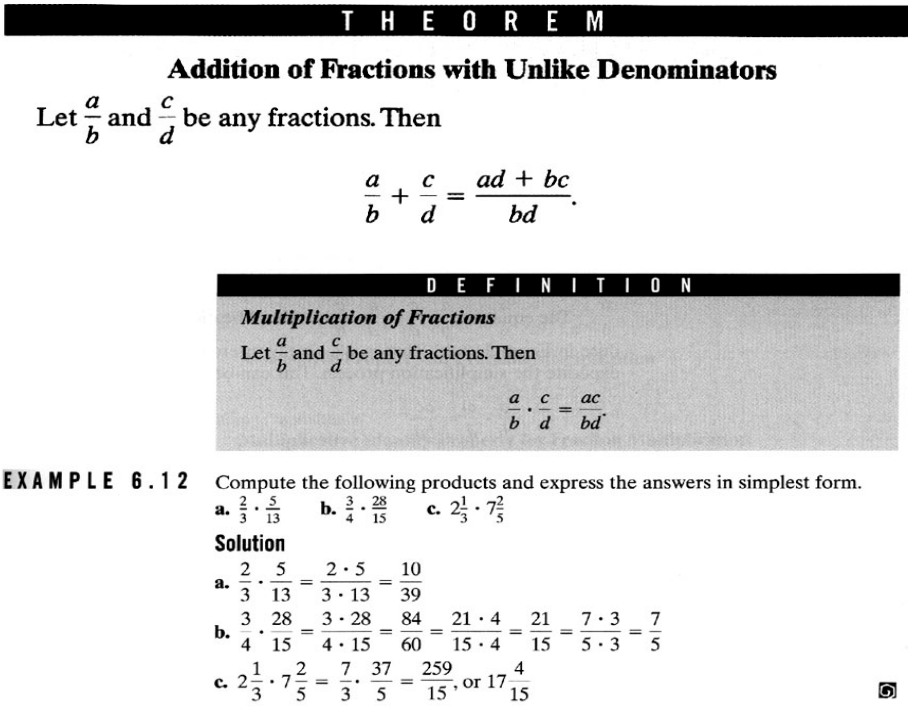
$\frac{34}{x} = \frac{85}{100} \quad \frac{85x}{85} = \frac{3400}{85} = \boxed{40}$

29. Bruce earns a commission of 12% on all of his sales. Last month his sales totaled \$6125. What was the amount of his commission? 29. 735

6125
12
12250
61250
73500

$\frac{12}{100} = \frac{x}{6125} \quad \frac{100x}{100} = \frac{73500}{100} = \boxed{735}$

FIGURE 4. 2nd Attempt of Student A: Departmental Competency Exam (Week 10).



THEOREM

Addition of Fractions with Unlike Denominators

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

DEFINITION

Multiplication of Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

EXAMPLE 6.12 Compute the following products and express the answers in simplest form.

a. $\frac{2}{3}\cdot\frac{5}{13}$ **b.** $\frac{3}{4}\cdot\frac{28}{15}$ **c.** $2\frac{1}{3}\cdot 7\frac{2}{5}$

Solution

a. $\frac{2}{3}\cdot\frac{5}{13}=\frac{2\cdot5}{3\cdot13}=\frac{10}{39}$

b. $\frac{3}{4}\cdot\frac{28}{15}=\frac{3\cdot28}{4\cdot15}=\frac{84}{60}=\frac{21\cdot4}{15\cdot4}=\frac{21}{15}=\frac{7\cdot3}{5\cdot3}=\frac{7}{5}$

c. $2\frac{1}{3}\cdot 7\frac{2}{5}=\frac{7}{3}\cdot\frac{37}{5}=\frac{259}{15}$, or $17\frac{4}{15}$

FIGURE 5. Text Statement of Definition of Multiplication of Fractions & Theorem of Addition of Fractions of Unlike Denominators.

Contrast the same student’s work on the basic skills exam with her work on Unit II exam questions.

Question 5:

Consider the set of numbers *A* = {5, 6, 7, 8} with a made-up operation called *. The rules for * are shown in the table below.

*	5	6	7	8
5	8	7	6	5
6	7	6	5	8
7	6	5	8	7
8	5	8	7	6

a) Is * commutative for set *A*? *Justify your response*.

Student response:

Yes. * is communtative.

5 * 5 = 8
5 * 6 = 7
5 * 7 + 6
5 * 8 = 5

6 * 5 = 7
6 * 6 = 6
6 * 7 = 5
6 * 8 = 8

7 * 5 = 6
7 * 6 = 5
7 * 7 = 8
7 * 8 = 7

8 * 5 = 5
8 * 6 = 8
8 * 7 = 7
8 * 8 = 6

5 * 6 and 6 * 5 both have the same result of 7.
7 * 5 and 5 * 7 both have the same result of 6.
5 * 8 and 8 * 5 both have the same result of 5.
6 * 7 and 7 * 6 both have the same result of 5.
6 * 8 and 8 * 6 both have the same result of 8.
8 * 7 and 7 * 8 both have the same result of 7.

Therefore, the set A is commutative since this is true for all.

Question 6: Does the expression $n^2 - n + 41$ give only prime numbers?
Explain your reasoning and justify your answer.

Student response:

No, this formula does not always generate a prime number. If n = 41, the square root of the sum is 41.

$$41^2 - 41 + 41 = 1681 \qquad \sqrt{1681} = 41$$

Student A was interviewed after the Second Unit Exam had been submitted. She was asked to explain how the answer to Question 6 was determined. She stated that both she and her partner had experienced an “AHHHH” moment. When asked to explain exactly what had they noticed that produced that AHHHH, Student A described their initial trial and error investigations, pulling out a copy of the spreadsheet she had created for testing various values of *n* in the equation. She provided the following explanation for the work:

> Our first approach to this problem was actualy computing the expression and plugging in numbers up to 108 for n to determine if the result was a prime number. Next we looked at the actual number 41 more closely and tried other prime numbers in its place in the expression above to see if it would also generate prime numbers. This did not occur when we replaced 41 with any other prime number, therefore we knew that there was something special about the number 41 in this particular expression. Having this information on the number 41 created an AHHHHH moment in our minds and we decided to plug in 41 for *n* to see if the expression worked. <u>This proved false and we found an example that proved that this expression was not true for all</u>.

Her articulate explanations of the strategies and reasoning, together with her typewritten responses, were taken as evidence of her present understanding. Further evidence of her present state of mathematical understanding was provided by her response to Question 11:

> *Question 11*:
>
> Find the smallest counting number that has the following properties: When it is divided by 2, the result is a perfect square number, and when it is divided by 3, the result is a perfect cube. Document all efforts and justify your conclusion.
>
> *Student response:*
>
> Because it is divisible by 2 it is an even number. Because the number is divisible by both 2 and 3, the number must be divisible by 6. Therefore “a” and “b” must be multiples of 6.
>
> $$n = b^2 * 2$$
> $$n = b^3 * 3$$
>
> Since 2^3 will have fewer mulriples of 6, we started by using 6.
>
> $$6^3 = 216 * 3 = 648$$
>
> Then we divided 648 by 2:
>
> $$648/2 = 324$$
>
> Finally we took the square root of 324 and got the result 18.

Student A’s ability to reason mathematically—to appropriately and effectively argue using a proof by exhaustion, to make and test conjectures, to recognize and represent relationships symbolically, and to clearly articulate what was done, both verbally and in writing—to make sense of the mathematics—provide a startling contrast to her work on the departmental competency exams.

The inability of preservice teachers to demonstrate competency of basic arithmetic computations at the beginning of the semester has not proven to be a valid measure of their ability to think mathematically—to make connections, justify their work, use mathematical arguments, represent problem situations, or to demonstrate competency. Since Fall 1996 and including the current Fall

2001 students, only 4 of 168 preservice elementary teachers have received a grade of 80% or higher on the departmental competency exam on their first attempt. The total enrollment through Spring, 2001 was 156 students, of which 34 took the competency exam initially but did not complete the course—a 78% completion rate. Of those who completed the course, 73% (89 of 122) were successful (grade of C or better and with a score of 80% or better on the competency exam).

A measure of student growth over the semester was determined using the *shift statistic*, defined, as follows:

$$shift = (\text{final test } \% - \text{initial test } \%)/(100\% - \text{initial test } \%)$$

We interpret *shift* as how much students have moved from their initial test result to their final test result. A student who has a relatively high initial test score does not have as much room for improvement as a student with a low initial test score. However, since the shift statistic takes into account a student's *potential for growth*, it is possible that any student can attain the highest shift, 1.00. The shift distribution (gain factor) for the 122 students is shown in Figure 7:

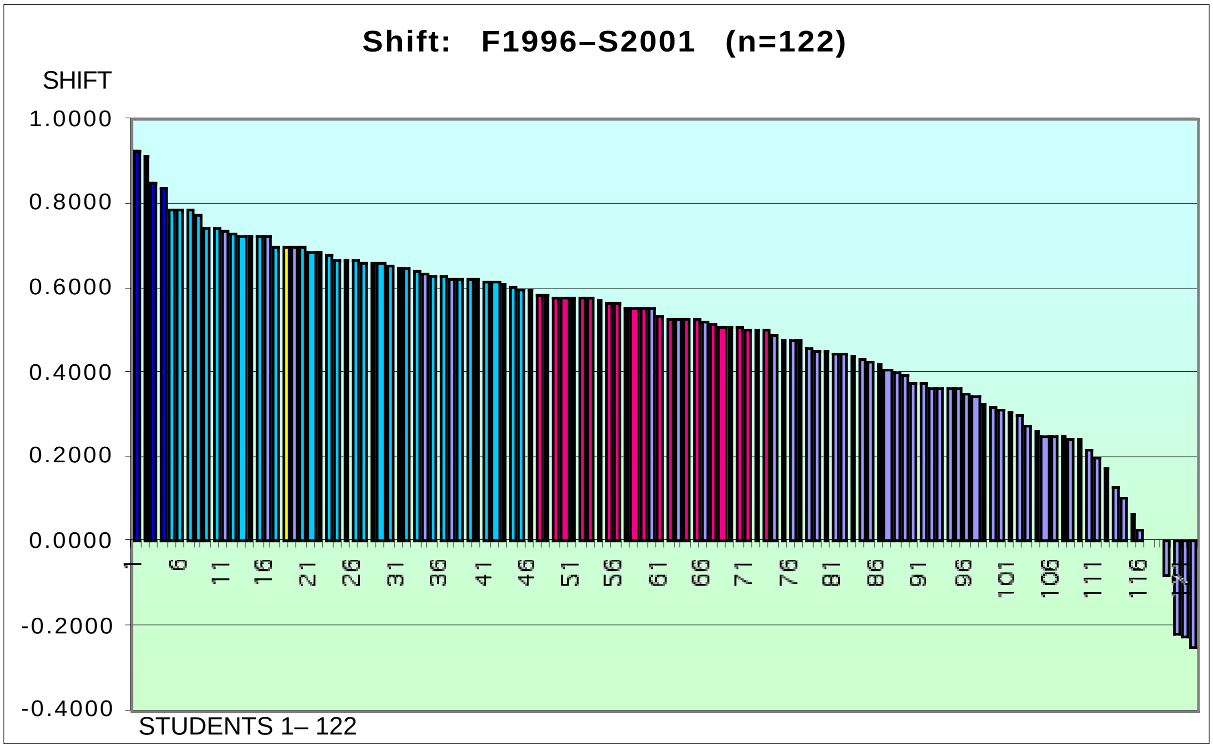


FIGURE 6. Student Growth as measured by the shift statistic: F1996 — S2001 (n = 122)

Eighty-eight students (72%) experienced a gain greater than 40%, with 73 (60%) of the 122 students having a gain greater than the mean shift of 0.4928. Two students had no gain over the semester. Four students experienced a negative shift.

By the end of the sixteen-week course, many students express a different, more “relational” view of mathematics:

> “I know that I don’t truly understand a math problem until I understand the algorithm, have correct computation, and am able to justify my answer.”
>
> “What it means to learn mathematics is finding patterns and connections from other problems to help you through your reasoning of the next problem. This includes flexible thinking, being able to think about the problem in many different ways to adapt what you already know to what you don’t.”
>
> “Before taking this class, I would think mathematics is about only dealing with solving the problems using numbers. When I took this class, I realize it is not just dealing with numbers, but the actual meaning of all the problems.”
>
> “The meaning of arithmetic operations is quite simple once all the connections are understood.”
>
> “Without a solid understanding of the concept behind the algorithm, a student hasn’t really learned anything, just memorized an equation.”

Their final self-evaluations document their growing understanding of justification and proof:

> “... ‘how do you know’ you have the right answer? How do you justify your work? I was annoyed at first with this question in class. I wanted to yell, ‘I know because this is the formula, and it produces the same answer over and over again!’ Well, this is not the correct answer to our question. There is proof by exhaustion and proof by induction.”
>
> “In justifying their work, students must be careful of the wording of their argument. The words ‘all’ or ‘any’ should present a ‘red flag’.”
>
> “We had to build as many towers 4 high and five high as we could, and at the same time, have mathematical proof that we had found all of them. This was where we learned proof by exhaustion. The idea was to argue by cases, showing that all possibilities were there. For example: zero red; one red; two reds; three reds; four reds, and all reds.”

They begin to appreciate the usefulness of definitions:

> “One issue I have always had problems with in mathematics is definitions. I can physically work through a math problem, but to try to put my efforts into words is a challenge. Definitions in mathematics play a vital role in building a solid base of one’s knowledge and abilities.”
>
> “The mistakes our class made in defining even numbers were (a) we assumed that we were working in base ten and (b) we tried to define even numbers by using the word ‘even’!”
>
> “If definitions are the base of our mathematical foundation, then algorithms are the bricks in the bridge of our mathematical path.”

Like many other students who have taken the course previously, Student A is a perplexing puzzle we are attempting to understand. On the one hand, these students struggle to un-learn computational algorithms which are fixed habits of automatic, incorrect responses learned previously that cause enormous anxiety—particularly under stressful exam conditions. On the other hand, many of these students demonstrate the ability to develop more flexible ways of thinking and make sense of mathematics as they investigate problems and reflect on their work during the semester. They recognize that, in order to make connections, they "have to have something to connect." Their language gradually becomes more precise and mathematical marks become truly symbolic.

Liping Ma, in her book, *Knowing and Teaching Elementary Mathematics*, makes the point that a profound understanding of elementary mathematics is developed *over years of teaching experience and on-going professional discussions with colleagues.* If, as Liping Ma suggests, the development of profound understanding requires years of experience, how do we determine what "deep understanding" looks like in multi-course programs that articulate across institutions—in the general education mathematics content courses for preservice elementary teachers at two-year colleges, in subsequent mathematics and methods courses in the teacher preparation programs at the four-year institutions and during the first few years of teaching?

How much time and consistent reinforcement throughout their academic careers are required to effect changes in attitudes and to develop more appropriate skill competencies which will not crumble under the stress of examinations? How do we align our assessment practices with instructional goals that promote the growth of deep conceptual learning in a process that involves many constituencies—particularly when the growth of mathematical reasoning and sense-making are coupled with algorithms used incorrectly and/or inappropriately which have become automated over several years? What are the appropriate kinds of instructional context, modeling and guidance necessary to facilitate these changes? These issues of implementation generate basic questions which need to be addressed in open forums in which discussion leads to the development of broad-based consensus about "deep understanding" and it's assessment.

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