Mathematical Practice, High School

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The work of teaching mathematical practice

There are certain mathematical proficiencies—things like perseverance, sense making, using precision, and seeking structure—that are different from the proficiencies associated with technical expertise in a particular content area. Common Core State Standards (CCSS) include Standards for Mathematical Practice that make these proficiencies an integral part of what teachers are expected to teach in the coming years. Quoting from CCSS:

> *The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years.*

The work of helping students become proficient in mathematical practice is situated inside the teaching of specific content, and hence teachers are faced with the difficult task of teaching a course in, say, trigonometry, in a way that helps students understand the body of results and methods while fostering their development of mathematical habits of mind.

Elsewhere in this report, we describe some subject-specific "faces" of mathematical practices; seeking structure in algebraic expressions is somewhat different from doing the same in geometric diagrams, and both of these are different from seeking and using structural similarities among algebraic systems. Here, we concentrate on some of the dispositions involved in teaching mathematical practice in any domain. In the work of teachers who raise mathematical practice to the same level of importance as the results of that practice, one often finds features like these:

- **Classrooms that put experience before formality.** Worked-out examples and careful definitions are important, but in classrooms where students are learning to do mathematics, they grapple with ideas and problems *before* these things are brought to closure.

- **Lessons that have textured emphasis.** Teaching mathematical practice requires one to separate matters of convention and vocabulary from matters of mathematical substance. Both are important, but students come to understand each in very different ways.

- **An overall parsimony of methods.** A focus on mathematical practice leads to a small number of general purpose tools and methods that are employed in seemingly different contexts. Teachers who focus on mathematical practice don't use different methods for mathematical domains that are connected in basic ways; the skills of building equations for lines and circles differ in technical details, but the same ideas underlie both.

- **Mathematical coherence at every level of detail.** Teachers who work to build mathematical practice in their students have an eye towards how everything they teach fits into the larger landscape of mathematics as a scientific discipline. They develop completing the square as a special case of removing terms from a polynomial via a linear substitution. They use a small set of examples—maximizing area for a given perimeter, for example—over and over, each time showing how different parts of mathematics can be used to gain insight into the example.

- **Explicit attention to mathematical thinking.** There is a striking lack of reliance on memorized formulas in classes that focus mathematical practice. Students do use formulas, but, for example, they are just as likely to use the Pythagorean theorem to find the distance between two points as they are to use the "distance formula;" indeed, they see the latter as a special case of the former. Students evaluate their expertise on the basis of what they can figure out rather than on what they already know.

Key understandings to support this work

A basic prerequisite for teaching mathematical practice is to have engaged in the practice of mathematics: almost all teachers who use the styles of work described in the previous section have had a personal and intense experience of doing mathematics for themselves.

The understandings one gains from such an experience are somewhat different from what one gets in a typical undergraduate or professional development setting, where the goal is to help one understand established mathematical terrains. Key understandings for teaching mathematical practice include all the things that mathematicians do in their work, things like

- Performing thought experiments

- Generalizing from examples

- Reasoning by continuity

- Seeking invariants
- Using deduction as a research technique
- Changing variables to hide complexity
- Extending operations to preserve rules for calculating
- Seeking and using structure

Higher education mathematics faculty are ideally suited to help prospective and practicing teachers understand and develop these and many other mathematical practices; after all, these habits capture the styles of work that are second nature for mathematicians. But precisely because they are habits, they are often not made explicit. One of the recommendations of this report is to put these mathematical practices front and center in courses, seminars, workshops, and mathematical experiences for teachers.

Illustrative examples

Mathematical Coherence. A standard exercise in elementary algebra involves the simplification of the expression

$$(a+b)^2 - (a-b)^2$$

This can be carried out by expanding and collecting like terms or by seeing it as a difference of squares that leads to the identity

$$(a+b)^2 - (a-b)^2 = 4ab$$

This identity reminds one of a geometric context. By writing it as

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

the identity shows that a square maximizes the area among all rectangles of the same perimeter (as a bonus, it shows by how much the square's area exceeds that of the rectangle). And the identity can be mined for more: It contains a proof of the arithmetic-geometric mean inequality for two variables. Hence a simple exercise from elementary algebra can be pulled throughout high school to form a coherent thread.

Experience before formality. Students are given a half dozen points and asked to find, of all lines of slope 3, the line that best approximates the data. Few calculators can do this, but the problem amounts to minimizing a quadratic function, something squarely inside a first-year algebra course. With very little machinery, students can find the best line of slope 3. Then they can do it for other slopes. Graphing all of their "best" lines, the get a nice surprise—the lines are all concurrent in the centroid of the data. This is a jumping off point for a teacher embark on a thorough development of regression.

Explicit attention to mathematical thinking The class is trying to find a formula for the area of a trapezoid by dissecting the trapezoid into a parallelogram and keeping track of what happens to the relevant dimensions. Students are asked to put their formulas on the board, producing a collection like this:

$$A = \frac{h}{2}(b_1 + b_2), \quad A = h\left(\frac{b_1 + b_2}{2}\right), \quad A = \frac{b_1 h}{2} + \frac{b_2 h}{2}, \quad A = \frac{1}{2}\left((b_1 + b_2)h\right)$$

Students are then asked to pick a formula in a different form from their own and to find a dissection algorithm that would lead one to that particular form. Their algorithms are compared with the ones of the students who originated each form.