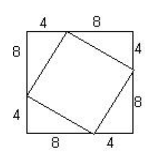
Chapter 5

Recommendations for High School Teacher Preparation

College and university programs for preparation of teachers often require mathematics coursework similar to that of a liberal arts mathematics major and education courses that emphasize teaching and learning of mathematics. Unfortunately, those programs do not appear to be attracting enough students to meet the national demand for new high school mathematics teachers. Furthermore, there is widespread concern that the mathematics courses in these programs do not provide prospective teachers with the depth and breadth of knowledge needed to teach high school mathematics well.

Recent recommendations for high school mathematics curricula and teaching demand even more of teacher preparation programs: that they provide their prospective teachers with knowledge of new mathematical topics, better understanding of the topics they will teach, and new teaching skills. In addition, future teachers need guidance in taking advantage of the increasingly sophisticated technological tools that permit more computationally involved applications and can give insights into theory. For example, computer software is now available that can be used to introduce high school students to discrete dynamical systems by iterating simple quadratic functions. The vignette that follows suggests the kind of mathematical breadth and depth that a high school teacher needs, illustrating new and longstanding concerns about teacher knowledge:[1]

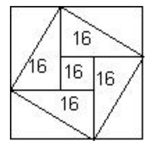
Ms. Liddell: The outside figure below is a 12 by 12 square. What is the area of the tilted square within it?



In response to this task two students came up with different answers. Reconciling their ideas led the class in quite unexpected directions.

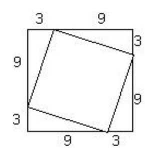
Julie: It looks to me as if the answer is 80. I wanted to see the rectangles—the ones the triangles are half of—so I drew the lines across and down like this:

[1]This vignette is based on material from Virginia Bastable's 1996 chapter "A Dialogue About Teaching."



Then I saw the tilted square was made up of four triangles and an even smaller square. The triangles are 16, which is half of 32. The little square is 4 by 4, so it is 16, too, for the four triangles and the little square, makes 80.

Bill: I got a different answer, but I see now that I didn't draw the figure right. I split the sides into sections of 3 and 9.



Alicia: Wouldn't the area be the same, anyway, no matter how you split up the sides?

Ms. Liddell: What do you think? If the point is moved from splitting the side of length 12 into sections of 4 and 8 to make other divisions, would the area of the tilted square stay the same?

While the students were considering that question, their teacher was thinking. She had to decide quickly whether to continue with her original plan—connecting the problem with irrational numbers—or to take advantage of the instructional opportunities raised by Bill's answer and Alicia's question: connecting the problem with the algebraic identity $(a+b)^2 = a^2 + 2ab + b^2$, asking her students for justification that the tilted figure is always a square, or to use a graphing calculator to explore the "tilted square area function" $y = 144 - 2(x)(12 - x)$.

In order to recognize the instructional opportunities described in the vignette, make a good decision, and implement it, a teacher needs a sound understanding of several areas of school mathematics and the connections among them. To recognize that Julie's diagram can be connected with $(a + b)^2 = a^2 + 2ab + b^2$ requires awareness of connections between algebra and geometry. To help students make this connection, a teacher needs the ability to identify particular correspondences between expressions and geometric objects. A teacher with an understanding of geometry and proof will notice that the implicit assertion in the problem statement that the inscribed figure is a square needs justification. A teacher with a knowledge of school curriculum will know if students have encountered geometric techniques which can be used for this justification. If students use a graphing calculator to explore the "tilted square area function" $y = 144 - 2(x)(12 - x)$, the teacher who recognizes that the graph of this function is a parabola, that it is symmetric, and that these facts have a geometric interpretation for this problem is better positioned to respond to the results of such explorations. All of these possibilities require a sound understanding of school mathematics, but an understanding that goes beyond the competence expected of a high school graduate. And, the need for this kind of understanding is not restricted to situations that require quick decisions. Teachers need a sound knowledge of mathematics to make curriculum decisions, plan lessons, and understand their students' work.

Current teacher preparation programs often do not explicitly focus on the kind of connections illustrated above for algebra, geometry, and functions—topics which are traditionally part of high school curricula—much less for newer topics such as data analysis or discrete mathematics. To meet these needs and to address the concerns discussed above, the education of prospective high school mathematics teachers should develop:

- Deep understanding of the fundamental mathematical ideas in grades 9–12 curricula and strong technical skills for application of those ideas.

- Knowledge of the mathematical understandings and skills that students acquire in their elementary and middle school experiences, and how they affect learning in high school.

- Knowledge of the mathematics that students are likely to encounter when they leave high school for collegiate study, vocational training or employment.

- Mathematical maturity and attitudes that will enable and encourage continued growth of knowledge in the subject and its teaching.

This report recommends two main ways that mathematics departments can attain these goals. First, core mathematics major courses can be redesigned to help future teachers make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching. Second, mathematics departments can support the design, development, and offering of a capstone course sequence for teachers in which conceptual difficulties, fundamental ideas and techniques of high school mathematics are examined from an advanced standpoint. Such a capstone sequence would be most effectively taught through a collaboration of faculty with primary expertise in mathematics and faculty with primary expertise in mathematics education and experience in high school teaching.

This chapter outlines the skills and understandings that prospective high school teachers should acquire in their mathematical educations. These objectives are presented in five sections that correspond to major areas of high school curricula—algebra and number theory, geometry and trigonometry, functions and analysis, statistics and probability, and discrete mathematics—with important connections indicated where appropriate. Each section suggests ways in which mathematics major courses can provide useful learning experiences for prospective high school teachers and the kind of material that would be appropriate in capstone courses specifically for teachers. Each point in this chapter is elaborated in Chapter 9 in Part 2 of this report.

Algebra and Number Theory

The algebra of polynomial and rational expressions, equations, and inequalities has long been the core of high school mathematics. Current school mathematics curricula connect algebra to topics in functions and analysis, discrete mathematics, mathematical modeling, and geometry. Graphing calculators, spreadsheets, and computer algebra systems can encourage and facilitate those connections.

To be well-prepared to teach such high school curricula, mathematics teachers need:

- Understanding of the properties of the natural, integer, rational, real, and complex number systems.
- Understanding of the ways that basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities.
- Understanding and skill in using algebra to model and reason about real-world situations.
- Ability to use algebraic reasoning effectively for problem solving and proof in number theory, geometry, discrete mathematics, and statistics.
- Understanding of ways to use graphing calculators, computer algebra systems, and spreadsheets to explore algebraic ideas and algebraic representations of information, and in solving problems.

Calculus and linear algebra courses provide an opportunity to give undergraduates extensive practice with algebraic manipulation. Making this an explicit goal for these courses helps to assure that future teachers have technical "know how" in high school algebra. Upper division courses in abstract algebra and number theory examine mathematical structures that are the foundation for number systems and algebraic operations. These courses should assure that future teachers "know why" the number systems and algebra operate as they do. Unfortunately, too many prospective high school teachers fail to understand connections between those advanced courses and the topics of school algebra.

Prospective teachers can be helped to make these connections in courses for mathematics majors and the capstone sequence. Both number theory and abstract algebra courses can be infused with tasks that ask for specific instances of these connections, for example, to show explicitly how the number and algebra operations of secondary school can be explained by more general principles. Assignments in a number theory course might ask for the use of unique factorization and the Euclidean Algorithm to justify familiar procedures for finding common multiples and common divisors of integers and polynomials. Assignments in an abstract algebra course might ask for each step in the solution of a linear or quadratic equation to be justified by a field property; and to show how each extension of the number system, from natural numbers through complex numbers, is accompanied by new properties. To make connections of collegiate and high school mathematics more natural, it also makes sense to place greater emphasis in the college courses on the ring, integral domain, and field structures that are fundamental in high school algebra.

Algebraic connections between high school and college courses can be an explicit focus of the capstone sequence for teachers. For example, this sequence could profitably examine the historical evolution of key concepts in number theory and algebra and it could trace the development of key number and algebra ideas from early secondary school through contemporary applications. It could examine the crucial role of algebra in use of computer tools like spreadsheets and the ways that computer algebra systems might be useful in exploring algebraic ideas. Each facet in such a capstone treatment of number and algebra would provide teachers with insight into the structure of high school mathematics, its uses in science and technology or in the workplace, and the conceptual difficulties in learning number and algebraic concepts.

Geometry and Trigonometry

High school geometry was once a year-long course of synthetic Euclidean plane geometry that emphasized logic and formal proof. Recently, many high school texts and teachers have adopted a mixture of formal and informal approaches to geometric content, de-emphasizing axiomatic developments of the subject and increasing attention to visualization and problem solving. Many schools use computer software to help students do geometric experiments—investigations of geometric objects that give rise to conjectures that can be addressed by formal proof. Some curricula approach Euclidean geometry by focusing primarily on transformations, coordinates, or vectors; and new applications of geometry to robotics and computer graphics illustrate how mathematics is used in the workplace in ways that are accessible and interesting to high school students.

To be well-prepared to teach the geometry recommended for high school, mathematics teachers need:

- Mastery of core concepts and principles of Euclidean geometry in the plane and space.
- Understanding of the nature of axiomatic reasoning and the role that it has played in the development of mathematics, and facility with proof.
- Understanding and facility with a variety of methods and associated concepts and representations, including transformations, coordinates, and vectors.
- Understanding of trigonometry from a geometric perspective and skill in using trigonometry to solve problems.
- Knowledge of some significant geometry topics and applications such as tiling, fractals, computer graphics, robotics, and visualization.
- Ability to use dynamic drawing tools to conduct geometric investigations emphasizing visualization, pattern recognition, conjecturing, and proof.

Current calculus and linear algebra courses for mathematics majors often give college students extensive experience with important geometric ideas and representations—especially Cartesian and polar coordinates, vectors, transformations, and trigonometry. Typical teacher preparation curricula require an advanced geometry course that examines Euclidean and rudimentary non-Euclidean geometry from an axiomatic point of view. This collegiate geometry course is an opportunity for teachers to deepen their understanding of Euclidean facts and principles and their skill in use of careful axiom-based reasoning. But future teachers should also be exposed to 20th-century developments in geometry.

This can be accomplished in two ways. First, the goals and objectives of standard college geometry courses can be reconsidered and greater emphasis given to modern approaches. For instance, the geometry of congruence and similarity can be developed from axioms about isometries and similitudes, and these developments can be connected to the algebra of matrices and complex numbers. Second, geometry course offerings can be extended to include examples of new topics and tools. For instance, many high school and college students are intrigued by artistic and scientific problems in computer graphics (which can be connected with matrix algebra) and by geometric investigations using dynamic drawing tools such as *Cabri Geometry* or *Geometer's Sketchpad* (leading to conjectures that require proof or disproof). Both of these topics emphasize the important role of visualization in mathematics and both provide opportunities for teachers to strengthen their skills in using coordinates and representations.

Knowledge of geometry for teaching can also be provided in the capstone sequence. For example, this sequence can explicitly trace the historical development of key ideas, identifying and exploring questions that will be as difficult for students as for the mathematicians who first encountered them. It can examine the interplay of exploration and proof. Such a course might also be an ideal venue for re-examination of trigonometric and closely related geometric ideas—the laws of sines and cosines, identities, the Pythagorean Theorem, and similarity—to assure that prospective teachers have the depth of understanding that is essential for effective instruction.

Functions and Analysis

The concept of function is one of the central ideas of pure and applied mathematics. For nearly a century, recommendations for school curricula have urged reorganization of school mathematics so that study of functions is a central theme. Computers and graphing calculators now make it easy to produce tables and graphs for functions, to construct formulas for functions that model patterns in experimental data, and to perform algebraic operations on functions.

Prospective high school mathematics teachers need to acquire deep understanding of the concept of a function and of the most important classes of functions (polynomial, exponential and logarithmic, rational, and periodic). For functions of one and two variables teachers should be able to:

- Recognize patterns in data that are modeled well by each important class of functions.

- Identify functions associated with relationships such as $f(xy) = f(x) + f(y)$ or $f'(x) = kf(x)$ or $f(x+k) = f(x)$.

- Recognize equations and formulas associated with each important class of functions and the way that parameters in these representations determine particular cases.

- Translate information from one representation (tables, graphs, or formulas) to another.

- Use functions to solve problems in calculus, linear algebra, geometry, statistics, and discrete mathematics.

- Use calculator and computer technology effectively to study individual functions and classes of related functions.

Undergraduate mathematics majors encounter functions in calculus, linear algebra, and various elective courses. However, most acquire only procedural facility in using formulas involving functions for calculations, not a deep understanding of functions and related concepts like limits or continuity. Thus, it is important for prospective teachers to revisit the elementary functions of high school mathematics from an advanced standpoint, in much the same way that they revisit algebraic and number system operations.

This sort of reflective look at functions and their unifying role in mathematics could be a prominent part of the capstone course sequence. The capstone study of functions might examine again the role of computers as computational and graphing tools in mathematical work. Such activities might lead to examination of relationships between explorations and proof—as well as experience in the kind of complex problem solving required by mathematical modeling.

In traditional college preparatory curricula, the primary goal is preparing students for study of calculus. Calculus is now commonly taught in advanced placement form at many U. S. high schools, so it is now even more important that prospective high school teachers gain an understanding of calculus that will allow them to make informed decisions about content and emphasis in preparatory courses. However, a beginning teacher is not expected to be well-prepared to teach an advanced placement course like calculus. Those high school mathematics teachers who will assume responsibility for teaching advanced placement calculus will need additional content beyond their initial preparation coursework.

Data Analysis, Statistics, and Probability

Over the past decades, statistics has emerged as a core strand of school and university curricula. The American Statistical Association's Quantitative Literacy project has encouraged inclusion of data-driven mathematics curriculum modules. The College Board's advanced placement examination in statistics is attracting a substantial number of high school students. The traditional school mathematics emphasis on probability has evolved to include more statistics, often in the context of using data analysis to gain insight into real-world situations.

Curricula for the mathematical preparation of high school teachers should include courses and experiences that help them appreciate and understand the major themes of statistics. Teachers need experience in:

- Exploring data: using a variety of standard techniques for organizing and displaying data in order to detect patterns and departures from patterns.
- Planning a study: using surveys to estimate population characteristics and designing experiments to test conjectured relationships among variables.
- Anticipating patterns: using theory and simulations to study probability distributions and apply them as models of real phenomena.
- Statistical inference: using probability models to draw conclusions from data and measure the uncertainty of those conclusions.
- Technology: using calculators and computers effectively in statistical practice.

Moreover, probability has important applications outside of statistics. Thus, prospective teachers should also:

- Understand basic concepts of probability such as conditional probability and independence, and develop skill in calculating probabilities associated with those concepts.

Despite the production of very interesting statistics materials for schools, it has been hard to find room for the subject in curricula dominated by preparation for calculus. Although the new advanced placement test in statistics may help to change this situation, at present prospective high school mathematics teachers commonly come to undergraduate study with very little prior work in statistics. Most programs for mathematics majors allow little time for statistics until, at best, an upper division elective. At that point, mathematics majors often find themselves thrust into a calculus-based mathematical statistics course, and are likely to miss many fundamental ideas and techniques that are at the heart of high school statistics and probability.

Because statistics is first and foremost about using data to inform thinking about real-world situations, it is critical that prospective teachers have realistic problem-solving experiences with statistics. Because modern statistical work depends on extensive calculations, the prospective teachers should also be able to use statistical computation software to organize, display, and analyze complex data sets. It is essential to carefully consider the important goals of statistical education in designing courses that reflect new conceptions of the subject. Such courses will be appropriate for most mathematics majors, as well as prospective teachers. There are recommendations from the MAA's Committee on the Undergraduate Program in Mathematics that provide helpful guidance in design of these courses.

Discrete Mathematics and Computer Science

The increasing application of mathematical methods in disciplines outside of the physical and engineering sciences has stimulated development and use of several key topics in discrete mathematics. Topics in discrete mathematics now appear in high school curricula. High school students have found them as accessible as some more traditional topics and many find their applications engaging.

High school mathematics teachers should get exposure to ideas, methods, and applications in the following areas:

- Graphs, trees, and networks.
- Enumerative combinatorics.
- Finite difference equations, iteration, and recursion.
- Models for social decision-making.

In each of these areas prospective teachers should have experience working on applied problems arising from real-world situations. Teachers should learn to reason effectively with mathematical induction, which is especially important in discrete mathematics. As in the case of statistics, most prospective teachers come to undergraduate study with very limited prior exposure to discrete mathematics.

The emerging importance of discrete mathematics is driven by the pervasive uses of digital computers and by problems involved in design of computer hardware, algorithms, and software. Computers are now used as problem-solving and learning tools throughout high school mathematics and other disciplines, but the underlying principles of computer science are seldom treated explicitly in any current high school mathematics course.

Prospective high school mathematics teachers commonly enter undergraduate studies with facility in using computers for a variety of information processing tasks, but they do not know much of the underlying theory. Teacher preparation programs commonly include a computer science requirement, though the nature of that requirement has been evolving as rapidly as the field of computer science itself. Future teacher preparation programs should include work in computer science and related mathematics such as:

- Discrete structures (sets, logic, relations, and functions) and their applications in design of data structures and programming.
- Design and analysis of algorithms, including use of recursion and combinatorics.
- Use of programming to solve problems.

Conclusion

The preceding recommendations outline a challenging agenda for the education of future high school mathematics teachers. Mathematics departments may not be able to meet this challenge by creating new courses required solely of prospective high school mathematics teachers. However, many of the recommendations above are probably as appropriate for all mathematics majors as they are for prospective teachers. All students in abstract algebra courses can profitably re-examine their knowledge of elementary algebra. Mathematics majors can gain insight into axiomatic methods and proof in a college geometry course, and most will be interested in newer topics recommended for inclusion in the geometry course for teachers. The basic themes of statistics, discrete mathematics, and computer science that have been suggested for inclusion are consistent with CUPM recommendations over the past two decades for all mathematics majors.

In addition to suggesting some new topics and emphases in existing mathematics courses, this report recommends creation of a new capstone course sequence aimed especially at future teachers. This sequence is an opportunity for prospective teachers to look deeply at fundamental ideas, to connect topics that often seem unrelated, and to further develop the habits of mind that define mathematical approaches to problems. By including the historical development of major concepts and examination of conceptual difficulties, this capstone sequence connects individual mathematics courses with school mathematics and contributes to the mathematical understanding and pedagogical skills of teachers.

In light of the current severe shortage of qualified high school mathematics teachers it might seem foolhardy to recommend preparation that is more ambitious than current standards. One might argue that teacher preparation should focus on very thorough grounding in a few core subjects. However, current high school curricula cannot be taught successfully by teachers with such limited preparation. As mathematics departments work to develop the kinds of courses needed to provide better preparation for future high school teachers, those efforts will also be useful in work with in-service teachers as well. In both cases, a teacher's preparation needs to be viewed as a foundation for a career of continuing professional development.