

## Chapter 3

### Recommendations for Elementary Teacher Preparation

Is elementary mathematics so simple that teaching it requires knowing only the “math facts” and a handful of algorithms? The premise of this chapter and its elaboration in Part 2 is that, quite to the contrary, this early content is rich in important ideas. It is during their elementary years that young children begin to lay down those habits of reasoning upon which later achievement in mathematics will crucially depend. Thus, for example, it is unrealistic to expect students who failed to develop early an understanding of how to manipulate arithmetic expressions to later manipulate algebraic expressions with confidence. And those students who have never had experience with decomposing and recomposing shapes in their early education are unlikely to attach meaning to the succession of assertions in typical proofs in Euclidean geometry.

When the goal of instruction is to help children attain both computational proficiency and conceptual understanding, teaching elementary school mathematics can be intellectually challenging. Consider the following vignette from a third grade classroom:<sup>1</sup>

The children have been working on multiplication, exploring what the operation *means*—the kinds of situations it models—in addition to learning their multiplication facts. Now, as they approach multiplication of two-digit numbers, their teacher wants to identify the ideas they bring to this new topic. She gives her students a problem—*There were 64 teams at the beginning of the NCAA basketball tournament. With 5 players starting on each team, how many starting players were in the tournament?*—and her students offer a variety of solution methods:

Laurel: That would be  $64 \times 5$ . I use one 10 because I know  $5 \times 10 = 50$ . Then you do that six times. That's 30, I mean 300. Then you add 4 five times, which is 25, no 20. I added it all together and got 320.

Chris: 64 means  $60 + 4$ . So I did 60 five times, for 300. Then  $4 \times 5$  is 20, so the answer is 320.

Jack: I split 64 into four parts—[First, I did] 20, 20, and 20. I did each one separately:  $20 \times 5 = 100$ ,  $20 \times 5 = 100$ ,  $20 \times 5 = 100$ . Then the last part,  $4 \times 5$ , is 20. All together, 320.

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<sup>1</sup>This vignette has been drawn from an actual event. More detail is given in Schifter et al., 1999, pp. 82-86.

The teacher now begins to distribute manipulatives for the next activity, but decides, on the spur of the moment, to give a new problem: *We have 18 kids here today and each needs 12 tiles for the next activity. How can we figure out the number of tiles to give out?*

Josh: That would be  $18 \times 12$ , and I know  $10 \times 10$  is 100 and  $8 \times 2$  is 16, so if you add them together it would be  $100 + 16 = 116$ .

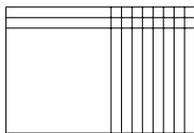
Dava: That's wrong. I did  $18 \times 10$  and got 180, but I thought at first I was wrong, so I double checked. I noticed that Josh didn't do  $8 \times 10$ , so my answer [for the sub-product,  $18 \times 10$ ] was right. I didn't do the 2 yet, so I do  $18 \times 2$ . Then you add it up— $180 + 36$ .

Presented with this scene from an elementary classroom, some readers may wonder why the teacher has solicited these children's ideas about how to multiply two-digit numbers before showing them the standard procedure. All the teacher really needs to do is take her students through the algorithm step by step—didn't we learn it that way? Relinquish, for a moment, memories of your own elementary-school experience and consider the opportunities for learning this classroom offers the children.

Implicit in the methods proposed by Laurel, Jack, Chris, and Dava, even Josh, are the associative and distributive properties and recognition of the flexibility gained by decomposing numbers into tens and ones. Thus, what appears to have been the initial step of Josh's strategy for multiplying  $18 \times 12$ —think of 18 as  $10 + 8$ , 12 as  $10 + 2$ —is sound reasoning. However, perhaps relying too exclusively on his understanding of additive relationships, his answer fails to account for all of the necessary sub-products. He has fallen victim to a mistake similar to one so common in college calculus classes that it has its own name, “the Freshman's Dream”: the belief that  $(a + b)(c + d) = ac + bd$ .

Now where is the teacher to go with all these ideas: compare the strategies of the children who got things right, explore Josh's procedure to see where he went wrong, or continue on to the next planned activity, perhaps later to come back to Josh and Dava? What should go into making such a decision, what must their teacher understand in order to work successfully with these children's ideas?

First, she must believe that mathematics is about ideas that make sense, rather than a collection of motiveless rules, and that her students have mathematical ideas that can be built upon; next, that there are many ways to solve a given problem. Then, she must be able to follow her students' thinking to determine which of the solution methods they propose are valid and identify the concepts upon which those methods are built. Too, she must recognize not only that Josh has made an error, but be able to subject the reasoning behind his error to investigation in a variety of ways. For example, what if 18 times 12 tiles were actually doled out? And how would this result compare to what an area representation of  $18 \times 12$  would show?



An area representation of  $18 \times 12$

More abstractly, how does the source of Josh's error connect to additive procedures, the distributive property, multiplication algorithms, multiplication of binomials, and so on? And how can these connections be expressed in a way that students can understand? A teacher who can formulate and weigh these questions for herself is in a position to decide whether to use Josh's error to further everyone's learning, because she knows how to do it.

Those who prepare prospective teachers need to recognize how intellectually rich elementary-level mathematics is. At the same time, they cannot assume that these aspiring teachers have ever been exposed to evidence that this is so. Indeed, among the obstacles to improved learning at the elementary level, not the least is that many teachers were convinced by their own schooling that mathematics is a succession of disparate facts, definitions, and computational procedures to be memorized piecemeal. As a consequence, they are ill-equipped to offer a different, more thoughtful kind of mathematics instruction to their students.

Yet, it is possible to break this cycle. College students with weak mathematics backgrounds can rekindle their own powers of mathematical thought. In fact, the first priority of preservice mathematics programs must be to help prospective elementary teachers do so: with classroom experiences in which *their* ideas for solving problems are elicited and taken seriously, their sound reasoning affirmed, and their missteps challenged in ways that help them make sense of their errors. Teachers able to cultivate good problem-solving skills among their students must, themselves, be problem solvers, aware that confusion and frustration are not signals to stop thinking, confident that with persistence they can work through to the satisfactions of new insight. They will have learned to notice patterns and think about whether and why these hold, posing their own questions and knowing what sorts of answers make sense. Developing these new mathematical habits means learning how to continue learning.

The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know—the mathematical ideas they hold, the skills they possess, and the contexts in which these are understood—so they can move from where they are to where they need to go. For their instructors, this requires learning to understand how their students think. The disciplinary habits of abstraction and deductive demonstration, characteristic of the way professional mathematicians present their work, have little to do with the ways each of us initially enters the world of mathematics, that is, experientially, building our concepts from action. And this is where mathematics courses for elementary school teachers must begin, first helping teachers make meaning for the mathematical objects under study—meaning that often was not present in their own elementary educations—and only then moving on to higher orders of generality and rigor.

The medium through which this ambitious agenda can be realized is the very mathematics these elementary teachers are responsible for—first and foremost, and still the heart of elementary content, number and operations; then, geometry, early algebraic thinking, and data, all of which are receiving increased emphasis in the elementary school curriculum.

This is not to say that prospective teachers will be learning the mathematics as if they were nine-year-olds. The understanding required of them includes acquiring a rich network of concepts extending into the content of higher grades; a strong

facility in making, following, and assessing mathematical argument; and a wide array of mathematical strategies.

Below is a summary of the major themes in the areas of number and operations, algebra, geometry, and data to be addressed in the three courses for elementary teachers recommended in Chapter 2. Chapter 7 in Part 2 of this document, built around a set of classroom scenes, illustrates some of the central topics of the elementary curriculum, considers the content teachers must know in order to successfully manage the mathematical issues these scenes raise (an elaboration of the points below), and offers examples of the insights and struggles of teachers learning this content.

### Number and Operations

To be prepared to teach arithmetic for understanding, elementary teachers, themselves, need to understand:

- A large repertoire of interpretations of addition, subtraction, multiplication and division, and of ways they can be applied.
- Place value: how place value permits efficient representation of whole numbers and finite decimals; that the value of each place is ten times larger than the value of the next place to the right; implications of this for ordering numbers, estimation, and approximation; the relative magnitude of numbers.
- Multidigit calculations, including standard algorithms, “mental math,” and non-standard methods commonly created by students: the reasoning behind the procedures, how the base-10 structure of number is used in these calculations.
- Concepts of integers and rationals: what integers and rationals (represented as fractions and decimals) are; a sense of their relative size; how operations on whole numbers extend to integers and rational numbers; and the behavior of units under the operations.

The study of number and operations provides opportunities for prospective teachers to create meaning for what many had only committed to memory but never really understood. It should begin by placing the mathematics in everyday contexts—e.g., comparing, joining, separating, sharing, and counting quantities that arise in one’s daily activities—and working with a variety of representations—e.g., number lines, area diagrams, and arrangements of physical objects. Instead of solving word problems by looking for “key words” or applying other superficial strategies, prospective teachers should learn to consider the actions the problems might posit. Learning to recognize that a single situation can be modeled by different operations opens up discussion of how the operations are related.

Future teachers must understand the conceptual underpinnings of the conventional computation algorithms as well as alternative procedures such as those commonly generated by children, themselves. (For example, in the vignette, Laurel, Chris, Jack, and Dava present various methods for calculating a product. An example of an addition calculation: a child explains her method for adding  $58 + 24$

is “Take 2 from the 24 and add it to the 58 to make 60;  $60 + 22 = 82$ .”) This process might begin by having teachers perform multidigit calculations mentally, without the aid of pencil and paper, to help loosen the hold of the belief that there is just one correct way to solve any mathematics problem. *As they become aware and then pursue their own ideas, they will recognize, often for the first time, that they do, indeed, have mathematical ideas worth following.* Similar exercises can be used to help teachers see how decimal notation allows for approximation of numbers by “round numbers” (multiples of powers of 10), facilitating mental arithmetic and approximate solutions.

Although most teachers are able to identify the ones place, the tens place, etc. and write numbers in expanded notation, they often lack understanding of core ideas related to place value. For example, future teachers should understand: how place value permits efficient representation of large numbers; how the operations of addition, multiplication, and exponentiation are used in representing numbers as “polynomials in 10”; and how decimal notation allows one to quickly determine which of two numbers is larger. Furthermore, they should be familiar with the notion of “order of magnitude.”

Having developed a variety of models of whole number operations, teachers are ready to consider how these ideas extend to integers and rational numbers. First they must develop an understanding of what these numbers are. For integers, this means recognizing that numbers now represent both magnitude and direction. And though most teachers know at least one interpretation of a fraction, they must learn many interpretations: as part of a whole, as an expression of division, as a point on the number line, as a rate, or as an operator. Teachers may have learned rules for comparing fractions, but now, equipped with a choice of representations, they can develop flexibility in determining relative size.

As with whole-number operations, placing operations with fractions in everyday contexts helps give meaning to algorithms hitherto regarded as mechanical devices. (Many college students see fractions only as pairs of natural numbers plugged into arithmetic procedures; hence, to them, adding two fractions is simply a computation with four integers.) Teachers must recognize that some generalizations often made by children about whole-number operations, e.g., a product is always larger than its factors (except when a factor is 0 or 1) and a quotient is always smaller than its dividend (unless the divisor is 1) no longer hold, and that the very meanings of multiplication and division must be extended beyond those derived from whole-number operations.

The idea of “unit”—that the same object can be represented by fractions of different values, depending on the reference whole—is central to work with fractions. In addition and subtraction, all the quantities refer to the same unit, but do not in multiplication and division.

Another area to be explored is the extension of place-value notation from whole numbers to finite decimals. Teachers must come to see that any real number can be approximated arbitrarily closely by a finite decimal, and they must recognize that the rules for calculating with decimals are essentially the same as those for whole numbers. Explorations of decimals lend themselves to work with calculators particularly well.

As with all of the content described in this document, the topics enumerated are not to be taught as discrete bits of mathematics. Always, the power comes from

connection—using the concepts and skills flexibly, recognizing them from a variety of perspectives as they are embedded in different contexts.

### Algebra and Functions

Although the study of algebra and functions generally begins at the upper-middle- or high-school levels, some core concepts and practices are accessible much earlier. If teachers are to cultivate the development of these ideas in their elementary classrooms, they, themselves, must understand those concepts and practices, including:

- Representing and justifying general arithmetic claims, using a variety of representations, algebraic notation among them; understanding different forms of argument and learning to devise deductive arguments.
- The power of algebraic notation: developing skill in using algebraic notation to represent calculation, express identities, and solve problems.
- Field axioms: recognizing commutativity, associativity, distributivity, identities, and inverses as properties of operations on a given domain; seeing computation algorithms as applications of particular axioms; appreciating that a small set of rules governs all of arithmetic.
- Functions: being able to read and create graphs of functions, formulas (in closed and recursive forms), and tables; studying the characteristics of particular classes of functions on integers.

Algebraic notation is an efficient means for representing properties of operations and relationships among them. In the elementary grades, well before they encounter that notation, children who are encouraged to recognize and articulate generalizations will become familiar with the sorts of ideas they will later express algebraically. In order to support children's learning in this realm, teachers first must do this work for themselves. Thus, they must come to recognize the centrality of generalization as a mathematical activity. In the context of number theory explorations (e.g., odd and even numbers, square numbers, factors), they can look for patterns, offer conjectures, and develop arguments for the generalizations they identify. And the arguments they propose become occasions for investigating different forms of justification. If, in this work, teachers learn to use a variety of modes of representation, including conventional algebraic symbols, the algebra they once experienced as the manipulation of opaque symbols can be invested with meaning.

Particularly instructive in work on word problems are comparisons of solution procedures using a variety of representations, illustrating how algebraic strategies mirror the actions modeled by other methods. As teachers become more confident of their skill in using algebra, they come to appreciate the advantages of its economy as against the cumbersomeness of other modes of representation, such as blocks or diagrams.

Although initially teachers' work in number and operations must be grounded experientially, now they are equipped to return to the study of computation, this

time to appreciate the algorithms on whole numbers, integers, or rationals as applications of commutativity, associativity, distributivity, identities, and (when it holds) inverses, the small set of rules governing all of arithmetic.

Especially important for teachers is recognition of how young children's work with patterns can be related to the concept of function—for example, that labeling the terms or units of a pattern by the natural numbers creates a function. As they pursue the study of functions, teachers learn to move fluently among descriptions of situations, tables of values, graphs, and formulas. And as they explore, they become familiar with certain elementary functions on integers: linear, quadratic, and exponential. They also learn to work with functions defined by physical phenomena, say, distance traveled by a runner over time, growth of a plant over time, or the times of sunrise and sunset over a year.

### Geometry and Measurement

For many years, the geometry curriculum for the elementary grades consisted of recognizing and naming basic two-dimensional shapes, measuring length with standard and non-standard units, and learning the formulas for the area and perimeter of a rectangle (and possibly a few other shapes). Because many students arrive in high-school geometry courses unprepared for its content, topics in geometry have recently been accorded a more prominent role in the curriculum of the lower grades. To most elementary teachers, their own encounter with high-school geometry notwithstanding, much of this material is new. In order to teach it to young children, they must develop competence in the following areas:

- Visualization skills: becoming familiar with projections, cross-sections, and decompositions of common two- and three-dimensional shapes; representing three-dimensional objects in two dimensions and constructing three-dimensional objects from two-dimensional representations.
- Basic shapes, their properties, and relationships among them: developing an understanding of angles, transformations (reflections, rotations, and translations), congruence and similarity.
- Communicating geometric ideas: learning technical vocabulary and understanding the role of mathematical definition.
- The process of measurement: understanding the idea of a unit and the need to select a unit appropriate to the attribute being measured, knowing the standard (English and metric) systems of units, understanding that measurements are approximate and that different units affect precision, being able to compare units and convert measurements from one unit to another.
- Length, area, and volume: seeing rectangles as arrays of squares, rectangular solids as arrays of cubes; recognizing the behavior of measure (length, area, and volume) under uniform dilations; devising area formulas for basic shapes; understanding the independence of perimeter and area, of surface area and volume.

A first goal in a geometry course for prospective teachers is the development of visualization skills—building and manipulating mental representations of two- and three-dimensional objects and perceiving objects from different perspectives. In exercises designed to cultivate these skills, teachers handle physical objects: build structures with cubes, create two-dimensional representations of three-dimensional objects, cut out and paste shapes. Such activities require that teachers work with projections and cross-sections, recognize rotations, reflections, and translations, and identify congruent parts.

From this work, teachers become familiar with basic two- and three-dimensional shapes: learn their names, learn to draw them, know their definitions and see how the shapes satisfy those definitions, recognize these shapes as parts of more complex configurations, and know some facts about them. They also develop different images of how shapes are composed: seeing a cube, say, as a stack of congruent squares or as an object whose surface unfolds into a net of six squares; or a tetrahedron as a stack of triangles decreasing in size or as an object whose surface unfolds into a net of four triangles.

In studying geometric shapes, teachers should cultivate technical vocabulary, developing an appreciation of the power of precise mathematical terminology as they work to communicate their ideas. Here, especially, the role of mathematical definition needs to be highlighted.

Prospective teachers are familiar with the concept of angle, but often only superficially. Teachers should understand the idea of angle, both as the figure formed by two rays sharing a vertex and as angular motion. They should understand that angles can be added, that the measure of the sum of angles is the sum of the measures (modulo  $2\pi$  or 360 degrees), and that the measures of the angles of a triangle sum to 180 degrees (a straight angle); and be able to prove that the measures of the angles of an  $n$ -gon sum to  $180(n - 2)$ .

Most prospective teachers understand the use of rulers, but few have had occasion to consider the conceptual issues involved in measurement. To measure an attribute, one must select a unit appropriate to that attribute, compare the unit to the object, and report the total number of units. Teachers should understand that measurements in the real world are approximations and that the unit used affects the precision of a measurement. They should be able to convert from one unit to another and be able to use the idea of conversions to estimate measure. In particular, teachers should know standard systems of units and approximate conversion rates from English to metric units and vice versa.

With regard to length, area, and volume, teachers should know what is meant by one, two, and three dimensions. (A common misunderstanding: perimeter is two dimensional since, after all, “the perimeter of a rectangle has both length and width.”) Many teachers who know the formula  $A = L \times W$  may have no grasp of how the linear units of a rectangle’s length and width are related to the units that measure its area or why multiplying linear dimensions yields the count of those units. An understanding of the volume of a rectangular solid involves seeing the relationship between layers of three-dimensional units and the area of its base. Formulas for the area and volume of some other kinds of objects can build from an understanding of rectangles and rectangular solids. The study of rectangles and rectangular solids can also lead to an understanding of how length, area, and volume change under uniform dilation.

### **Data Analysis, Statistics, and Probability**

Statistics is the study of data, and despite daily exposure to data in the media, most elementary teachers have little or no experience in this vitally important field. Thus, in addition to work on particular technical questions, they need to develop a sense of what the field is about. Prospective teachers need experience in:

- Designing data investigations: understanding the kinds of question that can be addressed by data, creating data sets, moving back and forth between the question (the purpose of the study) and its design.
- Describing data: understanding shape, spread, and center; using different forms of representation; comparing two sets of data.
- Drawing conclusions: choosing among representations and summary statistics to communicate conclusions, understanding variability, understanding some of the difficulties that arise in sampling and inference.
- Probability: making judgments under conditions of uncertainty, measuring likelihood, becoming familiar with the idea of randomness.

Teachers need to develop skill in the design and conduct of data investigations: to pose questions that can be addressed by data; design data collection procedures; collect and analyze those data; consider whether their initial questions have, indeed, been addressed; or, if necessary, revise both questions and data collection procedures and analyze the new data; and, finally, draw conclusions and communicate findings. Any of these steps can itself become an object of study. This includes understanding the kinds of questions that can be addressed by data; creating data sets; learning how to explore data through describing the shape, center, and spread of data distributions; and then using such descriptions to support conclusions. Teachers must have practice in analyzing the processes and causes of variability. In particular, they should understand that correlation does not imply causality.

In the early grades, children begin to explore the idea of making judgments under conditions of uncertainty; they talk about what is impossible or certain, more or less likely. Teachers should be able to extend these ideas to determine measures of likelihood: given equally likely outcomes, the probability of a particular event is equal to the ratio of the number of outcomes defined by the event to the number of total possible outcomes. A part of this study includes discussion of randomness and the difference between predicting individual events and predicting patterns of events.

### **Conclusion**

Too many students preparing for elementary teaching have been less than successful mathematics students, and even those with good grades often doubt their competence. Understandably, readers of this document may feel dismay at the prospect of working with such math-anxious, if not math-phobic, undergraduates.

However, those who work with them can testify that, once these prospective teachers experience their own capacities for mathematical thought, their anxiety is transformed into energy for learning.

In taking responsibility for the kind of instruction for elementary teachers envisaged here, mathematicians are invited, in effect, to re-enter the world of the naïve mathematical thinker. The recognition that the “unsophisticated” questions teachers pose do raise fundamental issues should inspire instructors to find contexts in which these can be addressed fruitfully. This means, at least initially, approaching the mathematics from a concrete and experientially based, rather than an abstract/deductive, direction.