Chapters 1–6 footnotes with hyperlinks

The footnotes from Chapters 1–6 are listed below and hyperlinked (when possible) to the references cited.

Many of the documents cited are freely available. National Research Council reports such as *Adding It Up* can be read on-line. They can be downloaded without charge as can documents from the Conference Board of the Mathematical Sciences and the Council of Chief State School Officers. In some cases, cited portions of documents can be seen via Google Books.

Mathematics education research journal articles are likely to require a subscription. At many academic institutions, these journals will be accessible via institutional subscription. Attempts to access a JSTOR link without such a subscription will get the response "Cannot download the information you requested."

Chapter 1

1. An overview of the CCSS structure appears as Appendix B of this report.

2. The full text of these standards appears as Appendix C.

3. Between 2004 and 2008, the Park City Mathematics Study Group (a group of research mathematicians) conducted discussions of school mathematics, including extended discussions with NCTM representatives. *Principles and Standards* and *Adding It Up* (published in 2000 and 2001) summarize findings from previous decades of research in mathematics education.

4. Such connections are outlined in the Progressions for the CCSS (see the web resources for this report).

5. Examples are given by Ma, *Knowing and Teaching Elementary Mathematics*, Erlbaum, 1999: change in number, p. 74; change in manipulative and problem context, p. 5.

6. For a summary (p. 400) and further examples of teaching tasks, see Ball et al., "Content Knowledge for Teaching," *Journal of Teacher Education*, 2008; also Senk et al., "Knowledge of Future Primary Teachers for Teaching Mathematics: An International Comparative Study," *ZDM*, 2012, p. 310.

7. See, e.g., the findings of the Teacher Education and Development Study in Mathematics (TEDS-M).

8. These are intertwined and occur on a variety of levels. For example, the institutional arrangement of having teachers share a room affords the professional practice of discussing mathematics. An institutionalized career hierarchy based on teaching shapes the professional activities of Chinese master teachers and "super rank" teachers described in *The Teacher Development Continuum*

in the United States and China, National Academies Press, 2010. In Japan, institutional arrangements afford the practice of "lesson study," allowing teachers to communicate with other teachers in their school or district, and with policy-makers (see Lewis, *Lesson Study*, Research for Better Schools, 2002, pp. 20–22).

9. Chapter 2 discusses this claim further, but note the findings of *Effects of Teacher Professional Development on Gains in Student Achievement*, Council of Chief State School Officers, 2009. Most successful professional development programs continued for 6 months or more, and the mean contact time with teachers was 91 hours.

10. For example, the Mathematics Common Core Coalition (comprised of professional societies and assessment consortia) addresses educators, teachers, teacher leaders, supervisors, administrators, governors and their staffs, other policy-makers, and parents.

11. The CBMS surveys (conducted every five years) consistently document large proportions of undergraduates enrolled in remedial mathematics courses (see, e.g., Table S.2 of the 2005 report).

12. The 2005 CBMS survey suggests that many mathematics departments do not have courses especially designed for elementary teachers (see Table SP.6). In 2010, Masingila et al. surveyed 1,926 U.S. higher education institutions that prepared elementary teachers. Of those who responded (43%), about half (54%) reported that requirements included two mathematics courses designed for teachers. See "Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey," *Journal of Mathematics Teacher Education*, 2012, Table 2. A more detailed picture for three states is presented by McCrory & Cannata, "Mathematics Classes for Future Elementary Teachers: Data from Mathematics Departments," *Notices of the American Mathematical Society*, 2011.

13. Chapter 2 gives an overview of teaching–learning paths.

14. In Masingila et al.'s survey less than half of respondents reported giving training or support to instructors of mathematics courses for elementary teachers.

15. For example, when surveyed in 2000, 86% of K–4 teachers reported studying mathematics for less than 35 hours over a period of three years, an average of less than 12 hours per year. See Horizon Research's 2000 National Survey of Science and Mathematics Education. More recent studies show large increases in elementary student mathematics achievement when their teachers receive content-based professional development. Student scores of teachers who do not receive such professional development do not show these gains (see the sections on curriculum-specific professional development in Chapter 2 and on mathematics specialists in Chapter 4). Thus, unsatisfactory student performance may suggest a greater need for content-based professional development.

16. The Association for Middle Level Education (AMLE) position statement notes, “in some states, virtually anyone with any kind of degree or licensure is permitted to teach young adolescents.” According to the AMLE web site, 28 states and the District of Columbia offer separate licenses for middle grades generalists. Separate licenses, however, do not necessarily imply the existence of separate preparation programs or different mathematics requirements. The 2005 CBMS survey found that 56% of mathematics departments at four-year institutions had the same mathematics requirements for K–8 certification in early and later grades (see Table SP.5). See also the discussion of opportunity to learn for U.S. prospective lower secondary teachers in Tatto & Senk, “The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics,” *Journal of Mathematics Teacher Education*, 2011, p. 127.

Chapter 2

1. These were: *Einleitung zur Rechen-Kunst* (*Introduction to the Art of Reckoning*), St Petersburg (vol. 1, 1738, vol. 2, 1740); *The Elements of Arithmetic*, London, 1830.

2. Hodgson points out that “one could even see the ICMI as having been formed on the very assumption that university mathematicians should have an influence on school mathematics.” See *The Teaching and Learning of Mathematics at University Level*, Kluwer, 2001, p. 503.

3. Murray discusses the polarization of teaching and research within the U.S. mathematical community in *Women Becoming Mathematicians: Creating a Professional Identity in Post–World War II America*, MIT Press, 2000, pp. 6–10. For examples of U.S. mathematician involvement (e.g., the founding of the International Commission on the Teaching of Mathematics (later ICMI) at the International Congress of Mathematicians) and social context of its diminution, see Donoghue, “The Emergence of a Profession: Mathematics Education in the United States, 1890–1920,” in *A History of School Mathematics,* vol. 1, NCTM, 2003. Changes in twentieth-century psychology research were also a factor, see Roberts, “E. H. Moore's Early Twentieth-Century Program for Reform in Mathematics Education,” *American Mathematical Monthly*, 2001.

4. *Teaching Teachers Mathematics* (Mathematical Sciences Research Institute, 2009) gives an overview of past and recent counterexamples.

5. In 2010, Masingila et al. surveyed 1,926 U.S. higher education institutions that prepared elementary teachers. Of those who responded (43%), less than half reported giving training or support for instructors of mathematics courses for elementary teachers. However, the authors write that “there appears to be interest in training and support as a number of survey respondents contacted us to ask where they could find resources for teaching these courses.” See “Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey,” *Journal of Mathematics Teacher Education*, 2012.

6. Quoted from Schoenfeld, "Learning to Think Mathematically" in *Handbook for Research on Mathematics Teaching and Learning*, 1992, p. 359. Note that these beliefs may not be explicitly stated as survey or interview responses, but displayed as classroom behaviors, e.g., giving up if a problem is not quickly solved. This discussion is not meant to exclude the possibility of exceptional mathematical talent, but focuses on the idea that K–12 mathematics can be learned in its absence.

7. Stevenson and Stigler documented similar beliefs among U.S. first and fifth graders, and their mothers, but found that their Japanese and Chinese counterparts focused more on effort rather than ability. See Chapter 5 of *The Learning Gap*, Simon & Schuster, 1992. See also *Data Compendium for the NAEP 1992 Mathematics Assessment for the Nation and the States*, National Center for Educational Statistics, 1993.

8. Note that such beliefs may vary according to domain, e.g., one may believe in a "math gene," but favor continued practice in order to improve sports performance.

9. For a brief overview of research in this area, including classroom studies, see Dweck, "Mind-sets and Equitable Education," *Principal Leadership*, 2010. For a review of research and recommendations for classroom practice, see *Encouraging Girls in Math and Science* (IES Practice Guide, NCER 2007-2003), Institute of Educational Sciences, 2007, pp. 11–13.

10. See Michalowicz & Howard, "An Analysis of Mathematics Texts from the Nineteenth Century" in *A History of School Mathematics*, vol. 1, NCTM, 2003, especially pp. 82–83.

11. Lambdin & Walcott, "Changes through the Years: Connections between Psychological Learning Theories and the School Mathematics Curriculum," *The Learning of Mathematics*, 69th Yearbook, NCTM, 2007. For discussion of current practices, see Ma, "Three Approaches to One-Place Addition and Subtraction: Counting Strategies, Memorized Facts, and Thinking Tools," unpublished.

12. This is a slight reformulation of Lampert, 1990 as quoted by Schoenfeld, "Learning to Think Mathematically" in *Handbook for Research on Mathematics Teaching and Learning,* 1992, p. 359. The surrounding text discusses research on school experiences that shape such beliefs.

13. For example, see Hiebert et al.'s study of eighth grade classrooms, *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study*, U.S. Department of Education, 2003.

14. See analyses of data from the TIMSS video studies of 1999 (Hiebert et al., pp. 73–75) and of 1995 (Manaster, *American Mathematical Monthly*, 1998).

15. Schmidt and Houang analyzed the content and sequencing of topics in grades 1–8 in the U.S. and other countries. See “Lack of Focus in the Mathematics Curriculum,” in *Lessons Learned*, Brookings Institution Press, 2007, p. 66. Examples of treatments of fractions and negative numbers that do not afford deductive reasoning are given by Wu in “Phoenix Rising,” *American Educator*, 2011.

16. For example, middle grades and high school teachers who participated in an MSP based on an immersion approach (involving intensive sessions of doing mathematics) reported changes in beliefs that affected their teaching, e.g., communicating that it is “OK” to struggle. See ,” *Focus on Mathematics Summative Evaluation Report 2009*, p. 73. Gains in student test scores are shown on p. 93 (high school) and p. 96 (middle grades).

17. For a snapshot from one such collaboration, see *Teaching Teachers Mathematics*, Mathematical Sciences Research Institute, 2009, p. 34; for descriptions of three Math Science Partnerships, see pp. 32–41.

18. Test quality can be a major limitation for this measure. An analysis of state mathematics tests found low levels of cognitive demand, e.g., questions that asked for recall or performance of simple algorithms, rather than complex reasoning over an extended period. See Hyde et al., “Gender Similarities Characterize Math Performance,” *Science,* 2008, 494–495.

19. See *Preparing Teachers: Building Evidence for Sound Policy*, National Research Council, 2010, p. 112. See also, Telese, “Middle School Mathematics Teachers’ Professional Development and Student Achievement,” *Journal of Educational Research*, 2012. Telese’s measure of student achievement was the Grade 8 National Assessment of Educational Progress, which includes items with a high level of cognitive demand. It found number of mathematics courses to be a strong predictor, but like many such studies, it did not have an experimental or quasi-experimental design.

20. Shulman, “Those Who Understand: Knowledge Growth in Teaching,” *Educational Researcher*, 1986.

21. On average, the prospective secondary teachers had taken over 9 college-level mathematics courses. Ball, “Prospective Elementary and Secondary Teachers' Understanding of Division,” *Journal for Research in Mathematics Education*, 1990.

22. Hill et al., “Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement,” *American Educational Research Journal*, 2005.

23. Cohen & Hill, “Instructional Policy and Classroom Performance: The Mathematics Reform in California,” *Teachers College Record*, 2000.

24. Blank & Atlas, Effects of Teacher Professional Development on Gains in Student Achievement, Council of Chief State School Officers, 2009.

25. Perry & Lewis, *Improving the Mathematical Content Base of Lesson Study: Summary of Results*, 2011.

26. Example from Ma, *Knowing and Teaching Elementary Mathematics*, Erlbaum, 1999, p. 15. Similar examples occur in other East Asian countries. Lewis et al. describe how Japanese teacher's manuals may support teachers' perceptions of paths in "Using Japanese Curriculum Materials to Support Lesson Study Outside Japan: Toward Coherent Curriculum," *Educational Studies in Japan: International Yearbook*, 2011.

27. Sarama & Clements, *Early Childhood Mathematics Education Research*, Routledge, 2009, pp. 352–363.

28. See special issue on learning trajectories, *Mathematical Thinking and Learning*, 2004.

29. See Duncan et al., "School Readiness and Later Achievement," *Developmental Psychology*, 2007; Claessens et al., "Kindergarten Skills and Fifth-grade Achievement: Evidence from the ECLS-K," *Economics of Education Review*, 2009; Siegler et al., "Early Predictors of High School Mathematics Achievement," *Psychological Science*, 2012. These studies examined large longitudinal data sets from the U.S. and other countries.

30. *Preparing Teachers: Building Sound Evidence for Sound Policy*, National Research Council, 2010, pp. 114–115.

31. This is made explicit for early childhood educators in *Mathematics Learning in Early Childhood*, National Research Council, 2009, pp. 3–4.

32. Ingersoll & Merrill, "Who's Teaching Our Children?," *Educational Leadership*, 2010.

33. Ingersoll & Perda, "Is the Supply of Mathematics and Science Teachers Sufficient?," *American Educational Research Journal*, 2010.

34. Ingersoll & May, "The Magnitude, Destinations, and Determinants of Mathematics and Science Teacher Turnover," Consortium for Policy Research in Education, 2010, pp. 44, 46.

35. *Preparing Teachers: Building Sound Evidence for Sound Policy*, National Research Council, 2010, pp. 34–39.

36. Ingersoll & Merrill, "Retaining Teachers: How Preparation Matters," *Educational Leadership*, 2012. See also Darling-Hammond, *Solving the Dilemmas of Teacher Supply, Demand, and Standards*, National Commission on Teaching and America's Future, 2000, pp. 17–19; *Tenth Anniversary Report*, UTeach, 2010, p. 16.

37. CAEP was formed by the merger of the National Council for the Accreditation of Teacher Education (NCATE) and the Teacher Education Accreditation Council (TEAC). Two of the MET II writers are engaged in the development of the CAEP standards.

38. *Key State Education Policies on PK–12 Education: 2008*, Council of Chief State School Officers, p. 22.

39. CBMS 2005 Survey, Table SP.3.

40. *National Impact Report: Math and Science Partnership Program*, National Science Foundation, 2010, p. 15.

41. In addition to the forthcoming CAEP standards, note the 2012 report *Supporting Implementation of the Common Core State Standards for Mathematics: Recommendations for Professional Development*, Friday Institute for Educational Innovation at the North Carolina State University College of Education.

42. For an overview of MSP outcomes, including increases in student achievement, see *National Impact Report: Math and Science Partnership Program*, National Science Foundation, 2010, pp. 6, 10–12.

Chapter 3

1. The recommendations for teacher preparation in this report are formulated in terms of courses and semester-hours, but this is not meant to exclude other ways of awarding credit or organizing teacher education. For example, collegiate institutions that do not follow a semester system with most courses earning 3 credit-hours will need to adapt these recommendations accordingly.

2. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons. Math teachers' circles focus primarily on giving teachers an experience to be learners and doers of mathematics. See the web resources for further information and examples.

3. See Recommendation 13 of *National Task Force on Teacher Education in Physics: Report Synopsis*, American Association of Physics Teachers, the American Physical Society, & the American Institute of Physics, 2010.

4. In the 2005 CBMS survey, special courses for K–8 teachers were offered by 11% of Ph.D.-granting and 33% of M.A.-granting statistics departments. Less than 0.5% of statistics departments surveyed reported that special sections of regular courses were designated for K–8 teachers. See Table SP.3.

5. For an overview, see *Key State Education Policies on PK–12 Education: 2008*, Council of Chief State School Officers, p. 22.

Chapter 4

Note that the MET II web resources at www.cbmsweb.org give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

1. As noted in Chapter 3, "Although elementary certification in most states is still a K–6 and, in some states, a K–8 certification, state education departments and accreditation associations are urged to require all grade 5–8 teachers of mathematics to satisfy the 24-hour requirement recommended by this report." Chapters 4 and 5 allow for a period of transition.

2. For example, "It is widely assumed—some would claim common sense—that teachers must know the mathematical content they teach" (*Foundations for Success: Reports of Task Groups of the National Mathematics Advisory Panel*, 2008, p. 5-6). "Aspiring elementary teachers must begin to acquire a deep conceptual knowledge of the mathematics that they will one day need to teach, moving well beyond mere procedural understanding" (*No Common Denominator*, 2008, National Council on Teacher Quality). "Mathematics courses for future teachers should develop 'deep understanding' of mathematics, particularly of the mathematics taught in schools at their chosen grade level" (Curriculum Foundations Project, 2001, Mathematical Association of America). See also *Preparing Teachers: Building Sound Evidence for Sound Policy*, 2010, National Research Council, p. 123.

3. An international comparison of prospective elementary teachers found that 48% of the U.S. teachers did not score above "Anchor point 2." Teachers with this score often had trouble reasoning about factors, multiples, and percentages. See Tatto & Senk, "“The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics," *Journal of Mathematics Teacher Education*, 2011, pp. 129–130. *Preparing Teachers* discusses concern about the adequacy of current teacher

preparation in mathematics, especially for elementary teachers. See Chapter 6, especially p. 124.

4. See the National Research Council report *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (2009) and the Counting and Cardinality Progression.

5. See the CCSS, pp. 88–89; or the Operations and Algebraic Thinking Progression for details and examples of situation and solution equations.

6. For examples of how teachers may construe the base-ten system, see Thanheiser, "Pre-service Elementary School Teachers' Conceptions of Multidigit Whole Numbers," *Journal for Research in Mathematics Education*, 2009.

7. Beckmann, "The Community of Math Teachers, from Elementary School to Graduate School," *Notices of the American Mathematical Society*, 2011.

8. For instance, a study of prospective elementary and secondary teachers found that many either did not know that division by 0 was undefined or were unable to explain why it was undefined. On average, the secondary teachers had taken over 9 college-level mathematics courses. Ball, "Prospective Elementary and Secondary Teachers' Understanding of Division," *Journal for Research in Mathematics Education*, 1990.

9. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons.

10. See this chapter's section on mathematics specialists for more discussion about their roles in professional development for teachers.

11. See, e.g., discussion of the use and organization of the blackboard in Lewis, *Lesson Study*, Research for Better Schools, 2002, pp. 97–98.

12. "Productive disposition" is discussed in the National Research Council report *Adding It Up*, pp. 116–117, 131–133.

13. See discussion of support in Masingila et al., "Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey," *Journal of Mathematics Teacher Education*, 2012.

14. See, e.g., Schoenfeld, "Working with Schools: The Story of a Mathematics Education Collaboration," *American Mathematical Monthly*, 2009, p. 202.

15. See Fennell, "We Need Elementary Mathematics Specialists Now, More Than Ever: A Historical Perspective and Call to Action," *National Council of Supervisors of Mathematics Journal*, 2011.

16. In general, a math specialist's roles and responsibilities are not analogous to those of a reading specialist.

17. Examples include the Vermont Mathematics Initiative (a Math Science Partnership), see *Teaching Teachers Mathematics*, Mathematical Sciences Research Institute, 2009, pp. 36–38. A 3-year randomized study in Virginia found that specialists' coaching of teachers had a significant positive effect on student achievement in grades 3–5. The specialists studied completed a mathematics program designed by the Virginia Mathematics and Science Coalition (also a Math Science Partnership) and the findings should not be generalized to specialists with less expertise. See Campbell & Malkus, "The Impact of Elementary Mathematics Coaches on Student Achievement," *Elementary School Journal*, 2011.

18. *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs*, Association of Mathematics Teacher Educators, 2009.

19. These examples come from Lee & Ginsburg, "Early Childhood Teachers' Misconceptions about Mathematics Education for Young Children in the United States," *Australasian Journal of Early Childhood*, 2009. This article summarizes research in this area and discusses possible sources of such beliefs.

20. See *Mathematics Learning in Early Childhood,* National Research Council, 2009, pp. 341–343.

Chapter 5

Note that the MET II web resources at www.cbmsweb.org give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

1. As noted in Chapter 3, "Although elementary certification in most states is still a K–6 and, in some states, a K–8 certification, state education departments and accreditation associations are urged to require all grade 5–8 teachers of mathematics to satisfy the 24-hour requirement recommended by this report." Chapters 4 and 5 allow for a period of transition.

2. See the listing at the Association for Middle Level Education web site.

3. See, e.g., Tatto & Senk, "The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics," *Journal of Mathematics Teacher Education*, 2011, p. 127; *Report of the 2000 National Survey of Mathematics and Science Education*, Horizon Research, p. 16.

4. See the Ratio and Proportion Progression for further details, including examples of double number lines and tape diagrams, and discussion of unit rates. In the CCSS, "fractions" refers to non-negative rational numbers in grades 3–5. Note that distinctions made in the CCSS between fractions, ratios, and rates may be unfamiliar to teachers.

5. For descriptions of multiplication and division problem types, see the CCSS, p. 89 or the Operations and Algebraic Thinking Progression.

6. "I don't think it's equal because I think that would be confusing to kids to say that 99 cents can be rounded up to a dollar" and other examples of conceptions that teachers may hold about this equation are given in Yopp et al., "Why It is Important for In-service Elementary Mathematics Teachers to Understand the Equality .999. . . = 1," *Journal of Mathematical Behavior*, 2008. Note that undergraduates may use decimal notation in ways that suggest notions of nonstandard analysis, see Ely, "Nonstandard Student Conceptions About Infinitesimals," *Journal for Research in Mathematics Education*, 2010.

7. It is important for middle grades teachers to have an elementary teacher's perspective on this content because they may need to provide support and instruction for students who have not yet achieved proficiency.

8. Note that the CCSS use the term "number line diagram" instead of "number line."

9. Examples of these representations occur in the Progressions for the CCSS.

10. See the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A PreK–12 Curriculum Framework* of the American Statistical Association.

11. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons.

12. Math teachers' circles and immersion experiences focus primarily on giving teachers an experience to be learners and doers of mathematics. See Chapter 6 and the web resources for further information and examples.

13. See, e.g., discussion of the use and organization of the blackboard in Lewis, *Lesson Study*, Research for Better Schools, 2002, pp. 97–98.

14. See Kidwell et al., *Tools for Teaching Mathematics in the United States, 1800–2000*, Johns Hopkins University Press, 2008.

Chapter 6

Note that the MET II web resources at www.cbmsweb.org give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

1. Translation of the third edition, Macmillan, 1932, p. 1.

2. This line of research and its limitations are discussed in more detail in Chapter 2.

3. "On the Education of Mathematics Majors" in *Contemporary Issues in Mathematics Education*, Mathematical Sciences Research Institute, 1999, p. 13.

4. See Recommendation 13 of *National Task Force on Teacher Education in Physics: Report Synopsis*, American Association of Physics Teachers, the American Physical Society, & the American Institute of Physics, 2010.

5. For examples, see Wu, "Phoenix Rising," *American Educator*, 2011.

6. The CCSS standards for high school include standards marked with a +, indicating standards that are beyond the college- and career-ready threshold.

7. From a modern viewpoint, this is an application, but the notion of group arose in this context. See Grattan-Guinness's discussion of "irresolving the quintic" in *The Rainbow of Mathematics: A History of the Mathematical Sciences*, Norton, 1997.

8. See, e.g., Howe, "The Secret Life of the $ax + b$ Group" in the web resources.

9. These and other ideas are listed in Kleiner's "The Teaching of Abstract Algebra: An Historical Perspective" in *Learn From the Masters!*, MAA, 1995.

10. See Smith & Karpinski, *Hindu-Arabic Numerals*, Ginn and Company, 1911, pp. 136–137.

11. This distinction is illustrated by x2 and x vs. sq and rt (or square and root).

12. For details of previous and subsequent notations, see Cajori, *A History of Mathematical Notations*, Dover, 1993. A similarity between base-ten notation and symbolic algebra is that they are "action notations" in which computations can occur, rather than "display notations" that only record results. See Kaput, "Democratizing Access to Calculus," *Mathematical Thinking and Problem Solving*, Erlbaum, 1994, p. 101.

13. This description is based on the University of California at Berkeley courses 151, 152, 153.

14. This description is based on the University of California, Santa Barbara courses 101A-B, 102A-B, 103.

15. Rotman's *Journey Through Mathematics* has been used for such a course.

16. Part of an eighth grade standard is: "Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane."

17. For example, see the reports of Focus on Mathematics (a Math Science Partnership). Comments from teachers include: "Study groups have made 'asking the next question' a much more intriguing mathematical exploration than I previously had imagined or realized I could access," *Focus on Mathematics Summative Evaluation Report 2009*, p. 29.