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Teachers' Mathematics: A Collection of Content Deserving To Be A Field

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Introduction

Like the authors of the MET report, we believe that to teach well, a teacher should know a great deal of mathematics. The higher the level taught, the more the teacher needs to know. For a teacher of middle school or high school mathematics, this means knowing a good deal of algebra, geometry, analysis, statistics, number theory, computer science, and mathematical modeling. This mathematics constitutes the traditional background of a teacher who is considered to be well-prepared mathematically.

In recent U.S. National Assessment data at the 8th grade level (Hawkins, Stancavage, and Dossey 1998), students of teachers with a mathematics major or minor performed higher than students of teachers who had not majored or minored in mathematics. There are important reasons beyond student performance why teachers need to know more advanced mathematics: one of the main reasons we teach mathematics is to prepare students who need to have an advanced knowledge of mathematics for their careers. Teachers need to know the various ways in which the mathematics they teach is applied later and to distinguish those ideas that are fundamental from those that are enrichment. Deductive proof and statistical inference have some similarities, but also they have some fundamental differences. The discrete mathematics used in computer science is different from the mathematics covered in most engineering courses. And teachers need to know the different ways in which people who use mathematics approach problems.

Even though taking more and more mathematics would not seem to have any down side, it can create a problem. Often the more mathematics courses a teacher takes, the wider the gap between the mathematics the teacher studies and the mathematics the teacher teaches. The result of the mismatch is that teachers are often no better prepared in the content they will teach than when they were students taking that content. A beginning teacher may know little more about logarithms or factoring trinomials or congruent triangles or volumes of cones than is found in a good high school text. It may be because of this mismatch that some studies have found that the number of mathematics courses taken by a teacher does not influence the performance of the students of that teacher.

The HSMFAS Project

For the past 2.5 years, Dick Stanley of the University of California at Berkeley, Tony Peressini of the University of Illinois, Elena Marchisotto of Cal State - Northridge and I have been writing and testing mathematics materials for preservice and inservice high school teachers. Although no other members of the writing team are at this conference, four people who have taught from the materials are here: Michael Keynes of Purdue University Calumet (who taught the materials at Berkeley) and Apple Bloom from the University of Arizona taught parts of a first draft, and Abie Herzig of Rutgers and Gordon Woodward of the University of Nebraska are field testing the current version of the materials.

Our work is in line with recommendation 2(iii), found on page 8 of the MET report, that "Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics, that includes a 6-hour capstone course connecting their college mathematics courses with high school mathematics."

This *teachers' mathematics* includes at least the following aspects: (a) ways of explaining and representing ideas new to students, (b) alternate definitions and their consequences, (c) why concepts arose and how they have changed over time, (d) the wide range of applications of the mathematical ideas being taught, (e) alternate ways of approaching problems with and without calculator and computer technology, (f) extensions and generalizations of problems and proofs, (g) how ideas studied in school relate to ideas students may encounter in later mathematics study, and (h) responses to questions that learners have about what they are learning.

Teachers' mathematics is not merely a bunch of mathematical topics that might be of interest to teachers but a coherent field of study, distinguished by at least three broad ideas: *concept analysis* – the phenomenology of mathematical concepts (including aspects a, b, c, d, and h above); *problem analysis* – the extended analyses of related problems (including e and f above); and the *connections and generalizations* within and among the diverse branches of mathematics (including g above but also related to many other aspects).

We had the following goals:

1. Mathematics of relevance to the classroom. The content should emanate from the mathematics of the high school classroom (in contrast to emanating from college-level mathematics), out of the teaching and learning of mathematics, in a way analogous to the way that statistics emanates from real data and operations research emanates from problems of scheduling and optimization.

2. Mathematics, not methods. Yet, because we want the course to count as a mathematics course towards degree requirements, the course should not be a course in teaching methods. It should be rich enough mathematically that mathematicians not in mathematics education would support this course.

3. Junior-senior level for preservice; master's level for inservice. For the approaches we had in mind, the student needs to be able to do proofs and handle abstractions, so the materials are designed for a junior or senior mathematics major or a practicing teacher who

has gone through calculus and had at least one or two courses beyond that. These are not remedial materials.

4. More material than can be taught in one (or two!) courses. We promised our funders at least one course but wanted to have more content available than could be taught in a single course. This would give instructors some choice and illustrate that there is an entire field of content which is deserving of at least two courses in the mathematics education of teachers by the time they have a bachelor's degree, and further courses if a teacher is going on for a master's degree.

5. Movement towards a set of canonical courses in the field. Perhaps if we did our job well, and if there were others who agreed with some of the ideas we put forth, this experience for teachers would change from one that is in so many places a topics course involving the favorite mathematical topics and hobbies of the professor who happens to be teaching it to a course that has the concepts and perspectives of particular importance to teachers.

It turns out that we have more material than can be taught in two normal courses, and we know we have not in any way covered the field. This is consonant with the MET recommendation mentioned above, since to my knowledge nowhere does there exist a 6-hour mathematics course. More than one course is needed to implement the recommendation.

We first called these materials *High School Mathematics from an Advanced Standpoint* (HSMFAS). That also is the name of the project, which is funded by the Stuart Foundation. We have now changed the title of the materials to *Mathematics for High School Teachers: An Advanced Perspective* because some people misunderstood the first title and thought the materials were for very good high school students, and other people found that they could not convince their mathematics departments to give graduation credit to a course with a text with "High School Mathematics..." in its name.

In this paper, I offer one example of each of the broad ideas mentioned above, as they are found in these materals..

Concept Analysis – The concept of *parallel*

Possible Definitions

What does the word *parallel* mean? Such a question seems possible to answer merely by invoking a definition. For most teachers students *parallel lines* are lines in the same plane that do not intersect. But this is not the only possible definition, and it is restricted to lines. In James and James *Mathematics Dictionary*, the word parallel is defined as “equidistant apart”. This dictionary then describes parallel curves, parallel lines, parallel rays, parallel planes, parallel surfaces, parallel vectors, parallels of latitude, etc. For *parallel rays*, the entry states that sometimes it is required that they point in the same direction. So under some definitions, opposite rays would be parallel and other definitions they would not be parallel.

These simple cases show us three different characterizations of *parallel*:

do not intersect
equidistant apart
go in the same direction.

By picking one of these for the definition, we may lose sight of the other possibilities, and we may lose opportunities to use intuition.

Examining Instances

Draw a parallelogram. Do its opposite sides *feel* parallel to you because they are equidistant, or is it because they do not intersect, or is it because they go in the same direction? For most of us, being equidistant or going in the same direction is the predominant feel (particularly if the sides are not horizontal). Similarly, our intuition is that corresponding angles formed by parallel lines have the same measure because their sides (parallel rays!) go in the same directions, not because their sides are equidistant from one another or do not intersect.

Students may have difficulty with the idea of parallelism because the definition we choose for parallel does not fit the intuition used in instances of the idea. Concept analysis includes this kind of examination.

Generalizations

The results of concept analysis make us realize that mathematics is not as rigid as it is sometimes made out to be, and these results may suggest changes in our formalization of the idea. For instance, for a given function f, the graphs of $y = f(x)$ and $y = f(x–k)$ are reasonably viewed as parallel. James and James define *parallel curves in a plane* as two curves which have their points paired on the same normals, always cutting off segments of the same length on these normals. This formal definition in terms of normals (perpendiculars to the tangents at points of the curves) agrees with intuition. But it arises from thinking of parallels as equidistant rather than going in the same direction or not intersecting. In fact, under this definition, the graphs of $y = \sin x$ and $y = \cos x$ are parallel curves, yet these parallel curves can intersect.

Can a line be parallel to itself? Not according to the definitions in many schoolbooks. Yet two *vectors* u and v are parallel (again according to James and James) if and only if there is a nonzero scalar k for which $u = kv$, that is, if they lie along the same line when they are represented by arrows with the same initial point. Just as with parallel rays, sometimes k is required to be positive so that the vectors point in the same direction. And a vector is parallel to itself.

With the idea of two lines being parallel if and only if they go in the same direction, then a line is parallel to itself. Under this conception, two lines in the coordinate plane are parallel if and only if they have the same slope, even if the two lines are identical.

History

The concept of parallelism has played an important role in the history of geometry, due to the influence of the *Elements* of Euclid. In the last chapter of our materials, we examine Euclid's parallel postulate in some detail. One of the interesting points about this postulate is that it does not mention the word "parallel".

Applications

No analysis of a concept is complete without some examination of its applications. Railroad tracks are often presented as a prime example of parallel lines. This is true when the tracks are straight but not otherwise unless a broader definition of parallel is employed. We might think of two north-south streets as parallel, but in theory all north-south streets intersect at the poles of Earth. Examples on a smaller scale are easier to find: walls of a room, lines on a sheet of lined paper, opposite sides of trapezoids and many other geometric figures, and so on.

Mathematical definitions are formulated within a mathematical system so that deductions can be made from them. The theorems that are deduced add to our understanding of the idea and can sometimes become alternate definitions for the idea. However, these abstract formulations may not precisely represent the intuitive ideas that created the need for the definition in the first place. Knowledge of the variety of possibilities can assist teachers in knowing why students have trouble both in using their intuition and in applying the abstractions.

Mathematical Extensions and Generalizations

A second kind of mathematics that a teacher needs to know consists of extensions and generalizations of the content they teach. For an example of extension, let us begin with the well-known formula for the area of a triangle, $A = \frac{1}{2}bh$, where b is the base and h is the height of the triangle using that base. This is the most commonly taught formula for the area of a triangle even though the height of a triangle is not one of the sides or angles of a triangle, not one of the parts of a triangle used to prove congruence.

Almost all high school geometry textbooks mention a second formula, Hero's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, which has the advantage of giving a triangle's area in terms of its three sides. A proof of Hero's formula requires either quite a lengthy synthetic geometric argument, or some algebra beyond most geometry students, or trigonometry. For this reason, a proof of Hero's formula is seldom found in today's geometry textbooks even when the formula is presented. Teachers should see at least one proof.

Quite a bit less known, because trigonometry is studied after geometry, is that another area formula for a triangle is $A = \frac{1}{2}ab \sin C$. This gives the area of a triangle in terms of two sides and an included angle. From this formula, it is easy to prove that the maximum area of a triangle with sides of known lengths a and b is when they are legs of a right triangle. Teachers should know this formula, too, and the corresponding formula for the area of a parallelogram with adjacent sides a and b and included angle C, Area = ab sin C.

We can think of Hero's formula and the formula $A = \frac{1}{2}ab \sin C$ as SSS and SAS area formulas. They motivate us to look for an area formula for a triangle given AAS or ASA. One such formula is $A = \frac{1}{2}a^2\frac{\sin B \sin C}{\sin(B+C)}$. This derivation is a nice problem for teachers.

What benefit is there in knowing these formulas? This knowledge changes our view from $A = \frac{1}{2}bh$ as *the* formula for the area of a triangle or *the most important* formula to *one of many* formulas for the area. Knowing many formulas shows how the different combinations of elements that determine a triangle can be employed to determine its area. It gives us another reason to learn some trigonometry. This knowledge piques our curiosity for still more formulas. My own favorite triangle area formula is $A = \frac{abc}{4R}$, where R is the radius of the circle containing the three vertices of the triangle.

These triangle area formulas *extend* typical content. Here is an example of *generalizing* content. In the box below are a number of properties used in solving equations and inequalities. Students usually learn these separately over perhaps four years, from 8th to 11th grade. For all real numbers a, b, and c, in solving equations we teach these properties.

Addition Property of Equality:	$a = b \Leftrightarrow a + c = b + c$.
Multiplication Property of Equality:	If $c \neq 0$, then $a = b \Leftrightarrow ac = bc$. (and $c \neq 0$ is necessary!)
Squaring non-Property (!):	If you square both sides of an equation, or take the square root of both sides of an equation, you may gain or lose solutions.
Cubing:	$a = b \Leftrightarrow a^3 = b^3$
Exponential Equations:	If $c > 0$, then $a = b \Leftrightarrow c^a = c^b$.
Trigonometric Equations:	If $0 \leq a \leq 90°$ and $0 \leq b \leq 90°$, then $\sin a = \sin b \Leftrightarrow a = b$, and $\cos a = \cos b \Leftrightarrow a = b$, but one of these is not true if $0 \leq a \leq 180°$ and $0 \leq b \leq 180°$.

We say that you can add the same number to both sides of an equation, or multiply both sides of an equation by the same non-zero number, and the resulting equation is equivalent to the given one. But if you square both sides of an equation, you may gain solutions. Yet cubing both sides of an equation does not affect the real solutions. With exponential equations, you seem to be able to cancel bases. And trigonometry seems not to follow any rules. These results, which may seem to students to be inconsistent, suggest other questions. Does taking the absolute value of both sides affect solutions to an equation? How can one tell, in general, whether an operation applied to both sides of an equation will change the solutions to the equation?

An answer to this question is found by considering an equation in one variable as being of the form

$$f(x) = g(x).$$

Applying an operation to both sides is like applying a function h to both sides. This results in the equation

$$h(f(x)) = h(g(x)).$$

The general theorem is:

> On a particular interval, the equations $f(x) = g(x)$ and $h(f(x)) = h(g(x))$ are equivalent if and only if on that interval h is a one-to-one function on the ranges of $f(x)$ and $g(x)$.

Examining this theorem and its special cases unifies the solving of equations and gives new insight into the process of equation-solving.

The corresponding general theorem for inequalities explains not only when inequalities are equivalent but also why the sense of the inequality changes some times but not others.

> On a particular interval, $f(x) < g(x)$ is equivalent to $h(f(x)) < h(g(x))$ if and only if on that interval h is an increasing function on the ranges of $f(x)$ and $g(x)$. $f(x) < g(x)$ is equivalent to $h(f(x)) > h(g(x))$ if and only if on that interval h is a decreasing function on the ranges of $f(x)$ and $g(x)$.

I particularly like these theorems because they are significant applications of the idea of one-to-one-ness and of increasing and decreasing functions. Students usually view these ideas as mathematical formalisms of obvious things that have no simple non-obvious applications.

These are only a few of the many theorems that extend or generalize properties that high school students are expected to know. A whole host of natural generalizations are found in abstract algebra, and many years ago I wrote about them (Usiskin 1975). They are typically not encountered by teachers in their college mathematics courses because college mathematics courses tend to look ahead to what is needed for graduate school rather than look back to what would be helpful in understanding pre-college mathematics.

Problem Analysis – The test average question

A third kind of mathematics needed by teachers but rarely encountered by them, and different from concept analysis, is the analysis of problems. Problem analysis involves more than finding different ways of solving a problem. It includes looking at a problem after it has been solved and examining what has been done. Will the method of solution work for other problems? Can we extend the problem? And so forth.

The problem and its solution

We begin with a typical problem found in algebra books and on tests.

> Jane has an average of 87 after 4 tests. What score does she need on the 5th test to average 90 for all five tests?

To answer this question, the algebra student is expected to let a variable such as x stand for Jane's score on the 5th test and to solve an equation such as $\frac{4 \bullet 87 + x}{5}$ = 90. But most students (and most attending the sessions at this conference where this problem was presented) use arithmetic. One way is to notice that Jane has been 3 points down on each of 4 tests from the average she needs. So she is 12 points short. Thus she needs 90+12 on the 5th test. Another way is to begin by noting that to average 90 points on 5 tests means to have 450 points. Jane has 348 points, so that she needs 102 points. The algebra parallels this second way. Solving the equation, we find that x = 102, so that if the last test only allows 100 points, she cannot average 90. Since the problem can be so easily solved without algebra, students naturally wonder why algebra is needed. Thus, though one reason for presenting this problem in an algebra class is to show the power of algebra, the effect is just the opposite. If we stop here, then we have shown that algebra is not needed.

Generalizing the problem

We inquire into the relationship between Jane's score on the 5th test and her average. Then we cannot avoid algebra. Let A equal Jane's average for all 5 tests. Then $\frac{4 \bullet 87 + x}{5}$ = A. We can use this equation to determine what Jane needs to obtain any given average, not merely an average of 90. If we think of x as the domain variable and A as the range variable, this is a linear equation with slope $\frac{1}{5}$. It indicates that each point Jane gets on the 5th test adds $\frac{1}{5}$ to her average. In fact, any point Jane gets on any test contributes $\frac{1}{5}$ to her average. A graph of $\frac{4 \bullet 87 + x}{5}$ = A shows all the possible solutions. The A-intercept 69.6 is the average Jane has if she gets 0 on the 5th test; it is the lowest average possible for her.

Extending the problem

The problem analysis could end here, but one important aspect is to relate this problem to similar problems. Instead of the average being a fixed goal, suppose the goal itself is variable. We use an actual example. In April, 1998, the basketball players Michael Jordan and Shaquille O'Neal were vying for the individual scoring title in the NBA. At the beginning of the last day of the season, Michael had 2313 points in 81 games, for an average of 28.6 (customarily, averages in newspapers are rounded to the nearest tenth). Shaq had scored 1666 points in 59 games, for an average of 28.2 points per game. With one day left in the season, no one else had a chance to win the scoring title. We phrase the problem to show its similarity to Jane's test problem.

Shaq has 1666 points after 59 games. How many points does he need in his 60th game to have a higher average than Michael, who has 2313 points after 81 games and also has one game to play?

Algebratizing the new problem

An analysis could begin using only arithmetic. Just estimate how many points Jordan and Shaq score in their last games, and then calculate, on the basis of those guesses, who will win the scoring title. If there were only a couple of possibilities for the numbers of points scored, this arithmetic would give you rather quickly all of the possible outcomes. But there are many possibilities: For this reason it is useful to consider all the scenarios at one time, and again algebra is needed.

If Jordan scores j points in his last game, then he has a total of 2313 + j points in 82 games, for an average of $\frac{2313 + j}{82} \frac{\text{points}}{\text{game}}$. Similarly, if s is the number of points Shaq scores in his last game, he will then have 1666 + s points in 60 games, for an average of $\frac{1666 + s}{60} \frac{\text{points}}{\text{game}}$. Thus Jordan wins the scoring title whenever

$$\frac{2313 + j}{82} > \frac{1666 + s}{60} .$$

On the other hand, Shaq wins the scoring title whenever

$$\frac{2313 + j}{82} < \frac{1666 + s}{60} .$$

A graphical solution

Teachers must have some idea of what it means to *solve* inequalities like those on the preceding lines. We cannot simply say j equals this and s equals that, because there are infinitely many solutions even if one of the variables is fixed. So we graph the boundary to the inequalities (Fig. 1). That is the line with equation

$$\frac{2313 + j}{82} = \frac{1666 + s}{60} .$$

Solving for s in terms of j, $s = \frac{30j + 1084}{41}$.

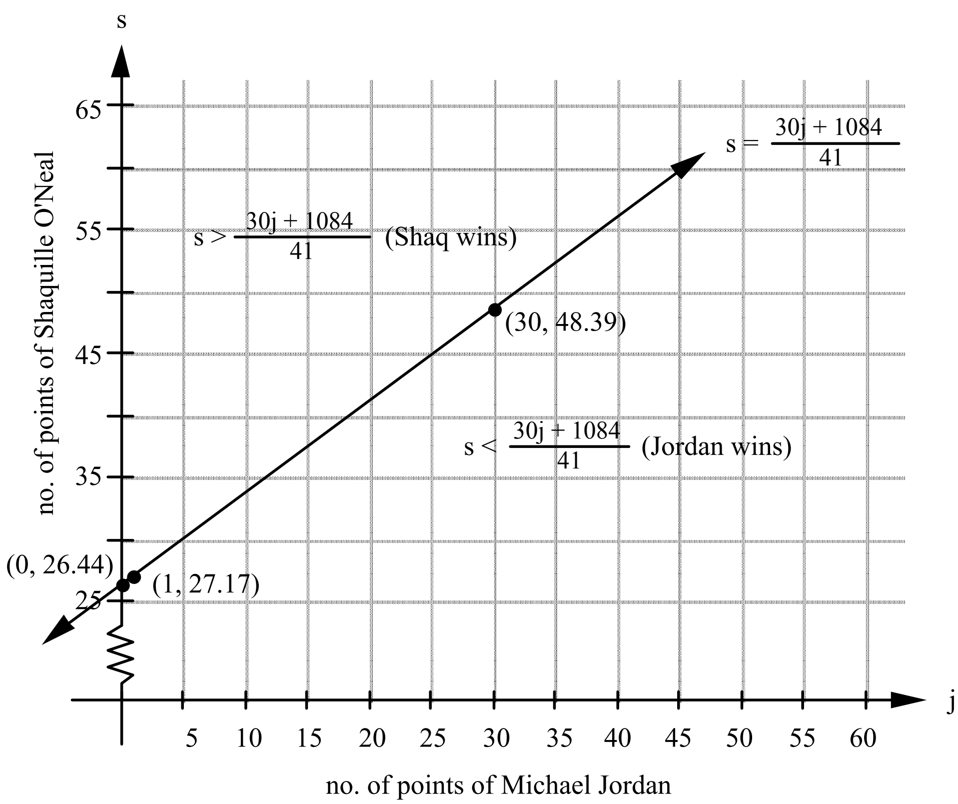

Fig. 1

Each of the points on the line has meaning. If Jordan scores 0 points, Shaq still needs to score over 26.44 points to win the title. That is, he needs 27 points or more. If Jordan scores 1 point, Shaq needs over 27.17 points – 28 points or more – to win. If Jordan scores 30 points, Shaq needs 49 points or more. We see that the lattice points *above* the line in the first quadrant offer all the possible ways in which Shaq wins. The lattice points *below* the line in the first quadrant show all the possible ways in which Jordan wins.

A surprise possibility

Notice that the line $s = \dfrac{30j + 1084}{41}$ has slope less than one. This means that it intersects the line s = j. Thus there are points above the line $s = \dfrac{30j + 1084}{41}$ that are below the line s = j. These points indicate situations in which Jordan can score more points than Shaq yet still lose the scoring title. This is an instance of Simpson's paradox.

It happened that Jordan scored 44 points in this last game of the regular season. When j = 44, s = 58.63..., which meant that Shaq had to score 59 points in his last regular season game to win the title. That is, Shaq would need a personal record for him to win the scoring title. Shaq scored 39 points, which is terrific scoring, but was not good enough to win the scoring title.

A function approach

Let us take the analysis one step further. Just as many students do not see why they need algebra for a problem that can be solved using arithmetic, many students do not see how functions might help in the interpretation of an algebra problem. In the Jordan-Shaq situation, two players are vying for the better average at the end of the season, rather than one student wondering what her test average would be. Suppose we let f(j) = Jordan's average if he scores j points in the last game, and g(s) = Shaq's average if he scores s points in the last game. Above we showed that $f(j) = \frac{2313 + j}{82}$, and $g(s) = \frac{1666 + s}{60}$. The equation of the line is when the two averages are the same, that is, it is f(j) = g(s). Since g is a linear function, it has an inverse, and so $g^{-1} \circ f(j) = g^{-1} \circ g(s) = s$. That is, $s = g^{-1} \circ f(j)$. Thus the equation that we graphed, $s = \frac{30j + 1084}{41}$, is the equation of $g^{-1} \circ f$.

Notice the meaning of $g^{-1} \circ f$. The function f maps Jordan's last game points onto Jordan's season average. The function g maps the number of points Shaq gets in the last game onto his average for the season. If the season averages are to be equal, then g^{-1} is mapping Shaq's average, which is Jordan's average, onto the number of points Shaq needs to get that average. Thus the function $g^{-1} \circ f$ maps the number of points Jordan scores in the last game onto the number of points Shaq needs to have the same average as Jordan. In this way, function composition and function inverses both generalize and provide an explanation for a common algebra problem.

Teachers' mathematics as applied mathematics

There is a huge amount of material that falls under the heading of teacher's mathematics – we have even in our materials only touched the surface. This material is usually picked up by teachers only haphazardly– through occasional articles in journals, by attending conferences, by reading the teachers' notes found in their textbooks, or by examining research in history and conceptual foundations of school mathematics. This mathematics is often not known to professional mathematicians. It covers both pure and applied mathematics, algorithms and proof, concepts and representations. The only way we will get this material to our teachers on a broad scale is if "teachers' mathematics" obtains its own place in the curriculum.

Teachers’ mathematics is a branch of applied mathematics, applied because it emerges directly from problems in the classroom. Teachers’ mathematics comes out of the teaching and learning of mathematics. The importance of teachers' mathematics thus goes well beyond the need of teachers to include all those who study the learning of mathematics and the mathematics curriculum.

Like other branches of applied mathematics, teachers' mathematics uses only a part of all known mathematics, and favors certain areas. Number theory, geometry, and the foundations of mathematics are important to teachers' mathematics in the same way that probability is important to actuarial science and graph theory is important in operations research. Also, like other branches of applied mathematics, knowledge in non-mathematical areas is very helpful. Thus, just as a financial analyst needs to know about the various investment possibilities available, a person studying statistics needs to know about sampling and the construction of tests, a teacher needs to know about learning theory, student motivation, and the effects of schooling and testing on student learning.

Teachers' mathematics encompasses a broad range of mathematics. It should include concepts from all the mathematical sciences that pertain to a given idea. Its essence is shaped by the various ways to approach a concept and by various ways of solving problems. It is the antithesis of a narrow research field.

Our thinking about teachers’ mathematics is not new. Mathematics courses for teachers have existed throughout the century. In the mathematics departments of most teacher-training institutions, there are courses especially designed for teachers. But it is almost always the case that no two of these courses are alike. Courses for mathematics teachers are mostly filled with material that is the hobby of their instructors.

What may be new in this work is our view of teachers' mathematics as a branch of applied mathematics, our view that this branch of mathematics is not watered-down content but more appropriate content, and our view that the body of knowledge represented in teachers' mathematics is huge and deserving of attempts by individuals and groups to structure it. In the past, we have often organized mathematics course programs for teachers by selecting those that seem to be most beneficial out of the vast array of courses offered for future pure and applied mathematicians. Teachers need to take some of these courses, but at least as important is the need to take a number of mathematics courses that start from the ground up, from the problems faced in the classroom.

Why is it so difficult to develop good mathematics teachers? A fundamental reason is that teaching mathematics is difficult to do. While some people seem to think that all that is needed to get students to understand a new theorem or new result is to present careful definitions and display a coherent logical argument proving the theorem, any schoolteacher knows that this is not the case. Students connect ideas in different ways, and the thoughtful teacher needs to know multiple pathways that students can take through the material. The teacher needs to be able to respond "on the spot" to a myriad of student questions concerning content and approach, from questions that represent misconceptions to questions that plow new ground. This complexity of teaching was mentioned by Roger Howe and elaborated upon by Deborah Ball and Hyman Bass at this conference.

There is a potential bonus in mounting courses in teachers' mathematics. Many students enter college liking mathematics and thinking about teaching it as a career but are subsequently turned off because the mathematics courses they take are not related to the mathematics they liked in high school. Though it is not easy, teachers enjoy teachers' mathematics because it is related to their work. By closing the gap between coursework and profession, we will increase the amount of mathematics that teachers take, we may increase the popularity of the major in mathematics teaching, and we may be able to make some headway into the shortage of mathematics teachers being trained.

We make no claim that our course or our approach is the answer to the issue of mathematics courses for teachers. But we think that the ideas in it should be encountered by all teachers at some time, and so we are trying to provide a framework that makes it easier for mathematics faculty to do that.

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