The work of teaching functions

Functions are general purpose tools that pervade all of mathematics, and hence teachers are faced with helping students build and use different kinds of functions in every course and in every mathematical area, from algebra to geometry to probability to calculus.

State standards and published curricula have concentrated on helping students interpret and use standard types of functions defined on the real numbers—linear, quadratic, exponential, and trigonometric. This involved using function notation and purposely transforming formulas that define functions to reveal certain information—extrema, end behavior, and so on.

The Common Core State Standards (CCSS) keep attention on these skills but adds explicit standards that require students to build functions of their own to model some situation or to express some perceived regularity. Teachers know that, for most students (and for most adults, as well), creating functions for a purpose is much harder than using “stock” functions. For example, many students who can compute the cost of a text messaging plan for a fixed number of messages find it very difficult to create an algebraic expression that defines a function that will do the calculation for any number of messages. Helping students develop the skill of expressing general relationships with functions is intimately tied up with the mathematical practice that CCSS calls expressing regularity in repeated reasoning.

Student-created functions are often not conceived in standard algebraic forms—they are often expressed as incomplete recurrences or as a mix of formal algebra and verbal descriptions. The work of teaching functions requires one to find ways to help students refine and make precise what they are trying to express about the regularity they see in a given situation. Once again, one of the CCSS standards for practice comes into play: the habit of using precise language to express ideas is useful when trying to shoe-horn an insight into a mathematical formalism.

Another aspect of this work is to help students become at home with the many ways to represent functions. The most common of these representations in high school is the Cartesian graph. There are all kinds of issues to deal
with in this representation. One of the most thorny is to understand the
effect of a linear change of variable on the corresponding graph: a common
misconception is that the graph of \( y = f(x - 3) \) is obtained from the graph of
\( y = f(x) \) by a translation 3 units to the left. Overcoming this misconception
takes considerable teaching skill.

In many applications, functions play a two-fold role: they express rela-
tionships between mathematical objects, and they are mathematical objects in
their own right. The standard example of this is in calculus where, for exam-
ple, ordinary functions become inputs to the operation of taking the derivative,
which may be viewed as a function in its own right, whose inputs and outputs
are themselves functions. But the idea shows up earlier—as soon as one talks
about equality or composition of two functions (as in geometry, for example,
when transformations are composed). It's a difficult task to help students
understand this reification of functions.

Key understandings to support this work

There are two kinds of background supports that help teachers carry out this
work: epistemological and mathematical.

The theories behind the abstracting regularity habit are useful when think-
ing about how the function concept develops in learners. This approach has
its roots in the genetic epistemology of Piaget. Several researchers have have
identified landmarks on the continuum of the levels of abstraction students
exhibit in their use of functions, calling them the action, process, and object
concepts of function. Briefly,

- Students who have an action concept of function see a function as a set of
isolated calculations. The output of the function \( x \mapsto 3x + 2 \) is calculated
by performing the multiplication, writing down the answer, and then
adding 2 to that. Students who have this point of view make extensive
use of the “=” button on their calculator, obtaining several partial results
on the way to doing a calculation.

- As students repeatedly perform a sequence of calculations, they begin
to chunk the individual steps together into coherent and self-contained
sequences or networks of calculations. They perform chain calculations
on their calculators (using the = key only when absolutely necessary)

- As processes are further interiorized and students further suppress the
details of the actual calculations that take place, students can begin to
encapsulate processes into objects, manipulating them as data (this is
very much in the spirit of how young children develop the concept of an
integer by encapsulating the counting process).

These levels of abstraction are markers on a continuum rather than discrete
stages. The action-process-object dimension is not the only one that can be
used to analyze the function concept. Another is to look at how functions are used in mathematics and at what people do with them in actual mathematical contexts:

- **Function as continuous change.** Classical analysis and physics were invented to study situations that vary continuously. Although continuity makes sense in more general situations, in school mathematics this point of view makes implicit use of the topology of the real line.

- **Function as algorithm.** Algorithms, in the sense of a sequence of explicit instructions for transforming one object into another, play an important role in many parts of algebra and number theory. Many highly theoretical investigations in algebra begin with a careful analysis of patterns that emerge from algorithmic calculations.

- **Function as mapping.** In combinatorics, it is quite common to set up an abstract correspondence between two sets in order to analyze cardinality questions. The functions under consideration are just pairings, and they are usually represented by arrow diagrams or by sets of ordered pairs.

Teacher preparation and professional development programs often don’t make these ideas explicit, but, with minor modifications, they could. For example, in number theory, Euclid’s algorithm can be viewed as a recursively defined function of two variables. Another example: in Galois theory, automorphisms of a field are both functions and elements of a group.

**Illustrative examples**

**Function as continuous change; domain; range.** Little congruent squares are cut out of the corners of a $5 \times 7$ rectangle, and the sides are folded up to make an open box. The volume of the box is a continuous function of the side-length of the cutout, and before they write down an explicit expression for the volume of the box, teachers can ask students some qualitative questions about the variation: what are the limits on the domain? Must there be a maximum volume? A minimum? Where might these extreme values occur?

**Function as algorithm; domain; equal functions.** Given an input-output table (say, with inputs 0, 1, 2, 3, 4) that can be matched by a linear function $f$ defined by $f(n) = 5n + 2$, many students see a pattern that can be modeled by a recursively defined algorithm: The output at $n$ is obtained from the output at $n - 1$ by adding 5. A complete model is

$$ g(n) = \begin{cases} 2 & n = 0 \\ g(n - 1) + 5 & n > 0 \end{cases} $$
Teachers can use these two models as a jumping off point for several productive discussions: Does \( f = g \)? Does \( f = g \) on all non-negative integers? How can you be sure?

**Function as mapping; function as object.** What’s the probability that, in a room of 25 people, two have the same birthday? There are many ways to think about this, but one is to count functions. A room-birthday sample can be represented by a mapping from the set \( \{1, 2, \ldots, 25\} \) to the set \( \{1, 2, \ldots, 365\} \). Combinatorial reasoning shows that there are \( 365^{25} \) such pairings. A double birthday occurs in the pairings that are not one-to-one. It’s not easy to count these, but students can count the functions that are one-to-one: there are

\[
365 \cdot (365 - 1) \cdot (365 - 2) \cdot \ldots \cdot (365 - 24)
\]

of these. Hence the number of non-one-to-one functions is \( 365^{25} \) minus this, and the desired probability is

\[
\frac{365^{25} - 365 \cdot (365 - 1) \cdot (365 - 2) \cdot \ldots \cdot (365 - 24)}{365^{25}} = 1 - \left(1 - \frac{1}{365}\right) \left(1 - \frac{2}{365}\right) \ldots \left(1 - \frac{24}{365}\right)
\]

Analysis of the structure of this expression, generalizations to rooms of other sizes, and numerical approximations can push this example quite far over several years in high school.