

Chapter 9

The Preparation of High School Teachers

There is a seductive plausibility to the “trickle down” philosophy underlying most current programs for the mathematical preparation of high school mathematics teachers—the idea that coursework in a standard mathematics major develops the reasoning skills necessary to teach high school mathematics well. This intuitively reasonable proposition has not been supported by research on teacher effectiveness. For example, Begle (1979) found that students’ mathematics performance was unrelated to either the number of college mathematics courses their teachers had taken or the teachers’ average grade in such courses. The number of courses taken by teachers was actually negatively correlated with their students’ performance in 15% of the studies. Begle also found that whether a teacher had majored in mathematics had a statistically significant impact in only 20% of cases studied. More recently, Monk (1994) found that each of the first four college mathematics courses taken by high school mathematics teachers was associated with a 1.4% increase in their students’ scores on an achievement test. But further mathematics courses and majoring in mathematics had negligible impact on teachers’ effectiveness. The number of mathematics education courses taken by teachers had a positive impact on their students’ achievement—comparable to that of the first mathematics content courses.

When content knowledge is measured by course-taking, this modest correlation between mathematics teachers’ content knowledge and students’ learning has been confirmed in other disciplines. In a review of related studies Darling-Hammond, Wise, and Klein (1995, p. 24) found “positive relationships between education coursework and teacher performance [that were] stronger and more consistent than those between subject-matter knowledge and classroom performance.” Researchers at the National Center for Research in Teacher Education (NCRTE) found that teachers who majored in the subject they were teaching often were not more able than other teachers to explain fundamental concepts of their discipline (NCRTE, 1991, p. i). Their investigations led to the conclusion that, “teachers need explicit disciplinary focus, but few positive results can be expected by merely requiring teachers to major in an academic subject. Studying subject matter in relation to subject matter pedagogy helps teachers be more effective. Teacher education programs that emphasize the underlying nature of the subject matter . . . more often result in knowledgeable, dynamic teachers with transformed dispositions and understandings of subject matter and pedagogy.”

Other research has helped to describe such knowledge more explicitly. It appears that instead of (or perhaps in addition to) acquiring knowledge of advanced

mathematics, what effective teachers need is mathematical knowledge that is organized for teaching—deep understanding of the subject they will teach; awareness of persistent conceptual barriers to learning; and knowledge of the historical, cultural, and scientific roots of mathematical ideas and techniques (Ma, 1999; Shulman, 1986).

The case for rethinking the content preparation of high school mathematics teachers is strengthened by considering changes in high school curricula. Responding to the growing role of data analysis, statistics, probability, and discrete mathematics in science, engineering, computing, and business, new high school curricula have broadened the typical high school curriculum to include generous amounts of material from statistics and discrete mathematics. Other proposals have suggested that new calculator and computer technologies—with powerful graphic and computational tools—will transform the traditional emphases of high school algebra and geometry courses (Heid, 1997).

Expectations for high school teaching are changing too. Research on teaching and learning suggests that carefully designed instruction that, for example, engages students in collaborative investigations rather than passive listening to their teachers, will produce deeper learning and better retention of mathematics as well as improved social and communication skills (Bransford, Brown, & Cocking, 1999; Springer, Stanne, & Donovan, 1999). Calculator and computer tools have suggested new ways of teaching school and collegiate mathematics, encouraging laboratory-style investigations of key concepts and principles.

All of these changes in high school curricula and teaching challenge conventional patterns of teacher education. To make intelligent curricular decisions for their students and to teach current school curricula, future high school teachers need to know more and somewhat different mathematics than mathematics departments have previously provided to teachers. Because they are being urged to teach in different ways, prospective teachers also need to experience learning mathematics in those ways themselves.

To meet these needs, the education of prospective high school mathematics teachers should develop:

- Deep understanding of the fundamental mathematical ideas in grades 9–12 curricula and strong technical skills for application of those ideas.
- Knowledge of the mathematical understandings and skills that students acquire in their elementary and middle school experiences, and how they affect learning in high school.
- Knowledge of the mathematics that students are likely to encounter when they leave high school for collegiate study, vocational training or employment.
- Mathematical maturity and attitudes that will enable and encourage continued growth of knowledge in the subject and its teaching.

Responding to the Challenge

Chapter 5 gave two main recommendations for ways in which mathematics departments can attain these goals. First, the content and teaching of core mathematics major courses can be redesigned to help future teachers make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching. Second, mathematics departments can support the design, development, and offering of a capstone course sequence for teachers in which conceptual difficulties, fundamental ideas, and techniques of high school mathematics are examined from an advanced standpoint.

As mathematics departments explore ways to respond to the needs of future high school teachers, they will naturally be concerned about the compatibility of teacher preparation initiatives and needs of other students majoring in mathematics. At the time this report is being written, the MAA Committee on the Undergraduate Program in Mathematics (CUPM) is engaged in a comprehensive review of the mathematics major curriculum. That study will give special attention to the needs of prospective teachers. Preliminary CUPM work is revealing that the mathematical needs of prospective teachers have more in common with those of other students majoring in mathematics than many faculty realize. Many mathematics courses that were originally designed with a focus on preparation for graduate school now serve a constituency of undergraduates who major in mathematics but do not plan to attend graduate school, as well as undergraduates from other majors. Thus, most of the suggestions here that are designed to better serve the needs of prospective teachers seem likely to appear in some form in the final CUPM report. The following outline of mathematics and supporting courses is one way to provide core knowledge for future high school teachers while satisfying many requirements in a standard mathematics major.

Year One: Calculus, Introduction to Statistics, supporting science.

Year Two: Calculus, Linear Algebra, and Introduction to Computer Science.

Year Three: Abstract Algebra, Geometry, Discrete Mathematics, and Statistics.

Year Four: Introduction to Real Analysis, Capstone, and mathematics education courses.

These courses can be enhanced in ways that will make them more useful for future teachers and other undergraduates as well. The capstone course idea is of particular value to future teachers. But its aim of giving broad historical and cultural perspectives, insight into mathematics learning, and applications of technology should serve other mathematics majors too. The balance of this chapter is an elaboration of the ideas outlined in Chapter 5—giving some specific examples of ways that standard mathematics major courses can be redesigned to be particularly useful for future teachers and themes that can be developed in the capstone sequence for teachers. The following sections describe goals for high school mathematics teacher preparation in five areas that correspond to major strands of the high school curriculum—algebra and number theory, geometry and trigonometry, functions and analysis, statistics and probability, and discrete mathematics. Each

section indicates conceptual connections among those strands, and the broad mathematical reasoning processes and the historical perspectives, that are of importance to teachers.

Algebra and Number Theory

Current school mathematics curricula connect algebra to topics in functions and analysis, discrete mathematics, mathematical modeling, and geometry. Graphing calculators, spreadsheets, and computer algebra systems encourage and facilitate those connections, and raise deep questions about the appropriate role of skill in algebraic manipulation.

To be well prepared to teach current high school curricula, mathematics teachers need:

- Understanding of the properties of the natural, integer, rational, real, and complex number systems.
- Understanding of the ways that basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities.
- Understanding and skill in using algebra to model and reason about quantitative relationships in real-world situations.
- Ability to use algebraic reasoning effectively for problem solving and proof in number theory, geometry, discrete mathematics, and statistics.
- Understanding of ways to use graphing calculators, computer algebra systems, and spreadsheets as tools to explore algebraic ideas and algebraic representations of information, and in solving problems.

Prospective teachers will enter undergraduate studies with some technical skill in algebra. They can manipulate familiar types of polynomial, rational, and exponential expressions to solve equations and inequalities and to transform given expressions into equivalent forms. They also have rudimentary knowledge of number theory concepts like primes, factors, multiples, greatest common divisors, and least common multiples, and they are proficient in operations with fractions, decimals, radicals, and complex numbers.

However, prospective teachers tend to have only limited understanding of the algebraic properties that characterize the various subsets of the complex number system, that justify methods for solving equations and simplifying expressions, and that guide operations in many other systems like polynomials, sets, logic, functions, or matrices. Their understanding of and skill in working with linear systems, matrices, and polynomials is often superficial.

Algebra and number theory in courses for mathematics majors. Typical programs for preparation of high school mathematics teachers have several components that contribute to their knowledge of algebra and number theory.

Calculus. Calculus courses are an excellent place to polish and extend all undergraduates' algebraic understanding and skill and to strengthen their facility with model building. Solving calculus problems requires a great deal of algebraic manipulation, and understanding related theorems can require further generalization and abstraction (for example, being able to write a particular power series and to determine whether or not it converges requires some algebraic manipulation. Being able to express a general power series requires abstraction and generalization of that process). Calculus also develops a more abstract sense of algebra and functions by expressing methods and results, such as integration by parts and the chain rule, in general notation. The algebraic aspects of calculus create many opportunities where an instructor's timely observations can give undergraduates new insights into algebra.

Linear algebra. After prospective teachers' skill and understanding with algebra has been enhanced in the relatively concrete setting of calculus, the stage is set to develop some theoretical understanding in a sophomore linear algebra course. The standard course in linear algebra gives extensive experience with linear systems and matrices, connections of those algebraic ideas to the geometry of vectors and transformations, and the beginnings of understanding and skill in algebraic proof. Most linear algebra courses strike a balance between the familiar \mathbf{R}^2 , \mathbf{R}^3 and associated matrices and the more general \mathbf{R}^n , abstract vector spaces, and linear transformations. For prospective teachers, it is particularly important for vector space ideas to be well understood in \mathbf{R}^2 and \mathbf{R}^3 , which are central to high school mathematics, before moving to abstract vector spaces. The challenge is that linear algebra courses have an increasing number of topics to cover. The study of \mathbf{R}^2 and \mathbf{R}^3 as vector spaces is a natural setting in which to develop students' understanding of connections between linear algebra and analytic geometry. One way to develop this area further is to devote several classes to applications of matrix-based analytic geometry in the growing area of computer graphics and visualization. This subject involves a variety of linear transformations of \mathbf{R}^3 onto \mathbf{R}^3 and projections of \mathbf{R}^3 onto \mathbf{R}^2 . For example, an extensive amount of three-dimensional analytic geometry, including cross products, is needed to determine which sides of a given tetrahedron are visible to a viewer at a certain position in space. Another important application of linear algebra that is especially useful for teachers is the role of projections in pseudoinverses and least-squares curve-fitting. Regression lines, which are based on least-squares projections, are becoming part of the statistics taught in high school. However, they are also of fundamental importance to undergraduates in many fields that require linear algebra.

Abstract algebra. Most mathematics majors take only one semester of abstract algebra, so it is important that such a course develop students' appreciation of the breadth and power of algebraic structures. The last comprehensive CUPM report on the mathematics major (MAA, 1981/1989) suggested a syllabus for abstract algebra that offers one way to balance depth with breadth. It proposed what might be called "visiting the algebraic structure zoo": Spend the first two or three weeks of the course in a tour of the whole zoo to get a basic introduction to all the major species. Then spend most of the rest of course studying a few algebraic species in some depth. At the end of the discussion of each algebraic structure devote time to connections with other areas of mathematics and to applications in other disciplines.

For all undergraduates, but especially for future high school teachers, such an abstract algebra course can effectively build on familiar algebraic structures encountered in high school and other college mathematics courses. Examples of rings, integral domains, and fields are familiar from high school, and the most useful for future high school teachers. The amount of time spent on group theory would be less than is often the case in the first semester of a two-semester algebra course and group theory would be closely connected to concrete examples such as isometry groups. There are now some abstract algebra texts for such a syllabus. A second semester of algebra, geared for, among others, mathematics majors considering graduate school, can focus on groups and deeper topics such as Galois theory.

It is natural for such study of algebraic structures to include exercises that explore the algebraic properties that underlie numerical and symbolic operations in school mathematics. It is important for prospective teachers to understand how most extensions of the number system, from natural numbers through complex numbers, are accompanied by new algebraic properties, and why the field axioms are so critical for arithmetic. Algebra courses also should make prospective teachers aware of the many algebraic structures that they have encountered but may not have noticed, such as the isometry groups of regular polygons.

Number theory. Number theory has always been a popular elective in the mathematics major, especially among prospective teachers. Numbers are the most familiar of mathematical objects. The subject is a concrete setting for strengthening algebraic and proof-building skills. It is useful here again to explicitly examine mathematics underlying number theory concepts used in school mathematics. For instance, students can be asked to use unique factorization and the Euclidean Algorithm to explain familiar procedures for finding common multiples and common divisors of integers. Modular arithmetic provides mathematically rich examples of algebraic structures that can be explored by students at various levels of sophistication from elementary and middle school through high school. There are also a number of contemporary applications of number theory to problems in coding and computing that high school students find accessible and intriguing.

Algebra and number theory in the capstone sequence. There are at least five themes that could be addressed in a capstone look at algebra and number theory.

Historical perspectives. Prospective teachers need to develop an eye for the ideas of mathematics that will be particularly challenging for their students. One very useful guide to such topics is the historical record showing how the ideas were first developed (Boyer, 1991; Katz, 1992, 2000; Swetz et al., 1995). The history of algebra is rich in stories that illuminate the challenge of expressing number patterns and properties in modern notation, of developing algorithms for solution of equations, of making sense of fractions, negative numbers, irrational numbers, and complex numbers.

For example, there is evidence that Babylonian mathematicians over 4,000 years ago were able to solve quadratic equations equivalent to $x^2 - x = 870$ using a method that was a special case of the quadratic formula now in common school use. However, the problems and solutions were expressed in rhetorical form and base 60 numeration. It was not until the 17th century that such problems and solutions were expressed in something like the modern symbolic notation that secondary school students are now expected to master quickly. Furthermore, it was not until

modern times that mathematicians were comfortable with the general quadratic $x^2 + px + q = 0$ where p and q are positive, because such equations can have negative or complex roots.

Investigation of these historical issues involves substantial mathematics. It also provides prospective high school mathematics teachers with insight for teaching that they are unlikely to acquire in courses for mathematics majors headed to graduate school or technical work.

Common conceptions. Although historical analysis shows the difficulties encountered in development of fundamental algebraic ideas and techniques, there are many aspects of algebra that seem to be persistent challenges in learning the subject. Experienced teachers come to know where students are likely to make mistakes in algebraic manipulation and reasoning. It makes sense to address these issues in the preparation of high school teachers and to analyze the mathematical and psychological factors that lead students to common errors. For example, it is not at all uncommon for students to overgeneralize algebraic laws like the distributive property to conclude that $(a + b)^2 = a^2 + b^2$, to be puzzled by expressions like 0^0 , or to have only limited understanding of the rationale underlying methods for solving equations and systems of equations. There is a rich body of research on the range and roots of such algebraic difficulties (Confrey, 1990; Kieran, 1992). Collaboration of mathematicians and mathematics educators could produce capstone activities that would prepare high school teachers with deep understanding of the issues from both mathematical and psychological perspectives.

Applications. Because algebra occurs throughout all branches of mathematics, it is easy for students and prospective teachers to get the impression that it is a tool in the service of other topics. Many applications of algebra do occur in the context of problems in geometry or analysis, but there are some very important applications that apply core algebraic topics directly. For example, linear programming problems make an excellent setting for illustrating the usefulness of core algebra topics like linear equations and inequalities. Cryptography is a direct application of both abstract algebra and number theory. Aspects of both linear programming and cryptography can be developed at levels that are appropriate for and interesting to high school algebra students.

Technology. College algebra and number theory courses are beginning to exploit and study the use of calculator and computer tools for doing and learning algebraic ideas. Calculators with powerful computer algebra systems (CAS) are now available at prices that make them accessible, and they are used more and more in schools. With only modest instruction, a high school student can use such tools to do most of the calculation problems that fill course assignments and examinations in algebra. The implications of this new-found calculating power are still being worked out in a variety of formal and informal classroom and curriculum experiments. High school teachers whose careers will cover the next 30 or 40 years need experience with these tools in their own learning and problem solving that will usefully inform their teaching. There is growing evidence that use of CAS technology can expand high school algebra and that intelligent users need common sense and mathematical habits of mind to use these tools wisely.

In addition to CAS software, computers offer other tools that require algebraic understanding. For example, spreadsheets are one of the most widely used computer tools, and designing a useful spreadsheet requires flexible ability to express

numerical relationships in algebraic notation. College mathematics courses not often incorporate spreadsheet tasks. However, high school computer literacy courses often introduce students to this tool, and there are important ways that spreadsheet work can illuminate important mathematical ideas. For example, spreadsheet formulas often implicitly define recursive procedures, so spreadsheets can be used to explore fundamental properties of arithmetic, geometric, and more general sequences and series.

The capstone sequence is a perfect place to explore these emerging technologies and their role in high school algebra curriculum and teaching. In addition to providing a new perspective on familiar algebraic concepts and techniques, the capstone can consider important issues about when and how technology should be used to maximum benefit. Once again, collaboration of mathematics and mathematics education faculty in designing this course would help to make examination of this perspective productive for future teachers.

Connections. Because algebra is the language in which so many relationships are expressed throughout mathematics, it is a natural topic to use in exploring important mathematical connections. Although it is tempting to assume that undergraduates will see those connections throughout their mathematical studies, experience suggests that a reflective overview of the mathematical landscape is of value.

For example, a first course in linear algebra only begins to develop an understanding of the interplay among matrices, systems of equations, and vectors or of applications like least-squares curve-fitting in data analysis, Markov processes in probability, or dominance matrices in graph theory. The first course in abstract algebra often uses symmetry groups of polygons as examples of groups, without connecting more generally to the geometry of transformations. Most abstract algebra courses do not connect solution of polynomial equations in radicals to the impossibility of classical Greek construction problems like doubling the cube. Each of these topics is rich in mathematics and in opportunities to give prospective teachers deeper understanding of the scope and important processes of the subject they will teach.

Geometry and Trigonometry

High school geometry was once a one-year course of synthetic Euclidean plane geometry that emphasized logic and formal proof. Recently, many high school texts and teachers have adopted a mixture of formal and informal approaches to geometric content, de-emphasizing axiomatic developments of the subject and increasing attention to visualization and problem solving. Many schools use computer software to help students make geometric experiments—investigations of geometric objects that give rise to conjectures that can be addressed by formal proof. Some curricula approach Euclidean geometry by focusing primarily on transformations, coordinates, or vectors, and new applications of geometry to robotics and computer graphics illustrate how mathematics is used in the workplace in ways that are accessible and interesting to high school students.

To be well-prepared to teach the geometry in high school curricula, mathematics teachers need:

- Mastery of core concepts and principles of Euclidean geometry in the plane and space.
- Understanding of the nature of axiomatic reasoning and the role that it has played in the development of mathematics, and facility with proof.
- Understanding and facility with a variety of methods and associated concepts and representations, including transformations, coordinates, and vectors.
- Understanding of trigonometry from a geometric perspective and skill in using trigonometry to solve problems.
- Knowledge of some significant geometry topics and applications such as tiling, fractals, computer graphics, robotics, and visualization.
- Ability to use dynamic drawing tools to conduct geometric investigations emphasizing visualization, pattern recognition, conjecturing, and proof.

Most prospective mathematics teachers enter undergraduate study familiar with basic properties of two- and three-dimensional figures (mainly triangles, quadrilaterals, circles, and related solids), and they have some facility in constructing proofs of elementary results for planar figures (generally relationships involving congruence, similarity, parallelism, and perpendicularity). Through pre-calculus study they will have gained some skill in elementary coordinate geometry and vector techniques. Despite the fact that measurement of geometric objects is a core subject in precollege mathematics, it is quite likely that even strong entering undergraduates will have only a formula-driven understanding of that topic.

Geometry and trigonometry in undergraduate mathematics courses.

The standard calculus and linear algebra courses for mathematics majors give students extensive experience with important geometric ideas and techniques—especially coordinate methods, vectors, transformations, and trigonometry. But the focal point of geometry for prospective high school teachers is usually a course in college geometry. There are several ways that these courses can help to prepare high school mathematics teachers.

Calculus. The analytic geometry component of current calculus courses is substantially smaller than a generation ago, with much of the familiar content moved into an already crowded pre-calculus course where it is not treated as fully. The traditional calculus topics of analytic geometry and trigonometry that have always been important for teachers and engineers are now of growing importance for computer scientists as well. It seems appropriate to restore some of that important material to a more central position in the college curriculum. Such a move would be very helpful in the preparation of high school teachers.

Linear algebra. One of the many ways in which linear algebra can be viewed is as generalized analytic geometry. There is an inherent tension between geometry and algebra in a linear algebra course, as well as between \mathbf{R}^n and abstract vector spaces. The key is to maintain a sensible balance. It is also important to constantly

view questions from both points of view, and to translate algebraic results into geometric language and vice versa. Attention to this interaction of geometry and algebra will be very useful for future high school teachers.

Geometry. Upper-division geometry courses for teachers typically involve re-examination of Euclidean geometry from an axiomatic point of view along with rudimentary non-Euclidean geometry. Some versions of this course include or even start from transformational or coordinate points of view, but the most typical approach is synthetic reasoning in the spirit of Euclid, with modern standards of rigor.

A major goal of a collegiate geometry course should be to deepen prospective teachers' understanding of standard Euclidean theorems and principles and their skill in use of axiom-based reasoning. A careful review of high school geometry can have substantial and valuable conceptual content. However, prospective teachers should also be acquainted with other aspects of geometry, in order that they understand that geometry is not restricted to axiomatic geometry and are prepared to teach high school topics such as computer graphics. Such topics also provide an opportunity to strengthen undergraduates' geometric and algebraic skills.

A geometry course for teachers could also examine the geometry of the sphere (Henderson, 2000) and the shapes of useful figures like conic sections. Another topic that often interests prospective teachers is the development of the geometry of congruence and similarity from axioms about isometries and similitudes. This development can be connected to the algebra of matrices and complex numbers in ways that appeal to high school students as well as prospective teachers.

Computer graphics provides a new viewpoint for re-examining a variety of topics in synthetic and analytic geometry. Artistic notions about perspective must be translated into mathematics in order for such perspectives to be represented on a computer screen. Dynamic geometry software permits experiments with geometric constructions that provide opportunities for students and teachers to explore the visual world mathematically. The graph theory component of discrete mathematics also contains several interesting topics of a geometric nature, such as planar graphs. Graph theory also provides an interesting approach to the classical topic of Platonic solids via Euler's formula.

A high school teacher who has some familiarity with aspects of modern geometry such as tiling and fractals and with applications such as computer graphics and robotics will convey a richer view of the subject to students. Fitting all of those topics into one college geometry course that also treats gives an in-depth axiomatic development of Euclidean geometry runs a clear risk of covering ground without developing depth of understanding. As is the case with abstract algebra, it seems promising to survey some topics quickly and then treat a selected few in depth.

Geometry and trigonometry in the capstone sequence. Knowledge of geometry for teaching can be provided in the proposed capstone sequence. This kind of course can explicitly trace the historical development of key ideas, identifying questions that were challenging for mathematicians and will be difficult for students. It can show ways that development of deep ideas can be started in high school courses, and it can examine thoughtfully the interplay of intuitive, exploratory work and axiomatic proof. The capstone sequence may also be an ideal setting for re-examination of key trigonometric ideas to assure that prospective teachers have the depth of understanding that is essential to effective curricular decision making

and instruction. Some specific examples illustrate the mathematical issues that can be addressed in such a course.

Historical perspective. College geometry courses often develop key ideas with some attention to the chronology of events beginning in Greek mathematics and culminating in formulation of non-Euclidean geometry during the 19th century. However, there is really much more to the subject than time allows in a typical formal geometry course. Tracing the roots of the Pythagorean Theorem alone reaches back to developments in Babylonian, Egyptian, and Asian mathematics and highlights the dramatic influence of Greek focus on axiomatic methods. Exploration of the geometric approach to number and algebra that dominated Greek mathematics will provide future high school teachers with geometric representations of algebraic ideas that can be used to help their students.

The visual side of geometry makes it an excellent place to explore the interplay of mathematics and cultural traditions. The visual arts of nearly every ancient and contemporary culture embody important geometric concepts and principles.

Common conceptions. Geometry doesn't have the collection of procedural skills associated with arithmetic or algebra, but there are still some important and predictable conceptual difficulties in learning the subject. Students have trouble with the logical issues involved in proof—often failing to understand the power of universally quantified statements, the role of counterexamples, the connections among propositions and their converses and contrapositives, and the class inclusions associated with definitions for parallelograms, rectangles, and squares. The connection between visual exploration of ideas and formal definitions or proofs is a persistent source of learning difficulties, with students often unable to free their minds from perceptions such as the belief that a pair of perpendicular lines must always consist of one horizontal and one vertical line. The natural human tendency to think in terms of linear proportionality very often leads students to difficulty in understanding the effects of similarity transformations on perimeter, area, and volume.

Teachers who understand these common mathematical and psychological challenges in learning geometry are prepared to plan instruction that heads off or corrects such student misunderstandings. Collaboration by mathematicians and mathematics educators in the design of a capstone course could help prospective teachers lay the foundation for development of this kind of deep content knowledge for teaching.

Applications. The connection of plane and solid geometry to objects in the physical world makes applications of fundamental principles very easy. However, the standard college geometry course seldom focuses on those topics that are so useful in the resource kit of high school teachers. The capstone sequence could profitably address both classical and modern applications of basic geometry—from the ubiquitous Pythagorean Theorem, the conic sections, and simple mechanics to modern topics in computer graphics, fractals, and robotics.

Technology. In the same way that computer algebra systems can be used to explore symbol manipulation, computer drawing tools provide powerful aids in geometric explorations. Tasks that require construction of figures with given properties help teachers (and will help their students) see the logical interdependence of key ideas. The dynamic grab-and-drag features of computer drawing tools illustrate

the universality of theorems in a way that goes far beyond typical paper and pencil explorations.

For example, after constructing a triangle and its medians with software like *Cabri Geometry* or *Geometer's Sketchpad*, one can grab a vertex and drag it across the screen to form an infinite number of new triangles. If the medians are constructed so that they satisfy the definition of median (rather than simply being drawn in approximately the correct position), all three will continue to intersect at the same point. Moreover, the ratios of the lengths of the two segments in which each median is cut will remain 2:1, even though the lengths of the medians themselves will vary. These kinds of geometric explorations lay a foundation for high school students' understanding of formal proof. They also illustrate an important aspect of creative mathematical work and of the way in which software can embody a mathematical definition. The capstone course can help make future teachers comfortable with use of such tools and also address the connections between experimental and deductive mathematics.

Connections. In the same way that algebraic notation provides a kind of universal language for representing quantitative information, geometric shapes provide visual representations of ideas. The connection between geometry and algebra provided by coordinate methods is among the most powerful tools of mathematics. The connection works both ways: Algebra allows computational methods to be used in reasoning about geometric objects, but geometry provides helpful visual images for algebraic calculations, limiting processes in calculus, reasoning about probability, and display of data in statistics.

Although there is a geometric basis for the subject of trigonometry, right-triangle, and periodic-function aspects of that topic have been traditionally taught in a separate high school course and as part of pre-calculus studies. This may be one reason why prospective high school mathematics teachers often have some technical proficiency in the trigonometry of right triangles when they come to undergraduate studies, but lack deep understanding of the core geometric principles that make trigonometry possible. The capstone sequence is a natural place to re-examine trigonometric ideas, several key identities (law of sines, law of cosines, Pythagorean Theorem), the addition formulas, and the general notion of identity—and to make or reinforce connections with geometry.

Functions and Analysis

The concept of function is one of the central ideas of pure and applied mathematics. For nearly a century, recommendations about school curricula have urged reorganization of high school mathematics so that study of functions is a central theme. Computers and graphing calculators now make it easy to produce tables and graphs for functions, to construct formulas for functions that model patterns in data, and to perform algebraic operations on functions. Prospective high school mathematics teachers must acquire deep understanding of the function concept in general and the most important classes of functions: polynomial, exponential and logarithmic, rational, and periodic. For functions of one and two variables, teachers should be able to:

- Recognize data patterns modeled well by each important class of functions.

- Identify function types associated with various relationships like $f(xy) = f(x) + f(y)$, or $f'(x) = kf(x)$, or $f(x+k) = f(x)$.
- Identify the types of symbolic representations associated with each class of functions and the way that parameters in those rules determine particular cases.
- Translate information from one function representation (tables, graphs, or rules) to another.
- Use function representations and operations to solve problems in calculus, linear algebra, statistics, and discrete mathematics.
- Use calculator and computer technology effectively to study individual functions and classes of related functions.

Students who study pre-calculus mathematics in high school or college will probably encounter the function concept and the major function families in a serious way. This introduction needs conscious reinforcement and extension in college mathematics courses.

Functions in courses. Future high school teachers meet functions in calculus, linear algebra, and various other courses of the mathematics major. It is easy for those encounters to focus on the procedural aspects of function notation, operations, and graphs. High school and college students often come to see functions as nothing more than alternate ways of expressing algebraic information. There are several ways that standard courses can convey a broader view of functions and their role throughout mathematics.

Calculus. Calculus instructors can provide a useful perspective for future high school teachers (and other undergraduates as well) by giving more explicit attention to the way that general formulations about functions are used to express and reason about key ideas throughout calculus. Its central concepts, the derivative and the integral, are conceptually rich functions. Many formulas in calculus, such as the chain rule and integration by parts, are posed in general form rather than formulated separately for each function. The study of Taylor series shows how all differentiable functions can be approximated by polynomials.

The calculus reform movement has suggested differences in the way functions are treated in instruction. For example, there is a greater attention to multiple ways of looking at functions by (i) comparing algebraic, graphical, and numerical information about a function; and (ii) by highlighting their role in mathematical modeling. Experience in reform calculus courses, with their emphasis on conceptual depth, has the potential to be especially useful for future teachers.

Differential equations. A course in differential equations is required in some mathematics majors. Although often oriented towards physical science applications, this subject is also a good “practical” setting to strengthen undergraduates’ understanding of functions, especially the role of functions in mathematical modeling. The whole notion of a differential equation is centered around an implicitly defined function that traces out a behavior governed by the differential equation. One sees solutions that are families of functions as in first-year calculus, but these

families may have more complicated relationships than simply differing by a constant. Wronskians are connected with the notion of linearly independent functions. Laplace and Fourier transforms are linear operators on functions—defined in a way that may require a novel view of functions for many undergraduates. Power series solutions to differential equations introduce, in a very practical way, another useful way of representing functions.

Differential equations are now commonly studied with the assistance of a variety of graphic and computational software. Exposure to these new ways of working with functions will be valuable experience for future teachers.

Linear algebra. Although called linear algebra, this course can be taught with a substantial amount of linear analysis. It is important that undergraduates notice that functions play two different roles: they can be both linear transformations on vector spaces and elements of vector spaces. There are a variety of insights into functions that can be obtained by studying various vector spaces and linear transformations. For example, studying the vector space of power series and linear transformations, such as differentiation can help undergraduates appreciate the power of linear algebra in analysis.

Advanced calculus/introduction to analysis. An advanced calculus course or an elementary introduction to real analysis is an opportunity to give a rigorous foundation for future teaching about functions and calculus. Informal notions about Euclidean space, functions, and calculus that undergraduates have used for several years can be given sound formal definitions. The sort of syllabus that would be most valuable for prospective high school mathematics teachers would cover elementary topology of open and closed sets on the real line, sequences and series, properties of functions such as continuity, a formal treatment of the derivative and the Riemann integral, and basic properties of metric spaces. Key results like the Intermediate Value Theorem and the Mean Value Theorem can be proved in a more rigorous way than in a first-year calculus course.

Probability/statistics and discrete mathematics. Courses in these newer areas in the curriculum can and should make important and deep use of functions. It is easy to forget that central concepts in probability and statistics are closely tied to functions. For example, a random variable is a function on a sample space, and a statistic is a function of a set of random variables. In discrete mathematics courses, generating functions and recurrence relations (on integer-valued functions) present two new, intellectually rich aspects of the concept of a function. Making explicit connections between these specific examples and the function concept that occurs in other areas as well should help to provide mathematical insight that will be of use to future teachers and other undergraduates as well.

Functions in the capstone sequence. Although mathematics majors use functions in nearly every undergraduate course, it is recommended that prospective teachers to revisit the elementary functions of high school mathematics from an advanced standpoint in much the same way that they revisit algebraic and number system operations from a structural point of view. This sort of reflective look at functions and their unifying role in mathematics could be a prominent part of the capstone content course. The capstone study of functions can examine again the role of computer numeric and graphic tools in mathematical work—an important

example of the connection between exploratory investigations and formal proof—and it can give undergraduates experience in the kind of complex problem solving required by authentic mathematical modeling.

Historical perspectives. Like many other mathematical ideas with broad applicability in modern mathematics, the function concept did not develop easily or early in the subject. The word “function” first appeared in a manuscript of Leibniz, and the first explicit definition of the term was by Johann Bernoulli who said, “One calls here a function of a variable a quantity composed in any manner whatever of this variable and of constants” (Siu, 1991, p. 105). However, many other distinguished mathematicians contributed to formulation of the concept, and it wasn’t until late in the 19th century that the full generality and subtlety of the modern concept of function was explored. Study of this history will certainly give high school teachers appreciation for their students’ difficulties in comprehending the concept that experienced mathematicians use almost without thinking. It also provides a case study in the way mathematical definitions evolve.

Common conceptions. Like early mathematicians, students very often come to believe that the only mathematical relations worthy of the name “function” are those that can be expressed by some sort of algebraic formula. They have difficulty imagining the variety of relationships that meet the formal definition of function, and few acquire the ability to think about functions as objects rather than processes for transforming inputs to outputs.

Recent research documents common cognitive difficulties in learning about and using functions (Dubinsky & Harel, 1992). The combination of mathematical and psychological issues involved provide yet another opportunity for mathematicians and mathematics educators to collaborate in capstone course activities that will help future teachers enter high school mathematics classrooms with deep personal understanding of this key concept and a sense of the instructional challenges it poses.

Applications. Undergraduate courses in calculus will give prospective teachers many encounters with ways that functions are useful in solving quantitative problems. They are less likely to provide a realistic experience in the mathematical modeling process that surrounds specific calculations. Since mathematical modeling has become a process that pervades problem solving and decision-making in the sciences, engineering, economics, and government, it is important for future high school teachers to have some experience with that kind of applied work.

Technology. Graphing calculators and computers are powerful tools in mathematical study and problem solving with functions, and the tools that are available become more impressive every year. Use of calculators and computers in pre-calculus mathematics and calculus is becoming very common in high school and collegiate mathematics courses. However, there is no consensus on the optimal way to introduce or use the technology. Activities in the capstone sequence for high school teachers should explore the variety of possible uses of calculators and computers in analysis—from numerical and graphic exploration and problem solving to formal symbolic operations in algebra, calculus, and linear algebra—and consider carefully the interplay of technology and formal reasoning methods.

Connections. Because the concept of a function has such general applicability in pure mathematics, its study offers opportunities to show the common mathematical structures in topics that are often perceived to be quite separate. For example,

functions are often encountered first as models of linear, quadratic, exponential, or periodic variation—relating numerical domains and ranges. However, the same underlying concept provides a dynamic way of thinking about geometry in terms of transformations, and the transformation concept returns the favor by providing a way of connecting algebraic forms like x^2 , ax^2 , $(x + b)^2$, and $a(x + b)^2 + c$ and similar variations on e^x , $\ln x$, or $\sin x$. Functions also provide models for probability distributions and useful data transformations like log-log plots. A capstone experience that highlights those kinds of connections via functions will provide valuable insight to prospective teachers.

In traditional college preparatory curricula, the primary goal is preparing for study of calculus. Calculus is now commonly taught in advanced placement form at many U. S. high schools, so teachers should understand calculus well enough to make informed decisions about the content and emphasis of courses intended to prepare students for calculus and to be prepared themselves to teach high school calculus. Prospective high school mathematics teachers often enter undergraduate studies with a year of high school calculus. But even if they do all of their calculus at the collegiate level, they typically acquire only a mechanical and unsure grasp of the subject. Even those who tackle the challenge of advanced calculus or real analysis will acquire a fairly fragile basis for teaching calculus to high school students.

Like curricula and instruction in K–12 mathematics, collegiate calculus is evolving in response to emerging technology, diverse applications, and new ideas about teaching/learning environments. There is considerable hope that these reforms will lead to deeper student understanding and skill in the subject. Such an improvement would be very helpful for prospective high school teachers. Because the concept of a function is a primary building block for future study of calculus, it will be valuable for prospective high school teachers to examine fundamental calculus concepts like limits, continuity, differentiation, and integration with an eye to the conceptual difficulties of those subjects.

Data Analysis, Statistics, and Probability

Statistics has emerged recently as a core strand of precollege curricula. For many years middle and high school mathematics textbooks have included units on finite probability and a small amount of work on graphic display of data, with data summaries limited to the familiar mean, median, and mode. The situation is now very different. There is an Advanced Placement statistics exam for high school students. The American Statistical Association's Quantitative Literacy Project has developed data-driven curriculum modules that offer fresh approaches to high school topics in statistics, algebra, and geometry. The traditional emphasis on probability-based statistical inference has given way to introductory statistics material that focuses on using data to “gain insight into real problems” (Moore & McCabe, 1999, p. xvii). This new conception of the content of probability and statistics is accompanied by commitment to teaching the subject in ways that reflect the practice of statistics. To be prepared to teach in this way, prospective teachers should have experience formulating questions, devising data collection protocols, and analyzing real data sets that result from their own investigations or from the data collection of others.

Readers are urged not to view the introduction of statistics topics as a watering down of the school mathematics curriculum. Probability and statistics are mathematically rich subjects that can be taught in a fashion that strengthens students' understanding of algebra and functions. Working with the concepts of the probability of an event, a random variable, and a statistic all involve solid mathematical reasoning and considerable thoughtfulness. Serious high school study in this area, including in a senior semester-long course for students headed for social science majors, has much to recommend it. Although data analysis has its simple descriptive side, such as constructing stem-and-leaf plots and calculating summary statistics, it also involves conceptually rich issues. There are deep and subtle questions in exploration of data distributions, modeling relationships among two or more variables, designing complex surveys or experiments to reduce bias and variability, and inferential reasoning (hypothesis testing and estimation through confidence intervals).

Curricula for the mathematical preparation of high school teachers must include courses and experiences that help them appreciate and understand the major themes of statistical practice:

- Exploring data: using a variety of standard techniques for organizing and displaying data in order to detect patterns and departures from patterns.
- Planning a study: using surveys to estimate population characteristics and designing experiments to test conjectured relationships among variables.
- Anticipating patterns: using theory and simulations to study probability distributions and apply them as models of real phenomena.
- Statistical inference: using probability models to draw conclusions from data and measure the uncertainty of those conclusions.

Probability has important applications outside of statistics. Thus, prospective high school teachers should also:

- Understand basic concepts of probability such as conditional probability and independence, and develop skill in calculating probabilities associated with those concepts.

Statistics is now widely acknowledged to be an extremely valuable set of tools for problem-solving and decision-making. But, despite the production of interesting statistics materials for schools, it has been hard to find room for the subject in curricula dominated by preparation for calculus.

Statistics and probability courses. Future high school mathematics teachers need at least two courses in probability and statistics. The most natural pair would be a calculus-based survey course in probability and statistics and a course in data analysis.

Survey of probability and statistics. The survey course is typically aimed at a broad audience of majors in mathematics, computer science and engineering.

Guidelines for such a course in the 1981 CUPM Recommendations for a General Mathematical Sciences program suggest starting with a survey of elements of data analysis. This survey might be reduced in light of the proposed separate data analysis course. Then there is an introduction to probability—sample spaces, independence, conditional probability and Bayes’ Theorem, common discrete and continuous probability distributions, and the Central Limit Theorem—followed by an introduction to mathematical statistics—common sampling distributions, point estimation, tests of hypotheses, and confidence intervals. Note that this course should spend only a modest amount of time on combinatorial probability problems, because combinatorial enumeration will be treated in the discrete mathematics course. Modern versions of this course use technology to demonstrate properties of probability distributions and inferential procedures.

Data production and analysis. This course should include elements of design of experiments and sample surveys, parametric inference (typically assuming a normal distribution of the population) on population means and proportions, the chi-square test for goodness-of-fit, association and homogeneity of proportions, regression analysis, and analysis of variance. If time permits, modern techniques for categorical data analysis (such as logistic regression) and some non-parametric procedures (such as the Wilcoxon rank sum test) might be introduced. Modern computer-intensive techniques such as the bootstrap could be introduced in such a course. The course should require a major data project from each undergraduate or from groups of undergraduates.

Probability and statistics in the capstone sequence. In addition to the two courses suggested above, a segment of the capstone sequence for teachers should address issues in data analysis, statistics, and probability. As is the case for algebra, geometry, and analysis, the capstone work could address history, common student conceptions, applications, technology, and connections.

Historical perspectives. In comparison with algebra and geometry, the history of probability and statistics is relatively short. The modern theory of probability began with the work of Fermat and Pascal in the mid-17th century and it was not until the 1930s that Kolmogoroff laid axiomatic foundations for the theory. Statistics also traces its roots to events in the 17th century, but formal inference is a 20th-century development and data analysis has been almost reinvented in the last several decades, as computers provided new tools for collection, summary, and representation of distributions. Some knowledge of this history of probability and statistics is a very useful resource for high school teachers.

Common conceptions. In contrast with other branches of mathematics which tend to focus on deterministic relationships, probability and statistics provide tools for reasoning about chance and variation. Research has revealed an interesting array of cognitive difficulties that most people have in dealing with this uncertainty. For example, it is well known that most people will judge that the outcome “HTTHHT” is more likely than the outcome “HHHTTT” in six tosses of a fair coin. Most people will judge that the outcomes “at least 4 heads in 6 tosses of a fair coin” and “at least 400 heads in 600 tosses of the same coin” are equally likely (Shaughnessy, 1992). Few people can accurately explain the meaning of confidence intervals or statistical significance, even after introductory courses in probability and statistics. Preparing high school mathematics teachers for their responsibilities in teaching statistics

and probability will require attention to both the mathematical and psychological aspects of these and many other common conceptions about the subject.

Applications. Current precollege teaching of statistics is generally based in analysis of data that arise from measurements in real or easily imagined situations. Statistical reasoning and arguments appear almost daily in news media and are available in classroom-usable form at a number of Web sites (e.g., http://www.dartmouth.edu/~chance/chance_news/news.html). Thus most future high school teachers won't need much help in connecting statistics and probability concepts to realistic teaching materials. At the same time, they probably do need help in conceptualizing the relationship between data and mathematical models that is at the heart of statistical practice (Moore, 1990, p. 99). This statistical modeling activity can serve to reinforce important concepts in algebra and functions.

Technology. Calculators and computers are now a fundamental part of doing and learning about probability and statistics. They have revolutionized the practice of statistics in the ways that they allow analysis and representation of huge data sets. They have transformed statistics education in much the same way. In preparing to teach probability and statistics to high school students, prospective teachers need to become proficient in the use of standard software tools for simulation, data analysis, data modeling, and inference. They also need to learn the common difficulties that arise when those tools are misused or trusted without careful consideration of the techniques being applied to a given problem situation. Savvy technology use will almost certainly not develop after one or even two beginning statistics courses, but it is a natural consideration in a capstone course for teachers.

Connections. The preceding discussions of algebra, geometry, and functions have indicated a number of ways that probability and statistics connect to those core mathematical topics. The most powerful of those connections is due to the fundamental role of data (which Moore describes as “numbers with context”) in statistical practice. Algebra and functions provide models for data patterns; interesting data sets provide settings for thinking about classes of functions and their symbolic representations. The connections in both directions are very effective means of engaging high school students in significant mathematical study. Teachers who understand and appreciate these connections will have a very useful teaching resource.

Discrete Mathematics and Computer Science

The increasing mathematization of disciplines outside of the physical and engineering sciences has included some impressive new applications of classical mathematics. More significantly, it has also stimulated development and use of a number of new areas that are often collectively referred to as discrete mathematics. Graph theory, enumerative combinatorics, finite difference equations, iteration, and recursion are core topics in this area. They play fundamental roles in computer science, operations research, economics, and biology. Ideas from the theory of games, fair division, and voting are in the part of discrete mathematics that provides tools for social decision-making (Kenney & Hirsch, 1991).

Many of the central ideas of discrete mathematics are as accessible to high school students as parts of classical curricula, and many students find the applications of discrete mathematics very engaging. Furthermore, discrete mathematics topics have important connections to other core strands of the high school curriculum. Recurrence relations provide a powerful strategy for representing and reasoning about functions, graph theory extends visual thinking and geometric reasoning, combinatorics has obvious applications in probability, and game theory/conflict resolution problems make use of matrices.

In light of the emerging importance of topics and methods in discrete mathematics, prospective high school teachers need a course that introduces them to the mathematical ideas, methods, and applications of topics in the following areas:

- Graph theory.
- Enumerative combinatorics.
- Finite difference equations, iteration, and recursion.
- Models for social decision-making.

An upper-division course addressing these topics would also serve majors in mathematics, computer science, and engineering. Except for the last topic, this proposed course is similar to the Discrete Methods course syllabus in the 1981 CUPM Recommendations for a General Mathematical Sciences Program. However, the proposed syllabus still represents a broad agenda, and instructors in the course will have to make choices of topics to survey and topics to develop in depth.

Given the large number of high school students who now enter careers that involve computing and the pervasive use of discrete mathematics in the design of computer hardware, algorithms, and software, it is incumbent on the mathematics community to give greater attention to the mathematical foundations of computer science and computer programming in the schools and in the preparation of future teachers. For prospective high school teachers, a solid one-semester introduction to computer science is essential for connecting mathematics to our computer-driven world.

Most prospective high school mathematics teachers enter college with rudimentary knowledge of computer programming and little knowledge of computer science. Teacher preparation programs commonly include a computer science requirement, often a programming course. Future high school mathematics teachers should have an introduction to computer science that addresses three themes:

- Discrete structures (sets, logic, relations, and functions) and their applications in design of data structures and programming.
- Design and analysis of algorithms, including use of recursion and combinatorics.
- Use of programming to solve problems.

Mathematical Thinking

The preceding descriptions of desirable content preparation for high school mathematics teachers comprise an ambitious agenda, extending and modifying the focus of typical current programs. Most of the mathematics outlined above appears now in one or another of the standard undergraduate courses. Each strand description suggested some new perspectives to be taken in presentation of that content so that students could use advanced mathematical study to gain deeper insight into the structure and uses of high school mathematics topics. But teachers of mathematics convey more than the technical skills of the subject.

By their choice of classroom mathematical tasks and activities and by their personal attitudes and behavior as mathematicians, teachers shape their students' perceptions of what it means to reason mathematically. They suggest and demonstrate the habits of mind that are characteristic of a mathematical approach to questions (Cuoco, Goldenberg, & Mark, 1998). Undergraduate experiences in mathematics should help future teachers understand and practice broadly applicable habits of mathematical thinking: (i) asking and exploring interesting mathematical questions; (ii) framing mathematical concepts and relationships in clear language and notation; (iii) constructing and analyzing proofs; (iv) applying mathematical principles in other disciplines.

In his introduction to the book *On the Shoulders of Giants: New Approaches to Numeracy*, Lynn Steen (1990) remarked that,

Mathematics is not just about number and shape but about pattern and order of all sorts. Number and shape—arithmetic and geometry—are but two of many media in which mathematicians work. Active mathematicians seek patterns wherever they arise.
(p. 2)

In their search for and application of patterns, mathematicians look for attributes like linearity, periodicity, continuity, randomness, or symmetry. They take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. They use abstractions like symbols, infinity, logic, similarity, recursion, and optimization. And they look for contrasts like discrete versus continuous, finite versus infinite, algorithmic versus existential, exact versus approximate, or stochastic versus deterministic patterns. In urging consideration of curricula organized around habits of mind, as much as lists of topics, Cuoco, Goldenberg, and Mark (1998) suggested some similar perspectives on the nature of mathematical work.

In rethinking the mathematical education of future high school mathematics teachers, it seems important to consider also such an agenda of habits-of-mind goals. Most mathematics faculty probably agree with such objectives and even argue that their courses include remarks or assignments designed to cultivate these desirable habits and dispositions. However, students often emerge from their undergraduate experiences with, at best, an unarticulated sense of what it means to be a mathematician. More explicit attention to this aspect of mathematical education may be needed in teacher preparation coursework.

Many mathematics programs also have a sophomore-level course with a title like Foundations of Higher Mathematics that focuses explicitly on developing mathematical thinking, especially skill in reading and writing proofs. These courses

were popular in the 1970s as the level of theory in the mathematics major was being raised. Then in the 1980s, many of the courses were dropped as attention focused on adding breadth in the mathematical sciences. In the past 15 years, in response to serious deficiencies in mathematical thinking among students in junior-level courses such as Abstract Algebra and Introduction to Real Analysis, many departments have reinstated or started such courses. The decision by so many mathematics departments to offer such a course is a strong argument in favor of its value. There are now several well-received texts for this course, although some faculty like to fashion their own individual course from selected readings. Some of these courses and texts focus on mathematical foundations in set theory, logic, relations, and number systems. Others cover number systems and some foundations material and then move on to selected problems in linear algebra, analysis and geometry.

Historical and Cultural Perspectives

Because of its enormous practical value, mathematics is frequently taught as a collection of technical skills that are applicable to specific tasks and often presented without reference to the intellectual struggles that led to contemporary understanding. The few personal names attached to particular mathematical results tend to suggest that mathematics is largely a product of European men.

Over the past several decades there have been important historical and cultural studies that reveal a rich and culturally diverse history of mathematics (Katz, 1992; Trentacosta & Kenney, 1997). We now know a great deal about the difficulty that very good mathematicians had with apparently simple ideas like negative or complex numbers, limits and continuity, and symbolic notation for algebra. We also realize that many non-European cultures have made sophisticated and significant contributions to mathematics.

Few prospective high school teachers come to undergraduate studies with much, if any, understanding of these issues. Some mathematics departments have the faculty expertise and undergraduate enrollment to offer history of mathematics courses. Some may choose to build historical background into the proposed capstone course. In either case, future high school teachers will be well-served by deeper knowledge of the historical and cultural roots of mathematical ideas and practices.

Experiences in Learning Mathematics

Typical collegiate mathematics instruction is associated with some implicit assumptions about learning. One should first hear a knowledgeable person explain the what and how and why of some new idea, then go off alone to see if one understood enough of the lecture to apply ideas and skills appropriately and effectively to tasks similar to those in the teacher's examples. The interaction between teacher and student in the typical college classroom is often limited to the silent thoughts of the students watching and listening. Chalk and talk are the primary instructional media. Emerging from extended experiences with this style of mathematics teaching and learning, prospective high school teachers frequently enter their methods courses and field experiences with with two main models of teaching: one from college and one from precollege schooling. Often, neither of these models suffices for the demands of current high school curricula.

The insight and personal enthusiasm of an inspiring lecturer often contribute to successful experiences for college students. However, in many cases the results are far from optimal. When even the most lucid explanations don't effectively communicate the big ideas of a mathematical topic, students are often reduced to memorization of problem-solving routines through imitation of worked examples in lectures and text materials.

Over the past 10 to 15 years, some instructors have experimented with ways to have undergraduates more actively involved in the intellectual work of class meetings. In such classes, students collaborate with each other on investigations that pose challenging problems with solutions that reveal important mathematical principles. They use calculators and computers as tools in those investigations and in composing reports of their findings. Such changes in teaching, if appropriately made, can result in better learning—more conceptual understanding, improved abilities to communicate orally and in writing, and to solve problems (see, e.g., Darken, Wynegar, & Kuhn, 2000; Harel & Sowder, 1998; Schoenfeld, 1985; Springer, Stanne, & Donovan, 1999). Moreover, such instruction gives prospective teachers experience with the kind of instruction they will be asked to use in their future careers.

Conclusion

The recommendations in this chapter offer a curriculum for the mathematical preparation of teachers that, compared to the chapters on elementary and middle school mathematics teachers, does not call for radical changes from past recommendations by groups such as the MAA Committee on the Mathematical Education of Teachers. The main new component recommended here is a year-long capstone course for prospective high school teachers that connects the major strands in courses for mathematics majors with high school mathematics. Other recommendations are: some refocusing of upper-division courses in abstract algebra and geometry, a new course in statistics, and some enhancement of lower-division courses.

Many of these recommendations may be as appropriate for all mathematics majors as they are for prospective teachers. However, the capstone sequence is aimed specially at future teachers. This course is an opportunity to look deeply at fundamental ideas of mathematics, to connect topics which students often see as unrelated, and to develop the important mathematical habits of mind. It connects logically to the coursework in mathematics education that also makes a strong positive contribution to the mathematical understanding and pedagogical skills of teachers (Monk, 1994). The design and teaching of this course is a natural point for collaboration of mathematics and mathematics education faculty who have joint responsibility for teacher preparation.

References

- Bransford, J., Brown, A., & Cocking, R. (Eds.). (1999). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.
- Begle, E. (1979). *Critical variables in mathematics education: Findings from a survey of empirical literature*. Washington, DC: Mathematical Association of America.

- Boyer, C. B. (1991). *A history of mathematics* (revised by U. C. Merzbach). New York: Wiley.
- Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. In C. B. Cazden (Ed.), *Review of research in education*, 16, 3-56.
- Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior*, 15(4), 375-402.
- Darken, B., Wynegar, R., & Kuhn, S. (2000). Evaluating calculus reform: A review and a longitudinal study. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (pp. 16-41). Providence, RI: American Mathematical Society.
- Darling-Hammond, L., Wise, A., & Klein, S. (1995). *A license to teach: Building a profession for the 21st century schools*. Boulder, CO: Westview Press.
- Dubinsky, E. & Harel, G. (Eds.). (1992). *The concept of function: Aspects of epistemology and pedagogy* (MAA notes no. 25). Washington, DC: Mathematical Association of America.
- Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Providence, RI: American Mathematical Society.
- Heid, M. K. (1997). The technological revolution and the reform of school mathematics. *American Journal of Education*, 106, 5-61.
- Henderson, D. (2000). *Experiencing geometry: In Euclidean, spherical, and hyperbolic spaces*. Englewood Cliffs, NJ: Prentice-Hall.
- Katz, V. (1993). *A history of mathematics: An introduction*. New York: Harper Collins.
- Katz, V. (Ed.). (2000). *Using history to teach mathematics: An international perspective*. Washington, DC: Mathematical Association of America.
- Kenney, M. J. & Hirsch, C. R. (Eds.). (1991). *Discrete mathematics across the curriculum, K-12*. Reston, VA: National Council of Teachers of Mathematics.
- Kieran, C. (1992). In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan Publishing.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.
- Mathematical Association of America. (1989). CUPM Recommendations for a General Mathematical Sciences Program. In *Reshaping college mathematics* (MAA notes no. 13). Washington, DC: Mathematical Association of America. (Original work published 1981)
- Mathematical Association of America. (1983). *Recommendations on the mathematical preparation of teachers*. Washington, DC: Author.
- Monk, D. A. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review*, 13(2), 125-145.

- Moore, D. S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95-138). Washington, DC: National Academy Press.
- Moore, D. S. & McCabe, G. (1999). *Introduction to the practice of statistics*. New York: Freeman.
- National Center for Research on Teacher Education. (1991). *Final report*. East Lansing, MI: Author.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Shaughnessy, M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: Macmillan Publishing.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Siu, M. K. (1991). Concept of function—its history and teaching. In F. Swetz et al. (Eds.), *Learn from the masters* (pp. 105-121). Washington, DC: Mathematical Association of America.
- Springer, L., Stanne, M. E., & Donovan, S. S. (1999). Effects of small-group learning on undergraduates in science, mathematics, engineering, and technology: A meta-analysis. *Review of Educational Research*, 69(1), 21-51.
- Steen, L. A. (1990). Pattern. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 1-10). Washington, DC: National Academy Press.
- Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.). (1995). *Learn from the masters*. Washington, DC: Mathematical Association of America.
- Trentacosta, J. & Kenney, M. J. (Eds.). (1997). *Multicultural and gender equity in the mathematics classroom: The gift of diversity*. Reston, VA: National Council of Teachers of Mathematics.