Functions, Grade 8 and High School

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The work of teaching functions

Functions are general purpose tools that pervade all of mathematics, and hence teachers are faced with helping students build and use different kinds of functions in every course and in every mathematical area, from algebra to geometry to probability to calculus.

State standards and published curricula have concentrated on helping students interpret and use standard types of functions defined on the real numbers—linear, quadratic, exponential, and trigonometric. This involved using function notation and purposely transforming formulas that define functions to reveal certain information—extrema, end behavior, and so on.

The Common Core State Standards (CCSS) keep attention on these skills but adds explicit standards that require students to *build functions of their own* to model some situation or to express some perceived regularity. Teachers know that, for most students (and for most adults, as well), creating functions for a purpose is much harder than using "stock" functions. For example, many students who can compute the cost of a text messaging plan for a fixed number of messages find it very difficult to create an algebraic expression that defines a function that will do the calculation for *any* number of messages. Helping students develop the skill of expressing general relationships with functions is intimately tied up with the mathematical practice that CCSS calls expressing regularity in repeated reasoning.

Student-created functions are often not conceived in standard algebraic forms—they are often expressed as incomplete recurrences or as a mix of formal algebra and verbal descriptions. The work of teaching functions requires one to find ways to help students refine and make precise what they are trying to express about the regularity they see in a given situation. Once again, one of the CCSS standards for practice comes into play: the habit of using precise language to express ideas is useful when trying to shoe-horn an insight into a mathematical formalism.

Another aspect of this work is to help students become at home with the many ways to represent functions. The most common of these representations in high school is the Cartesian graph. There are all kinds of issues to deal

with in this representation. One of the most thorny is to understand the effect of a linear change of variable on the corresponding graph: a common misconception is that the graph of $y = f(x-3)$ is obtained from the graph of $y = f(x)$ by a translation 3 units to the left. Overcoming this misconception takes considerable teaching skill.

In many applications, functions play a two-fold role: they express relationships between mathematical objects, and they are mathematical objects in their own right. The standard example of this is in calculus where, for example, ordinary functions become inputs to the operation of taking the derivative, which may be viewed as a function in its own right, whose inputs and outputs are themselves functions. But the idea shows up earlier—as soon as one talks about equality or composition of two functions (as in geometry, for example, when transformations are composed). It's a difficult task to help students understand this reification of functions.

Key understandings to support this work

There are two kinds of background supports that help teachers carry out this work: epistemological and mathematical.

The theories behind the abstracting regularity habit are useful when thinking about how the function concept develops in learners. This approach has its roots in the genetic epistemology of Piaget. Several researchers have have identified landmarks on the continuum of the levels of abstraction students exhibit in their use of functions, calling them the *action*, *process*, and *object* concepts of function. Briefly,

- Students who have an *action* concept of function see a function as a set of isolated calculations. The output of the function $x \mapsto 3x+2$ is calculated by performing the multiplication, writing down the answer, and then adding 2 to that. Students who have this point of view make extensive use of the "=" button on their calculator, obtaining several partial results on the way to doing a calculation.

- As students repeatedly perform a sequence of calculations, they begin to chunk the individual steps together into coherent and self contained sequences or networks of calculations. They perform chain calculations on their calculators (using the = key only when absolutely necessary)

- As processes are further interiorized and students further suppress the details of the actual calculations that take place, students can begin to *encapsulate* processes into *objects*, manipulating them as data (this is very much in the spirit of how young children develop the concept of an integer by encapsulating the counting process).

These levels of abstraction are markers on a continuum rather than discrete stages. The action-process-object dimension is not the only one that can be

used to analyze the function concept. Another is to look at how functions are used in mathematics and at what people do with them in actual mathematical contexts:

- **Function as continuous change**. Classical analysis and physics were invented to study situations that vary continuously. Although continuity makes sense in more general situations, in school mathematics this point of view makes implicit use of the topology of the real line.
- **Function as algorithm**. Algorithms, in the sense of a sequence of explicit instructions for transforming one object into another, play an important role in many parts of algebra and number theory. Many highly theoretical investigations in algebra begin with a careful analysis of patterns that emerge from algorithmic calculations.
- **Function as mapping**. In combinatorics, it is quite common to set up an abstract correspondence between two sets in order to analyze cardinality questions. The functions under consideration are just pairings, and they are usually represented by arrow diagrams or by sets of ordered pairs.

Teacher preparation and professional development programs often don't make these ideas explicit, but, with minor modifications, they could. For example, in number theory, Eulcid's algorithm can be viewed as a recursively defined function of two variables. Another example: in Galois theory, automorphisms of a field are both functions and elements of a group.

Illustrative examples

Function as continuous change; domain; range. Little congruent squares are cut out of the corners of a 5×7 rectangle, and the sides are folded up to make an open box. The volume of the box is a continuous function of the side-length of the cutout, and before they write down an explicit expression for the volume of the box, teachers can ask students some qualitative questions about the variation: what are the limits on the domain? Must there be a maximum volume? A minimum? Where might these extreme values occur?

Function as algorithm; domain; equal functions. Given an input-output table (say, with inputs 0, 1, 2, 3 ,4) that can be matched by a linear function f defined by $f(n) = 5n+2$, many students see a pattern that can be modeled by a recursively defined algorithm: The output at n is obtained from the output at $n-1$ by adding 5. A complete model is

$$g(n) = \begin{cases} 2 & n = 0 \\ g(n-1) + 5 & n > 0 \end{cases}$$

Teachers can use these two models as a jumping off point for several productive discussions: Does $f = g$? Does $f = g$ on all non-negative integers? How can you be sure?

Function as mapping; function as object. What's the probability that, in a room of 25 people, two have the same birthday? There are many ways to think about this, but one is to count functions. A room-birthday sample can be represented by a mapping from the set $\{1, 2, \ldots, 25\}$ to the set $\{1, 2, \ldots, 365\}$. Combinatorial reasoning shows that there are 365^{25} such pairings. A double birthday occurs in the pairings that are not one-to-one. It's not easy to count these, but students can count the functions that *are* one-to-one: there are

$$365 \cdot (365 - 1) \cdot (365 - 2) \cdot \cdots \cdot (365 - 24)$$

of these. Hence the number of non-one-to-one functions is 365^{25} minus this, and the desired probability is

$$\frac{365^{25} - 365 \cdot (365 - 1) \cdot (365 - 2) \cdot \cdots \cdot (365 - 24)}{365^{25}} = 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{24}{365}\right)$$

Analysis of the structure of this expression, generalizations to rooms of other sizes, and numerical approximations can push this example quite far over several years in high school.