Algebra, Grade 8 and High School

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The work of teaching algebra*

Teaching algebra intertwines teaching symbol manipulation skills with inculcating an understanding of the principles behind them. This starts with the introduction of symbols themselves, where it is easy for students to lose sight of the basic fact that the symbol stands for a number. Helping students keep a firm grasp on this fact enables them to see symbol manipulation as the result of properties of number operations.

For example, students have learned in earlier grades to explain $3 \times 29 = 3 \times 20 + 3 \times 9$ in terms of the distributive property. Now teachers help them see the distributive property for algebraic expressions as an expansion of that for whole numbers, so that a statement like $3(x + y) = 3x + 3y$ is viewed as a consequence of the distributive property, rather than as a mysterious new rule called "moving the 3 inside the parentheses," which refers more to the appearance of the expression on the page than it does to the meaning it expresses. Teachers understand how an area model showing $3 \times 29 = 3 \times 20 + 3 \times 9$ can be connected to one for $3(x + y) = 3x + 3y$ and they know that students have been introduced to the former in earlier grades.

Students have trouble making the transition from situations in which symbols are used to represent specific unknown numbers, for example the unknown quantity in a word problem, to situations where they are used to make universal statements in some domain (Kieran, *What do Students Struggle with When First Introduced to Algebra Symbols?*, NCTM, 2007). Surrounding expressions and equations with complete sentences helps make this distinction, e.g. "We are looking for a number p for which $p + (p + 5) = 47$" or "$2n$ is greater than n for any positive number n."

Precise language about expressions and equations, and later functions, helps students avoid common confusions about these objects. For example, when given an expression that is to be manipulated in some way, students might instead set it equal to zero. An important part of the work of teaching is continually bringing students back the roots of algebra in work with quantities and relationships between quantities. Graphs, tables, and appropriate contexts that engage students are the tools of the trade in this work.

*The main focus in this draft is on the early stages of algebra in late middle and early high school.

Key understandings to support this work

1. That symbols stand for numbers; expressions and equations do not stand by themselves, but live in sentences that ask questions or make statements about numbers.

2. That the rules for manipulating expressions are consequences of the properties of operations with numbers.

3. That the process of solving equations is a process of logical deduction, in which the equation is assumed to be true, and step by step consequences of that assumption are derived until the solutions are made manifest.

4. That equations in two variables represent relationships between two quantities; they can be represented by graphs and tables; finding the value of one variable that corresponds to a value of the other variable involves solving an equation in one variable.

5. How to represent symbolically problems external to mathematics, manipulate the symbolic representation purposefully, and thus solve the problem.

Illustrative examples

Manipulating expressions Students are considering a formula for the contribution of three test grades t_1, t_2, and t_3 to their final grade, which is

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right).$$

They are trying to figure out how much a 10 point increase in t_3 would change the final grade. One students says "you can just move the 3 over, so this is the same as

$$\frac{0.6}{3}(t_1+t_2+t_3) = 0.2t_1+0.2t_2+0.2t_3."$$

Why can you "move the 3 over"?

Solving equations A student writes the following solution up on the board:

$$\begin{aligned} x^2-3x-4 &= 0 \\ x^2-3x &= 4 \\ x(x-3) &= 4 \\ x=2, &\qquad x-3=2 \\ x=2, &\qquad x=5 \end{aligned}$$

Another student says "you made a mistake in the second line, you shouldn't have moved the 4 over." Did the first student make a mistake and where is it?

Solving equations graphically A student is trying to solve the equation $x^2 = 2x + 3$. Her mother suggestions graphing the equations $y = x^2$ and $y = 2x + 3$ and finding the x-coordinates of the points of intersection. The students says “why are those the solutions?” Give an answer which explains the mathematical reasoning in terms understandable by a high school student who has taken a CCSS-based course in middle grades.