Chapter 8

The Preparation of Middle Grades Teachers

The mathematics needed by prospective middle grades[1] teachers encompasses the mathematics needed by teachers in the lower grades, but extended in several important ways to reflect the more sophisticated mathematics curriculum of the middle grades. Work with rational numbers and operations builds on earlier work with whole numbers and number operation knowledge. Concepts of symmetry and similarity depend on knowledge of shapes acquired in earlier grades. Developing a deeper understanding of measurement and new types of measures uses previously learned counting skills and their applications to finding areas and volumes of simple shapes. Graphing and interpreting both discrete and continuous data in a variety of ways follows graphing simple sets of discrete data in earlier grades. Middle grades teachers need to have a thorough understanding of the mathematics of the middle grades so that they can instill in their students the belief that they can make sense of the mathematics they are learning and the confidence to seek it. "Good mathematics learners expect to be able to make sense of the rules they are taught, and they apply some energy and time to the task of making sense" (Resnick, 1986, p. 191). Sense-making should be a theme in all courses for prospective teachers.

This chapter expands on the discussion in Chapter 4, and provides foundations and further explanation for the recommendations found there.

Teaching for Mathematical Reasoning in the Middle Grades

As middle grades students mature in their ability to undertake more complex mathematical learning, they are simultaneously developing their ability to reason more deeply about mathematics. Middle grades teachers need to have opportunities to come to understand the types of reasoning middle grades students are able to undertake, and then be able to challenge their students in ways that will lead them to reason and make sense of mathematics. They need to provide their students with opportunities to explore, conjecture, provide counterexamples, and justify—preparation for formal deductive reasoning in later mathematics classes. Reasoning and seeking understanding must be recognized by teachers as critical aspects of middle grades mathematics. This will not occur unless university mathematics instructors model for prospective teachers ways in which these aspects of mathematical learning can be commonplace in the mathematics classroom, and

[1]The term "middle grades" is used in this chapter, as in Chapter 4, to refer to Grades 5–8 and "secondary" to refer to Grades 9–12.

consciously make reasoning and understanding salient features of learning for their students.

The commonsense notion that students in the middle grades can perform deeper, more complex forms of reasoning than younger students is supported by the work of developmental psychologists (e.g., Case, 1985). Prospective teachers are frequently unaware of the developing ability of middle grades students to reason in progressively more complex ways. Yet mathematics is learned best if reasoning and problem solving play integral roles in the learning process. Teachers at all levels, including mathematics instructors of prospective teachers, need to understand the role of choosing appropriate tasks to further develop reasoning and problem-solving skills (Hiebert et al., 1997; Knapp et al., 1995; Stigler & Hiebert, 1999).

A very important form of mathematical reasoning that students should develop in the middle grades is the ability to reason about proportions, often referred to as proportional reasoning. It is therefore incumbent on teachers to understand how proportional reasoning develops and how this development can be promoted. Proportional reasoning has been called the "capstone of children's elementary school arithmetic" and the "cornerstone of all that is to follow" (Lesh, Post, & Behr, 1988, p. 94). Research has shown that children develop proportional reasoning in increasingly more sophisticated stages and that their development depends on their instruction (Lamon, 1995).

Students who are unable to reason proportionally will often approach proportion problems using an additive strategy (e.g., asked to predict the height of a tree in an enlargement of a photo in which a man is 2 inches tall and a tree 5 inches tall to one in which the man is 5 inches tall, will use the difference in the heights in the first photo to predict that the tree will be 8 inches tall in the second photo) rather than recognizing that the ratio of man to tree must remain constant. Progression from using only additive strategies (in this case the difference of two quantities) to recognizing and using both additive and multiplicative strategies appropriately is a hallmark of middle grades students' mathematical development. There are many situations in which multiplicative reasoning can be appropriately used, such as situations involving ratios, proportions, linear relationships, both multiplication and division and knowing when each operation is appropriate, compound units such as problems involving person-hours, and counting situations where the Fundamental Theorem of Arithmetic applies. One way of approaching difficulties with reasoning multiplicatively, which in many cases involves proportions, is to present a variety of situations in which a ratio can be appropriately used as an index of measure, such as in the photo problem. Prospective teachers' past introduction to proportion may have been restricted to setting up the standard $a/b = c/d$ equation and solving for one unknown missing value (e.g., if 2 balls cost \$3.00, how much will 7 balls cost? would be found by solving $2/3 = 7/x$). If so, they will benefit from encountering problems and questions that lead them to think more deeply about what makes a situation proportional in nature. Given time and tasks aimed at exploring the inconsistencies that can arise when differences rather than ratios are found and compared, prospective teachers will recognize multiplicative situations and will begin to understand some of the difficulties inherent in teaching ratio and proportion. Proportional reasoning is inherent in many areas of middle grades mathematics. For example, regression and correlation in statistics and both theoretical and empirical probability have their basis in proportional reasoning.

Other forms of reasoning also play important roles in middle grades curricula and thus need to be part of the middle grades teacher's own mathematical understanding. Reasoning about quantities and quantitative relationships can form a bridge from arithmetic to algebra. For example, if solving word problems is approached by first identifying the quantities (e.g., distance over, distance back, speed over, speed back, time over, time back) in word problems without attaching numerical values, and these quantities are then analyzed in terms of their relationships (e.g., in each case, the distance is equal to the speed times the time; the distances are equal), then either numerical values or algebraic variables can be used to ultimately represent the problem. This approach allows students to see solving algebra word problems as simply an extension of the kinds of problems they solved earlier. Few prospective teachers have learned to make this connection in their own schooling, and they benefit from attention paid to reasoning about quantities and their relationships.

Spatial reasoning, that is, mentally visualizing and reasoning about geometric objects and their relationships, should also be a focus of middle grades mathematics. Without opportunities to develop this type of reasoning for themselves, it is unlikely that prospective teachers will be able to assist their future students in reasoning about shapes in space. For example, for teachers who have not tried to do so, identifying two- and three-dimensional shapes that have rotational symmetry is very difficult, but with practice it becomes easier. Instructors of prospective teachers will find that their students have a range of abilities to visualize objects in space, and some will need more practice than others. Many prospective teachers are surprised to learn that others' spatial reasoning ability is much better or much worse than their own, and that spatial reasoning ability is not always closely correlated with other mathematical abilities. This information will help them be more responsive to their future students' abilities and needs for assistance. Software programs are available that can assist teachers and students to reason spatially, if accompanied by appropriate instruction.

Prospective teachers need to develop statistical and probabilistic reasoning not only to prepare for their future teaching in middle grades, but also because this type of reasoning is increasingly important in dealing with daily life. Margins of error are usually stated in surveys reported in newspapers and on television; what does this tell about the results of these surveys? Some surveys use a volunteer sample. Can valid interpretations be made? How does the choice of scale affect interpretations of graphs that appears in newspapers and magazines? If a person is fired from a job because he tested positive for drug use, is that action justified? What is the probability of a false positive test result?

The Mathematical Content Needed by Prospective Teachers

Mathematics coursework designed for prospective middle grades teachers must allow them to revisit the mathematics they learned in the past. But now the mathematics should be approached in a manner that will strengthen their understanding to the extent that they will not only be able to teach it to others, but they will also know when their students have understood and what to do if students have not understood. The coursework described here also contains ideas that will be new to prospective teachers—here too they must also achieve a level of understanding necessary for teaching this content to middle grades students. This chapter provides an overview of the mathematics content needed by prospective middle grades teachers so that they can lead their students to make sense of mathematics, to be able to communicate effectively about mathematics, and be able to use mathematics appropriately.

Four areas are discussed in some detail in this chapter:

- Number and operation.
- Algebra and functions.
- Measurement and geometry.
- Data analysis, statistics, and probability.

Coursework in these areas is expected to require about 12 of the 21 semester-hours of mathematics recommended in Chapter 2. The following sections are not intended as syllabi for courses but rather to provide indications of the type and level of coursework needed by prospective middle grades teachers. In keeping with the spirit of developing in-depth understanding of the mathematics fundamental to the middle grades, only these four areas of mathematics are discussed in the following sections. The final section discusses some options for the design of courses that include the topics described here, and further mathematics courses appropriate for a middle grades preparation program of 21 semester-hours.

Number and Operations

Summary of number and operations content.

- Develop a deep understanding of rational numbers and operations on rational numbers:
  - recognize that fraction symbols are used to represent a variety of mathematical situations.
  - understand decimal notation as an extension of place value.
  - estimate calculations with fractions, decimals, percents.
  - compare relative sizes of rational numbers.
  - recognize, among proposed solutions to rational number problems, those that are unreasonable.
  - determine which operation or operations can appropriately be applied to a situation.
  - understand percent as a special case of ratio.
- Understand the structure of the rational number system and the real number system:
  - change repeating decimals to fractions and fractions to decimals.
  - establish the relationships among whole, integral, rational, irrational, and real numbers.
  - understand the number line as a representation of the real numbers.
  - understand and use field axioms.

- Understand the mathematics that underlies standard algorithms:
  - use place value to explain multiplication and division algorithms for whole numbers and rational numbers expressed as decimals.
  - understand the mathematics that underlies commonly used algorithms for fraction operations.
  - understand the different ways of interpreting a division remainder and when each is appropriate.
  - make sense of computation strategies devised by students and appreciate the number sense involved in their creation.
- Understand and explain fundamental ideas of number theory as they apply to middle grades mathematics:
  - use the Prime Factorization Theorem and relate it to algebra.
  - be able to make conjectures about odd and even numbers and about composite and prime numbers, and provide justifications that prove or disprove the conjectures.
  - be able to justify and use the Euclidean Algorithm.
- Make sense of large numbers:
  - relate large numbers to known quantities (e.g., think about the relative size of a million, a billion, and a trillion by asking how long ago was a million, a billion, or a trillion seconds).
  - express and calculate with large and small numbers using scientific notation.

Discussion.

> "Like common sense, number sense produces good and useful results with the least amount of effort. It is not mindlessly mechanical, but flexible and synthetic in attitude. It evolves from concrete experience and takes shape in oral, written, and symbolic expression. Links to geometry, to chance, and to calculation should reinforce formal arithmetic experience to produce multiple mental images of quantitative phenomena." (National Research Council, 1989, pp. 46–47)

Strengthening rational number knowledge and rational number sense are absolutely essential components of middle grades mathematics teachers' preparation. Some prospective teachers think that they know all there is to know about rational numbers because they can carry out algorithms for computations with rational numbers and feel that they could teach these procedures to others. But rational number sense means being able to think flexibly about rational numbers—to attach meaning to the symbols we use to represent rational numbers, to be able to move easily among these representations, to understand and compare the relative sizes of a rational numbers, to be able to estimate the results of calculations involving rational numbers, to undertake many such calculations mentally, and to recognize unreasonable solutions to such calculations. For example, if asked for the result of $^7/_8 \div {}^1/_4$, rather than automatically inverting and multiplying a person with number sense might ask, "How many fourths are in $^7/_8$?" then proceed to count them off:

three fourths in $3/4$, and then a half of a $1/4$ to reach $7/8$, so the answer is $3\ 1/2$. Some prospective teachers are "fraction avoiders" and habitually change all fractions to decimals before calculating. Although this behavior might not be harmful to them as individuals, as prospective teachers they will be less likely to develop the depth of understanding of fractions needed to teach in the middle grades. Reasoning with fractions (beyond learning algorithms for operations on fractions) has rarely been a curriculum topic in the background of prospective middle grades teachers, so it is not surprising that prospective teachers' knowledge of fractions is often limited to standard algorithms. Asking prospective teachers to write problems that can be solved by a particular arithmetic operation, for example, $2/3 \times 3/5$, helps them become aware that they need to know more than how to find the product of the two fractions.

Many people's understanding of fractions is limited to a part-of-a-whole interpretation, an interpretation that sometimes constrains their ability to think about fractions in other ways. For example, when asked to interpret the symbol $3/4$, many adults draw a circle, cut it into four equal parts and shade three (Silver, 1981). It is not surprising then that many middle grades students do not understand that a fraction symbol represents a quantity of something, that they do not interpret $3/4$ to mean 3 divided by 4, nor do they know that $3/4$ represents a point on the number line (Kerslake, 1986). Most have never thought of multiplication by $3/4$ as "shrinking" something to $3/4$ of its original size. An instructor of prospective teachers should ask these adults to interpret the symbol $3/4$ and find out how many of them can come up with these other meanings for the symbol $3/4$. Such an exercise will not only give the instructor insight into the understanding of these students, but can also be used as a basis for a class discussion of the meaning of the fraction symbol.

Admittedly, the part-whole relationship—3 of 4 parts of a whole as one way to think of $3/4$—is perhaps the most common way to interpret fraction symbols, and it is a useful one for thinking about fraction size, for example, that $8/9$ is a little less than 1, or that $5/9$ is a little more than half. This interpretation can then be extended to think about placement of fractions on the number line. Although comparing fractions in terms of size might seem trivial, only 35% of the eighth grade students taking the Seventh National Assessment for Educational Progress (Wearne & Kouba, 2000) were able to choose the correct ordering from smallest to largest of fractions such as $6/7$, $2/5$, and $1/2$. And in a study of middle grades teachers in a large midwestern city, only half correctly ordered $5/8$, $3/10$, $3/5$, $1/4$, and $1/2$ (Post, Harel, Behr, & Lesh, 1991). Both problems are quite simple for people with good number sense who are likely to think about these fractions in terms of their distance from 1, $1/2$, or 0 rather than using the procedure of finding a common denominator for the five fractions. (Of course, not all sets of fractions can be so easily ordered. Often students need only find a common denominator for a pair or two of the fractions to order the entire set.) Prospective middle grades teachers who are facile in using fractions will be more likely to help their future students develop this facility.

Too often instructors of prospective teachers don't spend sufficient instructional time on rational numbers because they mistakenly believe it to be review. There is evidence (e.g., Ball, 1990) that many teachers have never received instruction on operations on fractions beyond learning the algorithms and using them to solve simple word problems. Prospective middle grades teachers' knowledge of the mathematics they will be expected to teach is often superficial; they need to develop new mathematical understandings that will allow them to feel confident to teach rational number concepts and skills in a manner that will lead their students to develop robust understanding and strong skills. Courses for middle grades teachers should devote time to teasing out meanings for multiplication and division of fractions in particular. Answers to a problem such as "Is $3/4 \times 6/5$ greater than or less than $3/4$? Greater than or less than $6/5$?" can quickly suggest to an instructor whether or not a prospective teacher can think about fractions flexibly without needing to resort to algorithms that, in a case such as this, would give no insight into the question being asked. Prospective teachers need to think carefully about the unit to which each fraction in an operation refers, that is, in $2/3 \div 3/4 = 8/9$, the $2/3$ and $3/4$ each refer to some unit; $2/3$ of 1 and $3/4$ of 1, and the question asked is: How many $3/4$ of 1 are in $2/3$ of 1? But the referent unit for $8/9$ is the $3/4$; there is $8/9$ of one $3/4$ of 1 in $2/3$ of 1. Prospective teachers can learn this way of thinking about fractions and operations on fractions by identifying the referent unit for each fraction in word problems, such as "Miranda napped for $3/4$ hour. Shannon napped $2/3$ as long. How long did Shannon nap?" (What is the referent unit for each fraction, and for the solution?) Or, given a rectangle divided into 5 equal parts with 3 of the parts shaded, they can be asked questions such as: "Can you see $3/5$? If so, what is the unit? Can you see $2/3$? What is the unit? Can you see $3/2$? What is the unit? Can you see $5/3$ of $3/5$? What is the unit? Can you see $2/3$ of $3/5$? What is the unit?" (Thompson, 1995). Prospective teachers who work with manipulatives such as Pattern Blocks (available through many distributors) may find that they gain a great deal of insight into operations on fractions (and also on how concrete materials can be useful in teaching and understanding mathematics). Measurement situations lend themselves to working with fractions and with decimals.

Teachers usually introduce numbers represented as decimals in terms of their fractional equivalents and too rarely focus on extending place value understanding from whole numbers to decimals. But the middle grades student's understanding of how to represent a rational number as a decimal would be more useful if it were built on strong place value understanding developed for whole numbers in the earlier grades. When teachers themselves have developed a deep understanding of what decimal notation means in terms of place value, they have found that the additional time spent developing this understanding in their own students is gained back in later lessons focusing on comparing and operating on decimal numbers (e.g., Sowder, Philipp, Armstrong, & Schappelle, 1998; Wearne & Hiebert, 1988). This finding has implications for preparing middle grades teachers. Teachers also need to be aware of common conceptual difficulties middle grades students have when working with rational numbers. For example, some middle grades students think that 2.34 is larger than 2.4 because 34 is greater than 4; often they misplace the decimal point when multiplying decimal numbers. These difficulties are indications of the fragile knowledge of decimal notation that many middle grades students develop and never go beyond if their teachers do not have a good grasp of the role of place value in representing rational numbers using decimal notation and do not recognize students' difficulties in relating fractions, decimals, and percents.

Prospective teachers need to think carefully about the meaning of arithmetic operations, that is, determining which operation or operations will lead to a valid solution to a problem. Some exploration is needed, and often drawing diagrams can help teachers come to better understand operations. For example, some prospective teachers fall into the trap of thinking "Multiplication makes bigger and division makes smaller" (Graeber, Tirosh, & Glover, 1989). After multiplying to solve "If a pound of cheese costs \$3.35, how much does 1.5 pounds of cheese cost?" some will then solve the problem "If a pound of cheese costs \$3.35, how much does 0.89 pounds cost?" by dividing, because they know the answer must be less than \$3.35 and think that they must divide to obtain such an answer. Nor do many prospective teachers recognize the difference between problems such as "When $^7/_8$ of a yard of ribbon is used to make 3 bows, how long is the ribbon in each bow?" and "When $^7/_8$ of a yard of ribbon is available to make bows each of which requires $^1/_4$ of a yard of ribbon, how many bows can be made?" The first problem embodies the "sharing" (or measurement or quotitive) interpretation of division while the second problem builds on the "repeated subtraction" interpretation of division. Some arithmetic exercises, such as $6 \div {}^1/_2$, are usually easier if the repeated subtraction interpretation is used; how many halves are in 6 (e.g., how many times can $^1/_2$ be subtracted from 6)? Prospective teachers need to appreciate the difference in these interpretations in order to help their students make sense of division problems (Sowder, Philipp, Armstrong, & Schappelle, 1998). Research (e.g., Greer, 1992) has shown that multiplication and division situations are more complex than originally realized by curriculum developers and teachers. Recognizing how operations model situations demands a deeper understanding than simply viewing operations as computations.

Moving from fraction notation to decimal notation is straightforward given a good understanding of place value, but the reverse is more difficult. To have a full understanding of the rational number system, teachers must be able to move easily between these two representations of rational numbers. In particular, they need to understand how repeating decimals can be represented as fractions. (The notion that $0.99999\ldots = 1$ is a particularly difficult idea for prospective teachers to come to accept, even after an explanation of why this is so.) With the understanding of repeating decimals as a way of representing rational numbers, the question of non-repeating decimals should arise and can lead to a definition for irrational numbers.

By the end of fifth grade, students should have a good grasp of the multiplication and division algorithms used with whole numbers and decimal numbers, but not all do. The standard algorithms most of us learned were developed to be efficient, however complete descriptions may be lengthy and difficult to memorize without understanding. These algorithms are a challenge to teach because most teachers don't understand them themselves. An enormous amount of time in elementary and middle grades classrooms is spent on learning and practicing these procedures, especially that of long division, and is undertaken at the expense of other learning. The prospective teacher of middle grades needs to understand the rationale for the standard algorithms and how to teach these algorithms in a manner that will be understood and remembered by students without allowing the learning of these procedures to dominate the curriculum.

Prospective teachers and their students need to be able to interpret the answer to a long division problem in terms of the situation that led to the division. They often have particular difficulty dealing with remainders. When dividing 2664 by 84, is the answer 31 remainder 60? or $31\,^{60}/_{84}$? (or, as some children say, 31 remainder $^{60}/_{84}$?). Should the remainder be ignored, or should it be rounded to the next whole number? Should the division continue, resulting in a decimal number as a quotient, or should we stop here and present the remainder as a fraction? If we continue, what does "adding a decimal point and zeros" to 2664 mean? What now is being subtracted? Middle grades teachers should be helped to develop an appreciation of the role of such questions in leading to a deeper understanding of what quotients and remainders mean. Without this understanding silly but common mistakes occur such as, "11.5 buses are needed to take 322 children on a field trip, if 28 children can ride in each bus."

Although learning to use rational numbers demands a good portion of the middle grades mathematics curriculum, some whole number topics are also taught in these grades. Fundamental ideas of number theory, including the Prime Factorization Theorem, should be appropriately learned or reviewed by prospective teachers who have not had the opportunity to understand how these ideas prepare students for the learning of algebra. The Euclidean Algorithm is useful in finding the greatest common factor of two numbers, and discovering why it does so is a worthwhile exercise. Additionally, many number theory topics can be avenues that lead to practice with conjecturing and proving simple theorems about numbers, thus disproving for teachers the idea that proof is only relevant in geometry. Prospective teachers are sometimes surprised to learn that number theory has real-world applications, such as the manner in which large prime numbers are used in codes for electronic transfer of large sums of money.

Teachers should be able to work intelligently with very large numbers and very small numbers. Large and small are, of course, relative terms, but here we speak of those numbers that do not hold much meaning without developing some reference points, or benchmarks, particularly for large numbers. For example, "50,000 people would about fill the local stadium," "my city has a population of about 500,000," and "a million is a thousand thousands." If the national debt were paid off by charging every citizen an equal share, what would my share be? Rescaling in this manner offers another avenue to using proportional reasoning. Work with very large or very small numbers can motivate the need for and appreciation of scientific notation, which provides an effective and efficient way to symbolize and organize numbers. The structure of scientific notation needs to be understood, as does its relationship with significant digits. Teachers should be familiar with the manner in which scientific notation is represented on calculators.

Prospective teachers usually know how to carry out operations on integers, but are sometimes at a loss to explain the reasoning behind the algorithms. In particular, they cannot explain why "the product of two negatives is a positive." Multiplication of negative numbers has been called the first mathematical idea taught that cannot be explained by reverting to common sense, using models from the physical world. Teachers ought to know that mathematicians of the past also found it difficult to attach any meaning to negative numbers (Howson, 1996; Sfard, 2000). As work with rational numbers is extended to include negative rational numbers, teachers should extend their understanding of the number line to include numbers less than zero. After rational numbers and irrational numbers have been introduced, the real numbers can be defined and evidence provided that the number line can be used to represent both irrational and rational numbers. The rational numbers and the real numbers can now be discussed in terms of the field axioms, many or all of which would have been informally introduced previously and which are of course very important for middle grades teachers to understand and be able to use.

Mental computation and computational estimation should be included in the curriculum for middle grades teachers because these forms of computation call on and at the same time help develop the number sense these teachers need. Mental computation and estimation both call for flexibility in moving from one representation to another, on appropriate rounding (for estimation), and on knowledge of properties of operations. These skills need to be developed for whole numbers and for rational numbers. Discussing, for example, how to estimate the sum of 384, 7, and 6091 leads to a better understanding of significant digits; viewing 25×48 as $^{100}/_{4} \times 48$ or as $5 \times 5 \times 4 \times 12$ then as 20×60 makes this problem easier to compute mentally. There is evidence that prospective teachers avoid mental computation and computational estimation, and that in fact a very common way they approach these problems when requested to do so is to visualize the paper-and-pencil algorithm and attempt to carry out those steps mentally (Levine, 1982; Sowder, 1989). Developing flexibility with numbers through mental computation and estimation leads prospective teachers and middle grades students to feel a degree of confidence with numbers that they may never have experienced in the past. They also develop a better understanding of field axioms.

Algebra and Functions

Summary of algebra and functions content.

- Understand and experience the different roles algebra plays:
  - as a study of patterns.
  - as a symbolic language useful in many areas of life.
  - as a tool for problem solving.
  - as the study of functions, relations, and variation.
  - as generalized arithmetic.
  - as generalized quantitative reasoning.
  - as a way of modeling and understanding physical situations.
- Develop a deep understanding of variables and functions:
  - relate tabular, symbolic, and graphical representations of functions.
  - relate proportional reasoning to linear functions.
  - recognize change patterns associated with linear, quadratic, and exponential functions and their inverses.
  - draw and use "qualitative graphs" to explore meaning of graphs of functions.
  - understand the role of graphing calculators in the learning of algebra.
- Demonstrate skills connected to deep understanding:
  - represent physical situations symbolically.
  - graph linear, quadratic, exponential functions and their inverses and understand physical situations calling for each.

- solve linear and quadratic equations and inequalities.
- exhibit fluency in working with symbols.

Discussion. Algebra is a natural extension of arithmetic. It provides a symbolic language useful in representing quantitative relationships, it is a powerful tool for analyzing these relationships, and it provides models for decision making. Some educators describe algebra in terms of its roles in generalizing arithmetic reasoning and generalizing quantitative reasoning. When algebra is thought of as generalized arithmetic, its role as a language that encodes arithmetic properties is predominant. When thought of as generalized quantitative reasoning, algebra focuses on quantities as measurable aspects of a situation without necessarily attaching numerical values to those aspects. Both roles are useful. The first provides important links to what children already know. The second builds algebraic reasoning because it draws on aspects of growth and change, and because it focuses on relationships for the purpose of inference rather than for computation, thus providing a link to physical reality.

Prospective teachers whose previous algebra courses focused only on solving equations without understanding reasons for solution procedures and their relationships with arithmetic properties will understandably have a very limited notion of what algebra is about, and will be unequipped to address the curricular breadth now encompassed in school algebra. To prepare their students for future study of algebra and to be prepared themselves to teach algebra, teachers need to expand their own understanding of algebra. Coursework that allows students to begin to use algebra to model physical situations will allow them to see the usefulness of algebra as they become more skilled at expressing situations with symbols.

The teaching of algebra has been intensely examined over the past decade. Research studies have shown that the concepts of variable and function are poorly understood even by students studying calculus (e.g., see chapters in the book edited by Harel and Dubinsky, 1992). Thus it should not come as a surprise that prospective middle grades teachers often need experiences that will help them better understand functions. Computer technology and graphing calculators create new opportunities for them to learn about functions and graphing. Many educators say that the study of algebra ought to begin much earlier and extend throughout the grades, that algebra ought to be a curriculum strand rather than a course. There is no need to choose—algebra can be taught as a curriculum strand throughout the grades and as a course to follow arithmetic if that is the policy of the school district. The point here is that if prospective teachers think about algebra in its many roles they will begin to extend their understanding of algebra beyond its role as a body of procedures, and see how it can fit into many areas of the curriculum throughout the middle grades years. In addition, they should be prepared to teach a full year course in algebra, because some school districts are experimenting with requiring algebra in the eighth grade. Thus, they need to acquire both concepts and competencies in their algebra coursework.

One way of approaching the preparation of teachers of algebra is to enter this domain through the study of the mathematics of change. For example, situations can be presented in which it is necessary to plot and graph velocity as a function of time, distance as a function of time, velocity as a function of distance, acceleration as a function of time, and so on (see, e.g., Swan, 1985). This approach is quite new to most prospective teachers and forces them to come to grips with rate of change in different manifestations. "Qualitative" graphs allow them to focus on the variables and relationships as, for example, in drawing various graphs to represent a biker on flat ground traveling at a steady rate, then biking up a hill and slowing down, stopping at the top for a while, and speeding up when going down the hill, and finally slowing down at the bottom to a steady rate again, slower than when she approached the hill. Or graphs of functions without numbers on each labeled axis are given to students who then describe a situation portrayed by the graph. This type of study will prepare teachers to think more deeply about variables, functions, and graphs and what they represent.

Today, school algebra is often approached through investigations and problems, and the focus is on using algebra to make sense of the world around us. Both discrete and continuous quantities receive attention. Variables and relationships are identified in situations and used to model and understand the situations. There is a strong focus on being able to move easily from one representation to another—graphs, tables, and symbolic representations. Linear relations are given particular attention and pave the way for a later appearance of linear, quadratic, and exponential equations, and inequalities and their solutions. The study of linear functions should build on previous knowledge of proportional relationships. Teachers need to understand important ideas related to functions such as fixed points and asymptotes. They should understand how to use technology, particularly graphing calculators, to understand these ideas and how to illustrate them graphically, numerically, and symbolically. It is not that they will be teaching all of these ideas to their students, but rather that they need to know where what they teach leads. Technology is affecting our conceptions of what algebra is important to learn, and we must prepare teachers to think of algebra in different ways and to be able to use effectively a range of curriculum materials in their teaching.

Geometry and Measurement

Summary of geometry and measurement content.

- Identify two- and three-dimensional shapes and know their properties:
    - make conjectures about shapes and offer justifications for conjectures.
    - understand similarity and congruence of shapes.
    - be familiar with currently available software that allows exploration of shapes.
- Develop spatial reasoning through physical and mental activities:
    - manipulate mentally physical representations of two- and three-dimensional shapes.
    - determine the rotational and line symmetry for two- and three-dimensional shapes.
    - be familiar with interactive geometry software that allows movement of two- and three-dimensional drawings.
- Connect geometry to other mathematical topics, e.g., to:

   - algebra via work with rectangular coordinate systems and with transformations.
   - proportional reasoning via the study of similarity.

- Connect geometry to nature and to art.

- Understand measurement processes:

   - quantification of attributes of objects or ideas.
   - role of choice of measurement instrument and its influence on accuracy.
   - selection of unit of measure.

- Understand and use measurement techniques and formulas:

   - relate measurements within each of the two common systems of measure, English and metric.
   - estimate using common units of measurement.
   - develop and use formulas for measuring area and volume.
   - decompose and recompose non-regular shapes to find area or volume.
   - understand roles of π in measurement.
   - understand and use the Pythagorean Theorem.

Discussion. In current curricula for middle grades, students are expected to learn not only names of two- and three-dimensional shapes, but also to investigate characteristics of these shapes by visualizing, classifying, defining, conjecturing, and justifying or giving counterexamples to conjectures. A careful study of the meaning of congruence and of how congruence can be established should be included. Scaling is one approach to the study of similarity, which of course calls on proportional reasoning. Instructors of prospective teachers of middle grades will find that their students know names for two-dimensional shapes, remember (often rotely and sometimes incorrectly) some formulas for determining measurements of geometric shapes, and that some of their students remember standard proofs of theorems about triangles and parallelograms. One way to motivate prospective teachers to review two-dimensional shapes is to introduce three-dimensional shapes and describe them in terms of their two-dimensional faces. The names of less common three-dimensional shapes and identifying these shapes are often new to prospective middle grades teachers. The study of shapes should focus on properties and relationships of the shapes, and prospective teachers should be provided with opportunities to use software programs that allow for exploration of shapes to an extent not possible if the shapes must all be constructed with hand-held instruments. Circular shapes and shapes with circular cross-sections such as plates, cups, oatmeal boxes, and garbage cans lead to explorations of the special relationship of the perimeter and diameter of a circle. These types of explorations can lead to a level of conjecturing, proving, and disproving that enhances geometric reasoning and can lead to a deeper understanding of the role of proof in geometry.

The inability to reason spatially is a particular weakness of many prospective teachers who have never experienced a disciplined way of thinking about movement in space. For example, given different arrangements of six adjoining square regions, can they determine which arrangements could be folded along adjoining edges to form a cube? Or, can they envision how to slice a cube to get a cross-section that is a square? A non-square rectangle? An equilateral triangle? A trapezoid? Reasoning with two- and three-dimensional shapes can lead quite naturally to a study of geometric transformations and of symmetry. This background will allow, if an instructor so wishes, an excursion into the realm of tessellations of the plane and of space. Software that displays movement of three-dimensional objects can be especially useful in developing visualization ability.

Making connections between geometry and other areas of mathematics is an important aspect of preparing teachers to teach mathematics. The rectangular coordinate system, so often used to represent equations in two variables, lends itself well to investigating motions in the plane. The study of transformations allows prospective teachers insight into the role functions can play in geometry. The study of dilations leads to a study of proportions. Geometry should also be studied as it occurs outside of mathematics, such as in nature and in art. For example, a study of the "golden ratio" can lead to interesting explorations in art. Transformations also play a major role in artwork of many cultures—for example, they appear in pottery patterns, tilings, and friezes.

The different ways in which we measure attributes of geometric shapes is only one aspect of the study of measurement. The fact that we have found ways to quantify and measure so many aspects of our lives of work and of play is too little appreciated. Think about ways have people found to measure such things as blood pressure, atmospheric pressure, gum disease, the hotness of peppers, and the health of a new born child. Although teachers do not need to know the specifics of each of these ways of measuring, they should reflect on the role of measurement in advancing our knowledge. How do new forms of measure come about? Lord Kelvin (1824–1907), inventor of the Kelvin Temperature Scale, once said, "When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind." Prospective teachers can benefit from developing their own units for measuring quantities, such as sixth-grade students' answers and explanations to an open-ended problem given on a district test, to gain an appreciation of difficulties involved in developing measures.

Teachers in this country must understand, connect, and be able to teach both the English and the metric system of measurement. They need to understand the important role of having standard units of measure for each type of measurement, and how those units are related to units of other measures particularly in the metric system, which can be tied back to what students know about place value. They must also understand that measurements of continuous quantities are approximate, and that the need for greater or less accuracy influences the choice of the instrument selected for measuring. Teachers should learn to develop personal benchmarks for estimating common units of measure—the space from the shoulder to the fingertips of the opposite arm outstretched is usually about a meter but of course this unit will vary from person to person. A raisin weighs about a gram, a liter is slightly more than a quart, and so on. Prospective teachers should have opportunities to investigate relationships between types of measurements to think about how to design instruction that will lead to understanding measures, for example, how does holding area of rectangles constant affect perimeters of the rectangles? They need to have answers for questions such as: Why do the angles of a triangle always add up to 180°? What does π mean?[2]

Developing formulas for measuring area and volume should be undertaken in such a way that teachers realize that they can later redevelop a formula if it is not remembered. Activities involving surface area and volume can lead to better understanding of attributes and of units of measure. Decomposing complex geometric shapes into shapes that can be more easily measured is also a skill that develops with practice. Scaling and the study of similar figures call upon proportional reasoning. Situations calling for indirect measurement should be explored. The Pythagorean Theorem and at least one of its proofs, perhaps one that uses notions of composing and decomposing geometric shapes, should be included within a unit on measurement.

Many geometry activities, for example, problems involving transformations of two-dimensional figures, can be quite nicely presented via the coordinate plane, and illustrate the clear connection between geometry and algebra. The distance formula, based on the Pythagorean Theorem, offers a way of measuring the lengths of line segments in the coordinate plane.

The study of geometry and of measurement described here will prepare a teacher to be comfortable and proficient in teaching middle grades geometry as it now appears in many middle grades curriculum materials. It will not prepare teachers to teach a geometry course now typically offered in the secondary school, but prospective middle grades teachers should be familiar with the role of axioms, theorems, and proofs in the geometry curriculum of the secondary school. A course for middle grades teachers should include at least a brief foray into high school mathematics, at least to the extent that clarifies for middle grades teachers the geometry that follows and builds on that of the middle grades. Alternately, a course could be designed that prepares both middle grades and secondary teachers for the geometry appropriate in grades 7–12 curricula.

Data Analysis, Statistics, and Probability

Summary of data analysis, statistics, and probability content.

- Design simple investigations and collect data (through random sampling or random assignment to treatments) to answer specific questions:
  - formulate questions that can be addressed through data collection and interpretation.
  - make decisions on what and how to measure.
  - understand what constitutes a random sample and how bias is reduced.
  - understand how surveys are undertaken and what can be learned from them.
  - understand how statistical experiments are designed and what can be learned from them.

[2]See, e.g, Ma, 1999, p. 116.

- Understand and use a variety of ways to display data:
  - interpret and use bar graphs and pie charts for categorical data.
  - interpret and use histograms, line graphs, stem-and-leaf plots, and box plots for continuous data.
  - interpret and use scatter plots, regression lines, and correlations for bivariate data.
- Explore and interpret data by observing patterns and departures from patterns in data displays, with particular emphasis on shape, center, and spread:
  - recognize shapes of data distributions (e.g., symmetric, skewed, bimodal).
  - develop and use measures of center and spread.
  - identify misuse of cause-and-effect interpretations of correlations.
- Anticipate patterns by studying, through theory and simulation, those produced by simple probability models:
  - develop theoretical probabilities using information about equally likely outcomes.
  - develop empirical probabilities through simulations; relate to theoretical probability.
- Draw conclusions with measures of uncertainty by applying basic concepts of probability:
  - calculate and understand probabilities of independent and dependent events.
  - understand conditional probability and some of its applications.
  - understand sampling distributions, how they arise, and what can be learned from them.
  - understand margin of error and confidence intervals.
  - understand expected values.
- Know something about current uses of statistics and probability in many fields.

Discussion. Of all the mathematical topics now appearing in middle grades curricula, teachers are least prepared to teach statistics and probability. Many prospective teachers have not encountered the fundamental ideas of modern statistics in their own K–12 mathematics courses, and in fact need convincing that they need to learn this mathematics to be prepared to teach in the middle grades. Even those who have had a statistics course have probably not seen material appropriate for inclusion in middle grades curricula. Statistics is very much a context-rich subject. Interpretation, within the context of the problem, is very important. Prospective teachers must develop both skills for calculation and those for interpretation.

Teachers themselves need to learn to be critical consumers of data and statistical claims.

Currently, middle grades curricula emphasize exploratory analyses of data sets, some collected by the students and others found from outside sources such as the Web. The obvious bar graphs and pie charts for categorical data, which provide another opportunity to use proportional reasoning, are familiar to most prospective teachers. Moving from discrete data to continuous data motivates the need for graphs such as histograms, line graphs, stem-and-leaf plots, and box plots. After prospective teachers develop a few initial graphs on their own, much time can be saved by using a software program developed for teaching conceptual understanding rather than for undertaking tedious calculations. The time saved is better spent on learning to interpret the graphs and related summary statistics, such as the median, mean, and mode as measures of center and the interquartile range and standard deviation as measures of spread. Prospective teachers should have practice with and develop understanding of the role of conjecturing using sample data, and they should understand that conjectures as to why certain patterns appear in data are part of the exploratory process. They should encourage their future students to think about data in this way.

Association between two variables, represented by two univariate sets of data, is basic to most statistical investigations. To show possible associations, two-way tables are used for comparing the conditional distributions of two categorical variables; parallel box plots for comparing the shapes, centers and spreads of two (or more) measurement variables; and scatterplots, with attendant regression lines and correlations, for studying the bivariate relationship between two measurement variables. Lines of best fit allow for predictions and also connect with properties of linear functions learned in algebra. Examples of the misuse of statistical association to make cause-and-effect statements, particularly in the case of regression and correlation, can be brought into class discussions.

A statistical investigation is designed to answer specific questions posed by the investigator, and sound answers depend upon a well-formed question and a well-designed study that pay careful attention to randomization and proper measurement procedures. Teachers should understand that sample surveys involve the random selection of samples from a fixed and well-defined population for the purpose of estimating parameters of that population, such as the proportion of students in the school who plan to vote in the coming student elections. In sampling, teachers and students must have a clear understanding of the difference between a sample (and its statistics) and a population (and its parameters); of what constitutes a random sample; of the role of random selection in reducing bias in a survey; of the role of random assignment in reducing confounding of variables in an experiment; of why stratification may be important in the sample design, of how to determine a stratified random sample; and of how sample size is related to the precision of results. Experiments involve the random assignment of treatments to experimental units for the purpose of drawing conclusions about possible treatment differences, such as the possibility that water stays hot longer in Styrofoam cups than in paper cups. In experimenting, teachers and students must have a clear understanding of what constitutes a treatment and what constitutes an experimental unit; of how one randomly assigns treatments to experimental units; of how to set up a matched pairs experiment; and of why the possibility of replication (repeating each treatment on more than one experimental unit) is important.

Probability can be thought of as a way of thinking about the future. It is a way of describing the chance that an outcome will occur. Predictions can be made based on probabilities found theoretically (as can be the case of the probabilities of the various outcomes from tossing a fair die) or empirically (as in the case of collecting data by actually tossing a balanced die). Theoretical probabilities can be checked by collecting data to see if the observed relative frequencies agree with their theoretical models. The notion of a model providing a theoretical probability that can be compared to empirical results is fundamental to the study of the relative frequency concept of probability that is most useful to the study of statistics. The fact that, under random sampling, the empirical probabilities actually converge to the theoretical (the law of large numbers) can be illustrated by technology (computer or graphing calculator) so that an understanding of probability as a long-run relative frequency is clearly established. For example, simulations of the probability of getting a 5 when two balanced dice are tossed allows prospective teachers to visualize what happens to the empirical probability of the event after, say, 10 tosses, 100 tosses, and 1000 tosses.

The rules for calculating probabilities of compound events made up of independent or dependent events need many examples and much discussion by prospective teachers before they are fully understood. Tree diagrams and area models are quite helpful to many. This understanding requires a discussion of conditional probability, a topic rich with applications that leads prospective teachers to realize the usefulness (and the subtlety) of probability. For example, suppose 5% of the general population uses drugs and a drug test used is 95% accurate; then a person with a positive test result has only a 50-50 chance of actually being a drug user. Problems such as "If you know a family has two children, and a boy answers the door, what is the probability that the other child is a boy?" lend themselves nicely to simulation using coins or random numbers. Setting up a simulation can lead to better understanding of probability and the power to predict events. One goal of the course should be to consider carefully the many serious misconceptions people have about chance events, such as the "gambler's fallacy" (if a coin comes up heads for seven tosses in a row, it is certain to come up tails on the next toss), or the representativeness misconception (a hand of four cards of all queens is less likely that a hand with the 8 of hearts, the 2 of clubs, the 10 of clubs, and the 3 of diamonds).

As important as the actual calculation of probabilities for single events is, an understanding of probability distributions and how they arise, both empirically and theoretically, will lead to a much clearer understanding of likely and unlikely outcomes. For example, under the assumption 50% of a population will vote in the next election, the outcome of 40 voters in a random sample of 50 is unlikely not because its probability is small but, rather, because it is far out in the tail of the probability distribution of possible outcomes. (All individual outcomes have small probabilities in this scenario.) Distributions of sample statistics, like the proportion of voters, are called sampling distributions and are the basic building blocks of statistical inference. Summary measures for probability distributions are couched in terms of expected values, hence prospective teachers need some experience with expected value problems. Teachers in particular should have some understanding of the normal distribution, because much of the information they receive on testing is based on this distribution.

For properly designed studies, conclusions generally take the form of an estimate of a parameter or a decision as to whether to reject an hypothesized model. These forms of statistical inference have measures of error attached, either through the margin of error in an estimate or the chance of rejecting a true model. How far a sample statistic (like a proportion or mean) is likely to be from the target parameter can be assessed by looking at many different simulated sampling distributions of the statistic, under differing assumptions about the parameter and, perhaps, differing sample sizes. Teachers should understand the process of making inferences through simulated sampling distributions (which can be done effectively in middle grades) and its relationship with more mathematically based inference procedures taught at higher levels.

Proportional reasoning plays a crucial role in understanding both statistical and probabilistic ideas. Probabilities represent ratios, and dealing with ratios calls upon proportional reasoning. For example, if a mechanic says that he gets far more cars of type x to repair than cars of type y, does that mean the cars of type x are inferior? How can this problem be thought of as a proportion problem?

A Program to Prepare Middle Grades Teachers

Recommendation 11 of Chapter 2 advocates that mathematics in middle grades, from Grade 5 on, be taught by mathematics specialists. In recent years many states have developed credential programs for middle grades teachers, but some universities have been slow to instantiate them. Elementary teachers seldom have a mathematics background appropriate or sufficient for teaching middle grades mathematics, and most secondary programs do not examine the middle grades mathematics content in a manner needed by teachers of these grades. For example, neither elementary nor secondary teachers are likely to have examined the conceptual underpinnings of the rational number system in the manner needed to teach this topic in middle grades. Prospective middle grades teachers need coursework designed with their needs in mind. This coursework might also serve other populations, depending on how the courses are structured.

The actual manner in which these courses are designed will depend on how courses are already (or in the process of being) designed for elementary and for secondary teachers. Courses for middle grades teachers should strengthen these prospective teachers' own knowledge of mathematics and broaden their understanding of the mathematical connections between one educational level and the next. The following is an example of one 12 semester-hour option in which courses are designed to meet the needs of multiple audiences.

> A course offered for elementary teachers that focuses on understanding numbers and place value would be a valuable course to require of middle grades teachers, who not only need to understand the fundamental role of place value in the decimal number system and computations with rational numbers expressed as decimals, fractions, and percents, but also need familiarity with the curriculum of grades K–4.

> A full semester of study of geometric shapes and their properties together with measurement of shapes and other quantities might serve both elementary and middle grades teachers, or a one-semester course could be designed to serve both middle grades and secondary teachers.
>
> A course that focuses on conceptual understanding of probability and statistics (rather than on statistical procedures) might already be available among course offerings, but if not, one should be designed for this population or for middle grades and secondary teachers.
>
> A course that examines the structural properties of integer, rational, real, and complex number systems and the ways in which algebraic reasoning and number theory are used in contemporary applications of mathematics could serve both middle grades and secondary teachers.

These four courses would include all the basic content thus far discussed. Many examples of appropriate problems and tasks can be found on the Web site of the National Council of Teachers of Mathematics, in the Principles and Standards of School Mathematics. New examples are being added to this Web site on an ongoing basis.

Additional coursework that allows prospective middle grades teachers to extend their own understanding of mathematics, particularly of the mathematics they are preparing their students to encounter, will also be required. We suggest that this second type of coursework contain at least one semester of calculus if a course exists that focuses on concepts and applications. (The usual calculus designed for engineers and mathematics majors would probably not have this focus because of its emphasis on connections with physics and engineering.) Number theory and discrete mathematics can offer teachers an opportunity to explore in depth many of the topics they will teach. A history of mathematics course can provide middle grades teachers with an understanding of the background and historical development of many topics in the middle grades curriculum. A mathematical modeling course, depending on the level and substance of the course, can provide prospective teachers with understanding of the ways in which mathematics can be applied. If the prospective teachers are likely to teach algebra, then coursework in linear algebra and modern algebra would be appropriate. If, in addition, the teachers might be expected to teach a full-year course in geometry, then they should have the same geometry coursework as prospective secondary teachers. These last options might require more than 21 semester-hours.

It is recommended that mathematicians designing courses for middle grades teachers work closely with colleagues from the college or school of education to develop a program for middle grades specialists. These programs are likely to vary from institution to institution. Currently, in some cases, the education faculty offer a methods course for middle grades mathematics teachers, but in other cases prospective teachers are offered only an elementary methods course or a secondary methods course. In such cases, mathematicians might be able to fill in some of the gaps by courses designed for middle grades teachers. Ideally, these mathematics courses will be combined with opportunities, arranged with colleagues in education, to observe middle grades teaching and to tutor small groups of middle grades students.

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