Operations and Algebraic Thinking, K – 5

The work of teaching operations and algebraic thinking

Teachers of Kindergarten through Grade 5 teach the meanings of the operations of addition, subtraction, multiplication, and division, the types of situations in the world these operations solve (mostly presented as word problems), the algebraic properties of these operations, including their use in strategies for solving and working toward fluency with operations beginning with operations on single-digit numbers, and other patterns and rules that can be explored with simple arithmetic.

Teachers help students learn to determine when to use addition, subtraction, multiplication, or division to solve a problem. The most basic meanings of the operations arise from the simplest kinds of situations in which these operations can be used, but the taxonomy of word problem types for the operations is more detailed and elaborate than most people who are not extensively involved with elementary school math may realize. For example, addition arises not only from "add to" situations, and subtraction arises not only from "take from" situations. Addition and subtraction also arise in situations involving parts composed physically or conceptually to make wholes (and vice versa) and situations involving comparisons. Furthermore, in all these types of situations, different quantities can be the unknown amount that is to be found to solve the problem. For example, in an "add to" situation, the initial amount, the change amount, or the final amount can be the unknown quantity that is to be determined. Some problem types are harder for children than others. The situation is similarly complex for multiplication and division.

As children work on word problems, teachers also help them develop and learn strategies for solving numerical addition, subtraction, multiplication, and division problems. Much is known from research in mathematics education about how children progress along a learning path of methods of different levels of conceptual complexity toward fluency first with single-digit additions and associated subtractions and later for single-digit multiplications and associated divisions. Importantly, learning the single-digit calculations is not simply a matter of rote memorization of a bunch of "facts."

As students work towards fluency with single-digit calculations, teachers help them learn to apply – often informally – the commutative and associative properties of addition and multiplication and the distributive property. For example, students learning single-digit additions might calculate 8+7 by breaking 7 into 2 and 5, combining the 2 with 8 to make a ten, and then adding the 5, in effect using the associative property: $8+7=8+(2+5)=(8+2)+5=10+5=15$. Students learning single-digit multiplications might calculate 7*8 by viewing it as five eights plus two more eights, thus using the distributive property, $7*8=(5+2)*8=5*8+2*8=40+16=56$.

Teachers help students connect addition with subtraction and multiplication with division. For example, when teachers teach young children to view subtraction problems as unknown addend addition problems, the children are able to solve many more subtraction problems efficiently and correctly than by counting backward (for example, viewing $12 - 9 = ?$ as $9 + ? = 12$ and counting forward is much easier than counting back 9 from 12).

The case of division has some special limitations and considerations that teachers must know about. Teachers help students understand that when dividing a whole number by another (nonzero) whole number, there are usually several ways to represent the result. The exact quotient can be expressed as a decimal, fraction, or mixed number, or the result can be expressed as the largest whole number quotient and a remainder. Students must be able to select and interpret a form that is appropriate for a context.

Teachers help students notice and use patterns. For example, teachers may ask students to look for patterns in the multiplication table, which can help students learn these multiplications. Investigating repeating patterns and some growing sequences of designs or numbers can provide contexts for children to explore and use whole number quotients and remainders.

Key understandings to support this work

- Know and be able to pose the different types of word problems for addition, subtraction, multiplication, and division with whole numbers, decimals, fractions, and mixed numbers and how to represent the problems with objects, with drawings, and with equations.
- Know which problem types are typically less difficult or more difficult for students.
- Have an initial understanding of children's learning paths in single-digit calculations, the different levels of strategies within them, and how to help children move along these paths.
- Know the commutative and associative properties of addition and of multiplication, and the distributive property of multiplication over addition, know how to see why these properties are valid for whole numbers (such as with arrays), and know how to use these properties as calculation aids for deriving single-digit calculations from other single-digit calculations.
- Know how addition and subtraction are connected and how multiplication and division are connected. Recognize that these connections are not just important theoretically, but also computationally.
- Interpret calculations and numerical answers in terms of a problem context (for example, interpret a remainder in a division problem) and recognize that there can be a distinction between representing or solving a numerical problem arising from a word problem and solving the word problem. Explain why the result of dividing a whole number by a (nonzero) whole number can be represented as a decimal, a fraction (or mixed number), or as the largest whole number quotient and a remainder and choose a form appropriate to a context.
- Be able to pose and solve a variety of multi-step word problems.
- Be able to pose and solve problems about sequences of designs, shapes, and numbers and understand that sequences are special kinds of functions.

Illustrative Examples

These examples illustrate how prospective teachers can engage with mathematics they will teach and how this mathematics is a foundation for mathematics coming up in later grades.

Before their coursework, prospective teachers may not appreciate how surprising it is that multiplication is commutative. They can be asked to consider why it won't be obvious to elementary school students that 3x5 and 5x3 have the same value. Young learners will initially understand these expressions as the total in 3 groups of 5 versus the total in 5 groups of 3, which have different meanings. The teachers can then use arrays to explain why the commutativity of multiplication makes sense. Note the difference between this approach to commutativity and the use of other operations, such as matrix multiplication or transformations of the plane, to demonstrate that commutativity is not obvious.

Courses for prospective elementary teachers can help the teachers recognize and reason clearly about the many subtleties involved with division, such as why one can't divide by 0 and what it means when we express the answer to a whole number division problem as a whole number quotient with a remainder.

Courses for prospective elementary teachers can help the teachers recognize how elementary school students' problem solving methods can be connected to symbolic algebra. For example, elementary school students can use tape diagrams to help them solve multi-step word problems. Prospective teachers can see how tape diagrams correspond to algebraic equations and how the steps used with a tape diagram often correspond directly to the steps used in solving the problem using symbolic algebra.

Explorations and topics such as the following might be used in developing prospective teachers' mathematical habits of mind or in deepening teachers' knowledge of the mathematical ideas related to the domain.

- Explore how the cases of dividing nonzero numbers by 0 and dividing 0 by 0 are different.
- Study why and how the base-ten system creates various patterns, such as why multiples of 2 always end in 0, 2, 4, 6, or 8 and why the sum of the digits in a number divisible by 9 is also divisible by 9.
- In creating a sequence of equilateral triangles in which each one is larger than the one before it, the pattern 1, 4, 9, 16 … emerges from counting the number of "unit" equilateral triangles in each larger shape. This can be connected to functions by examining the relationship between the input (the ordinality of the shape in the sequence) and output (the number of unit triangles in the shape) in order to obtain a function that will produce the output for any given input.