Mathematical Practice, Elementary Grades

The work of teaching mathematical practice

This section provides brief sketches of the Common Core State Standards for Mathematical Practice as they apply to teaching in elementary school.

1 Make sense of problems and persevere in solving them. Young children are eager for challenges and are problem solvers by nature. A challenge for elementary teachers is to help children maintain their enjoyment for engaging with problems. Teachers can help their students explore, investigate, and persevere in solving problems by creating a nurturing classroom environment. It is important for teachers to convey that everyone can learn math and that it takes active effort and thinking to do so. It is also important for teachers to convey that by thinking hard, we can actually increase our intelligence. Research on motivation indicates that supporting autonomy, competence, and relatedness supports internal motivation and leads to better outcomes than environments that are experienced as highly controlling. Elementary school teachers often want to make mathematics "fun" for students and shelter them from the difficulty of learning mathematics, which frequently leads to activities that have little mathematical substance.

2 Reason abstractly and quantitatively. Much of elementary teachers' work is helping young students connect their observations about quantities in the world with the abstract mathematical symbols we use to describe, record, and reason about relationships among quantities. Objects or simple math drawings (e.g., drawings that show tens and ones) can be especially helpful in making such connections, so elementary teachers need to be well prepared to use and discuss such drawings or objects in their instruction.

3 Construct viable arguments and critique the reasoning of others. Mathematics is about ideas, so mathematical arguments and lines of reasoning are an important part of mathematics, even at elementary school. For example, elementary school students need to give arguments for why strategies for adding, subtracting, multiplying, and dividing whole numbers work, and similarly with fractions. Students cannot make sense of mathematical arguments unless they think actively about them, which includes making their own arguments and carefully listening to and evaluating other people's arguments.

4 Model with mathematics. In elementary school, modeling with mathematics often involves writing an equation for a situation and then solving the equation to solve a problem about the situation. Students also model with mathematics when they draw a quadrilateral to show a route that started and ended at the same location and had four turns. At elementary school, modeling with mathematics is often *mathematizing*—which means focusing on the mathematical aspects of a situation and formulating it in mathematical terms. For example, students may notice shapes in objects around them, such as triangular bracing in chairs or quadrilaterals in collapsible gates. Teachers also need to help students notice math in the world around them.

5 Use appropriate tools strategically. Math drawings, such as drawings of tens and ones or hundreds, tens, and ones, can be an especially valuable tool in elementary school. Students use math drawings strategically when they use them to make sense of numerical work and not just for rote calculations. Elementary school students also need to learn to use tools such as rulers and protractors. Teachers should know how to guide students to make strategic use of these tools. Technology tools can also be used effectively in elementary schools, and teachers need to think carefully about how and when to use such tools. A common assertion in elementary schools is that children should not use calculators until they have memorized all of the number combinations for the four operations. However, elementary school students can use calculators effectively for problem solving to tackle mathematics for which they understand the operation but do not yet have facility with the computation. For example, children who understand the concept of multiplication but who have not yet mastered a means for multiplying multi-digit numbers can use calculators to determine how many times their heart beats in a day, week, month, or year after counting the number of beats in a minute. Thus, teachers need to have opportunities to think carefully about using technology tools strategically.

6 Attend to precision. Elementary school students attend to precision when they take care to make math drawings and carefully coordinate them with numerical work, such as when they show how to decompose a rectangle into component parts that correspond to the partial products in a multiplication problem. They also attend to precision when they describe a line of reasoning with care, attending to the key points and choosing their words to say exactly what they mean.

7 Look for and make use of structure. At elementary school, looking for structure usually involves the following:

- unitizing—finding or creating a unit, such as seeing one hundred as ten groups of ten, viewing a unit fraction such as 1/5 as a new unit, or making a car shape out of several smaller shapes and then repeating the car shape to show traffic on a road*;*
- decomposing and composing, such as decomposing 1 ten into 10 ones or viewing a rectangular prism as composed of equal layers made of unit cubes;
- relating and ordering, such as putting a collection of sticks in order by length or reasoning that 1/11 is greater than 1/12 because when a cake is cut into 12 equal pieces the pieces are smaller than when it is only cut into 11 equal pieces; and
- looking for patterns and structures and organizing information, such as noticing the repeating pattern of ones digits in the multiples of a number or that when you make quadrilaterals out of sticks so that the opposite sides are the same length, the opposite sides are also always parallel.

8 Look for and express regularity in repeated reasoning. Elementary school students require repeated opportunities to reason about and make sense of strategies and methods. It is through repeated reasoning that students will be able to make sense of calculation methods and use them with understanding.

Key understandings to support this work

Preparation for teaching the CCSM mathematical practice standards should include regular opportunities to engage deliberately in mathematical practices. Almost all teachers whose reflect the work described in the previous section have had substantial opportunities to experience learning in similar environments. In particular, it is important for teachers to experience the enjoyment and satisfaction of working hard at solving a problem so that they realize that this sort of intellectual work can also be "fun" (albeit in a different sort of way than more "playful" activities are fun). Key understandings for teaching mathematical practice in the elementary grades can be achieved if teachers have learning experiences such as:

- Solving challenging problems
- Engagement in mathematical arguments and lines of reasoning
- Active listening to and evaluating other people's arguments
- Being precise and deliberate when discussing one's own reasoning
- Understanding the importance of careful definition and learning to use mathematical terminology correctly
- Reflecting on and looking for examples of unitizing, decomposing and composing, relating and ordering
- Looking for patterns and structures and organizing information
- Generalizing from examples

Illustrative Examples

The following are sample problems that can provide elementary teachers with experiences such as those described in the previous section.

Which whole numbers can be expressed as the difference of two perfect squares? Provide an explanation that justifies your answer.

There's a famous fast-food restaurant where you can order chicken nuggets in boxes of 6, 9 and 20. By combining boxes of one or more size, there are many orders you can place. For example, if you order a box of 20 and two boxes of 6, you get 32 chicken nuggets. What is the largest number of chicken nuggets that you cannot order exactly? Explain why you can't order that number but can order a combination equal to any larger number.

A cube with edges of length 2 centimeters is built from centimeter cubes. If you paint the faces of this cube and then break it into centimeter cubes, how many cubes will be painted on three faces? How many will be painted on two faces? On one face? How many will be unpainted? What if the edge has a length different from 2? What if the edge length of the large cube is 3, 4 or 5 cm? What if the edge length of the cube was 20 cm? Can you generalize this to an edge length of *n* cm?

Heather went to the store to purchase ink pens. She found three kinds of pens. The first cost $4 each; the second was 4 for $1; and the cost for the third kind was 2 for $1. She bought 20 pens and she bought at least one of each kind. The cost was $20. How many of each kind did I buy? Provide an explanation that shows how you solved the problem. Is there more than one answer?

Mathematical Practice, High School

13 September 2011

The work of teaching mathematical practice

There are certain mathematical proficiencies—things like perseverance, sense making, using precision, and seeking structure—that are different from the proficiencies associated with technical expertise in a particular content area. Common Core State Standards (CCSS) include Standards for Mathematical Practice that make these proficiencies an integral part of what teachers are expected to teach in the coming years. Quoting from CCSS:

> *The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years.*

The work of helping students become proficient in mathematical practice is situated inside the teaching of specific content, and hence teachers are faced with the difficult task of teaching a course in, say, trigonometry, in a way that helps students understand the body of results and methods while fostering their development of mathematical habits of mind.

Elsewhere in this report, we describe some subject-specific "faces" of mathematical practices; seeking structure in algebraic expressions is somewhat different from doing the same in geometric diagrams, and both of these are different from seeking and using structural similarities among algebraic systems. Here, we concentrate on some of the dispositions involved in teaching mathematical practice in any domain. In the work of teachers who raise mathematical practice to the same level of importance as the results of that practice, one often finds features like these:

- **Classrooms that put experience before formality.** Worked-out examples and careful definitions are important, but in classrooms where students are learning to do mathematics, they grapple with ideas and problems *before* these things are brought to closure.

- **Lessons that have textured emphasis.** Teaching mathematical practice requires one to separate matters of convention and vocabulary from matters of mathematical substance. Both are important, but students come to understand each in very different ways.

- **An overall parsimony of methods.** A focus on mathematical practice leads to a small number of general purpose tools and methods that are employed in seemingly different contexts. Teachers who focus on mathematical practice don't use different methods for mathematical domains that are connected in basic ways; the skills of building equations for lines and circles differ in technical details, but the same ideas underlie both.

- **Mathematical coherence at every level of detail.** Teachers who work to build mathematical practice in their students have an eye towards how everything they teach fits into the larger landscape of mathematics as a scientific discipline. They develop completing the square as a special case of removing terms from a polynomial via a linear substitution. They use a small set of examples—maximizing area for a given perimeter, for example—over and over, each time showing how different parts of mathematics can be used to gain insight into the example.

- **Explicit attention to mathematical thinking.** There is a striking lack of reliance on memorized formulas in classes that focus mathematical practice. Students do use formulas, but, for example, they are just as likely to use the Pythagorean theorem to find the distance between two points as they are to use the "distance formula;" indeed, they see the latter as a special case of the former. Students evaluate their expertise on the basis of what they can figure out rather than on what they already know.

Key understandings to support this work

A basic prerequisite for teaching mathematical practice is to have engaged in the practice of mathematics: almost all teachers who use the styles of work described in the previous section have had a personal and intense experience of doing mathematics for themselves.

The understandings one gains from such an experience are somewhat different from what one gets in a typical undergraduate or professional development setting, where the goal is to help one understand established mathematical terrains. Key understandings for teaching mathematical practice include all the things that mathematicians do in their work, things like

- Performing thought experiments

- Generalizing from examples

- Reasoning by continuity

- Seeking invariants
- Using deduction as a research technique
- Changing variables to hide complexity
- Extending operations to preserve rules for calculating
- Seeking and using structure

Higher education mathematics faculty are ideally suited to help prospective and practicing teachers understand and develop these and many other mathematical practices; after all, these habits capture the styles of work that are second nature for mathematicians. But precisely because they are habits, they are often not made explicit. One of the recommendations of this report is to put these mathematical practices front and center in courses, seminars, workshops, and mathematical experiences for teachers.

Illustrative examples

Mathematical Coherence. A standard exercise in elementary algebra involves the simplification of the expression

$$(a+b)^2 - (a-b)^2$$

This can be carried out by expanding and collecting like terms or by seeing it as a difference of squares that leads to the identity

$$(a+b)^2 - (a-b)^2 = 4ab$$

This identity reminds one of a geometric context. By writing it as

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

the identity shows that a square maximizes the area among all rectangles of the same perimeter (as a bonus, it shows by how much the square's area exceeds that of the rectangle). And the identity can be mined for more: It contains a proof of the arithmetic-geometric mean inequality for two variables. Hence a simple exercise from elementary algebra can be pulled throughout high school to form a coherent thread.

Experience before formality. Students are given a half dozen points and asked to find, of all lines of slope 3, the line that best approximates the data. Few calculators can do this, but the problem amounts to minimizing a quadratic function, something squarely inside a first-year algebra course. With very little machinery, students can find the best line of slope 3. Then they can do it for other slopes. Graphing all of their "best" lines, the get a nice surprise—the lines are all concurrent in the centroid of the data. This is a jumping off point for a teacher embark on a thorough development of regression.

Explicit attention to mathematical thinking The class is trying to find a formula for the area of a trapezoid by dissecting the trapezoid into a parallelogram and keeping track of what happens to the relevant dimensions. Students are asked to put their formulas on the board, producing a collection like this:

$$A = \frac{h}{2}(b_1 + b_2), \quad A = h\left(\frac{b_1 + b_2}{2}\right), \quad A = \frac{b_1 h}{2} + \frac{b_2 h}{2}, \quad A = \frac{1}{2}\left((b_1 + b_2)h\right)$$

Students are then asked to pick a formula in a different form from their own and to find a dissection algorithm that would lead one to that particular form. Their algorithms are compared with the ones of the students who originated each form.

Operations and Algebraic Thinking, K – 5

The work of teaching operations and algebraic thinking

Teachers of Kindergarten through Grade 5 teach the meanings of the operations of addition, subtraction, multiplication, and division, the types of situations in the world these operations solve (mostly presented as word problems), the algebraic properties of these operations, including their use in strategies for solving and working toward fluency with operations beginning with operations on single-digit numbers, and other patterns and rules that can be explored with simple arithmetic.

Teachers help students learn to determine when to use addition, subtraction, multiplication, or division to solve a problem. The most basic meanings of the operations arise from the simplest kinds of situations in which these operations can be used, but the taxonomy of word problem types for the operations is more detailed and elaborate than most people who are not extensively involved with elementary school math may realize. For example, addition arises not only from "add to" situations, and subtraction arises not only from "take from" situations. Addition and subtraction also arise in situations involving parts composed physically or conceptually to make wholes (and vice versa) and situations involving comparisons. Furthermore, in all these types of situations, different quantities can be the unknown amount that is to be found to solve the problem. For example, in an "add to" situation, the initial amount, the change amount, or the final amount can be the unknown quantity that is to be determined. Some problem types are harder for children than others. The situation is similarly complex for multiplication and division.

As children work on word problems, teachers also help them develop and learn strategies for solving numerical addition, subtraction, multiplication, and division problems. Much is known from research in mathematics education about how children progress along a learning path of methods of different levels of conceptual complexity toward fluency first with single-digit additions and associated subtractions and later for single-digit multiplications and associated divisions. Importantly, learning the single-digit calculations is not simply a matter of rote memorization of a bunch of "facts."

As students work towards fluency with single-digit calculations, teachers help them learn to apply – often informally – the commutative and associative properties of addition and multiplication and the distributive property. For example, students learning single-digit additions might calculate $8+7$ by breaking 7 into 2 and 5, combining the 2 with 8 to make a ten, and then adding the 5, in effect using the associative property: $8+7=8+(2+5)=(8+2)+5=10+5=15$. Students learning single-digit multiplications might calculate $7*8$ by viewing it as five eights plus two more eights, thus using the distributive property, $7*8=(5+2)*8=5*8+2*8=40+16=56$.

Teachers help students connect addition with subtraction and multiplication with division. For example, when teachers teach young children to view subtraction problems as unknown addend addition problems, the children are able to solve many more subtraction problems efficiently and correctly than by counting backward (for example, viewing 12 – 9 = ? as 9 + ? = 12 and counting forward is much easier than counting back 9 from 12).

The case of division has some special limitations and considerations that teachers must know about. Teachers help students understand that when dividing a whole number by another (nonzero) whole number, there are usually several ways to represent the result. The exact quotient can be expressed as a decimal, fraction, or mixed number, or the result can be expressed as the largest whole number quotient and a remainder. Students must be able to select and interpret a form that is appropriate for a context.

Teachers help students notice and use patterns. For example, teachers may ask students to look for patterns in the multiplication table, which can help students learn these multiplications. Investigating repeating patterns and some growing sequences of designs or numbers can provide contexts for children to explore and use whole number quotients and remainders.

Key understandings to support this work

- Know and be able to pose the different types of word problems for addition, subtraction, multiplication, and division with whole numbers, decimals, fractions, and mixed numbers and how to represent the problems with objects, with drawings, and with equations.
- Know which problem types are typically less difficult or more difficult for students.
- Have an initial understanding of children's learning paths in single-digit calculations, the different levels of strategies within them, and how to help children move along these paths.
- Know the commutative and associative properties of addition and of multiplication, and the distributive property of multiplication over addition, know how to see why these properties are valid for whole numbers (such as with arrays), and know how to use these properties as calculation aids for deriving single-digit calculations from other single-digit calculations.
- Know how addition and subtraction are connected and how multiplication and division are connected. Recognize that these connections are not just important theoretically, but also computationally.
- Interpret calculations and numerical answers in terms of a problem context (for example, interpret a remainder in a division problem) and recognize that there can be a distinction between representing or solving a numerical problem arising from a word problem and solving the word problem. Explain why the result of dividing a whole number by a (nonzero) whole number can be represented as a decimal, a fraction (or mixed number), or as the largest whole number quotient and a remainder and choose a form appropriate to a context.
- Be able to pose and solve a variety of multi-step word problems.
- Be able to pose and solve problems about sequences of designs, shapes, and numbers and understand that sequences are special kinds of functions.

Illustrative Examples

These examples illustrate how prospective teachers can engage with mathematics they will teach and how this mathematics is a foundation for mathematics coming up in later grades.

Before their coursework, prospective teachers may not appreciate how surprising it is that multiplication is commutative. They can be asked to consider why it won't be obvious to elementary school students that 3x5 and 5x3 have the same value. Young learners will initially understand these expressions as the total in 3 groups of 5 versus the total in 5 groups of 3, which have different meanings. The teachers can then use arrays to explain why the commutativity of multiplication makes sense. Note the difference between this approach to commutativity and the use of other operations, such as matrix multiplication or transformations of the plane, to demonstrate that commutativity is not obvious.

Courses for prospective elementary teachers can help the teachers recognize and reason clearly about the many subtleties involved with division, such as why one can't divide by 0 and what it means when we express the answer to a whole number division problem as a whole number quotient with a remainder.

Courses for prospective elementary teachers can help the teachers recognize how elementary school students' problem solving methods can be connected to symbolic algebra. For example, elementary school students can use tape diagrams to help them solve multi-step word problems. Prospective teachers can see how tape diagrams correspond to algebraic equations and how the steps used with a tape diagram often correspond directly to the steps used in solving the problem using symbolic algebra.

Explorations and topics such as the following might be used in developing prospective teachers' mathematical habits of mind or in deepening teachers' knowledge of the mathematical ideas related to the domain.

- Explore how the cases of dividing nonzero numbers by 0 and dividing 0 by 0 are different.
- Study why and how the base-ten system creates various patterns, such as why multiples of 2 always end in 0, 2, 4, 6, or 8 and why the sum of the digits in a number divisible by 9 is also divisible by 9.
- In creating a sequence of equilateral triangles in which each one is larger than the one before it, the pattern 1, 4, 9, 16 … emerges from counting the number of "unit" equilateral triangles in each larger shape. This can be connected to functions by examining the relationship between the input (the ordinality of the shape in the sequence) and output (the number of unit triangles in the shape) in order to obtain a function that will produce the output for any given input.

Numbers and Operations in Base Ten, Grades K - 5

The work of teaching Numbers and Operations in Base Ten.

These standards concern representing, comparing, and calculating with numbers in base ten. The base-ten place value system provides for the remarkably efficient representation of numbers and for efficient, compact methods of calculation and of comparison. But this efficiency and compactness hides the underlying meaning, so teachers must explicitly bring meaning to the fore. The standards make clear that students are expected to understand the meaning of numbers represented in base ten and the rationale and logic underlying calculation methods. Much is known about how children can learn calculation methods with understanding, about the difficulties that must be surmounted and the common errors that occur, and about representations (such as drawings to show tens and ones) that can help children reason about and make sense of base-ten calculation methods.

At each grade, teachers help students make progress in their understanding of how the base ten system uses units of ten. For young children, understanding the whole numbers 11 through 19 as standing for 1 unit of ten and some ones is a major advance. Even older children may treat numbers in base ten as concatenations of digits rather than as involving units of ones, tens, hundreds, etc., unless they have strong instruction that highlights these base-ten units and the meaning of numbers written in base ten.

Efficient base-ten calculation methods for addition, subtraction, multiplication, and division rely on breaking numbers into their base-ten place value components and applying properties of arithmetic, including the commutative and associative properties of addition (often informally) and the distributive property, to decompose the calculation into parts. To help students learn calculation methods and make sense of the calculation methods they use, teachers will lead discussions with their class about the meaning and rationale of calculation methods. They help students learn to use and explain efficient methods with understanding, culminating in fluency.

Because of the uniform structure of the base-ten system (the value of each place is 10 times the value of the place to its right), comparisons (more/less than) and calculations with decimals follow the same logic as comparison and calculations with whole numbers. Teachers help students understand this common structure and use it in calculations, including understanding the reasoning behind the placement of the decimal point in calculations with decimals.

Key understandings to support this work

- Understand how the base ten system uses place value and bundling in units of ten to represent numbers. Use visual supports such as drawings, objects, and cards to show base-ten structure. Recognize that the way we say numbers in English does not always reveal base-ten structure.

- Understand the rationale and logic underlying efficient base-ten calculation methods (algorithms) and know ways to reveal and discuss the underlying rationale and logic with the aid of visual representations that show the quantities involved.
- Understand and write equations to show how properties of arithmetic, such as the distributive property, are used in justifying efficient base-ten calculation methods (algorithms). Know how to coordinate the equations with visual representations.
- Examine and evaluate hypothetical or actual student calculation methods to determine if they are mathematically valid or if they could be modified to become valid.
- Explain how to extend the base ten system to decimals and how to represent decimals on number lines.
- Know ways of explaining the rationale for placement of the decimal point in calculations with decimals.
- Know some common base-ten calculation errors, such as errors involving zeros that arise in division and in subtraction and errors that arise in interpreting and comparing decimals.

Illustrative examples

Before their coursework, prospective elementary teachers often do not realize that the common calculation algorithms can be explained in terms of more fundamental ideas. In other words, the calculation algorithms are frequently taken as "given." Considerable course time should be devoted to a careful development of the algorithms.

Prospective teachers can examine hypothetical or actual student calculation methods and decide if the methods are valid or not. For example, if a student calculates 23 x 45 by calculating 20 x 40 and 3 x 5 and adding the two results, is that method legitimate or not, why or why not, and if not, how could it be modified to become correct?

Problem: Examine this proposed method for rounding a number to the nearest hundred. Is it valid? 2367 => 2370 => 2400

Number and Operations – Fractions, Grades 3 - 5

The work of teaching number and operations – fractions

These standards concern the meaning of fractions, comparing fractions, and calculating with fractions. Fractions are well known to be difficult to teach and learn, but the study of fractions is an important springboard to algebra. It is therefore especially important for grades 3 - 5 teachers to have solid conceptions of fractions as numbers, to know the progression for teaching fractions described in the standards and to be able to help students represent fractions, use representations to reason about fractions, and explain the logic underlying equivalent fractions, comparing fractions, and calculating with fractions.

Teachers help students understand that fractions are numbers, that unit fractions are the basic unit for fractions, and that every fraction is a sum of unit fractions. Students learn that the operations of addition, subtraction, multiplication, and division retain their meanings from whole numbers, although extending these meanings to the case of fractions is not automatic or easy for students. Students learn that multiplication and division with fractions less than one reverses the size relationships students are used to with whole numbers (e.g., multiplying a number by a whole number yields a larger product but multiplying a number by a fraction less than one yields a smaller product). Teachers must be aware that the connection between fractions and division requires investigation and discussion. For example, the fraction ¾ is initially defined as a sum of three ¼ and then as 3 times ¼, but 3 divided by 4 may be thought of as the amount in one share when 3 equal things are divided equally among 4 shares. Only teachers with a good understanding of fractions can facilitate the reasoning that is required to connect this situation of 3 divided by 4 with the fraction ¾.

The use of number lines to represent fractions requires special attention by teachers because of difficulties that are known to occur with this representation. In particular, teachers must know the importance of drawing attention to intervals on the number line because of the tendency for students to attend only to tick marks and not to lengths of intervals.

Key skills and understandings needed to begin teaching this domain

- Understand that fractions are numbers, that unit fractions are the basic units for fractions and be able to use number lines to represent fractions.
- Know how to represent fractions and to use representations to reason about fractions and to facilitate discussions about equivalent fractions.
- Know several approaches to comparing fractions and be able to explain why “cross multiplying” works as a way of comparing fractions.
- Understand how to convert between fractions, decimal fractions and percents and know which fractions correspond to finite decimals.
- Know how to add, subtract, multiply and divide fractions and realize that the basic rules of arithmetic (e.g. the distributive property) holds true for fractions.

Illustrative Examples

Use the meaning of fractions to explain why $\frac{6}{8} = \frac{9}{12}$, providing an explanation that is appropriate for an elementary classroom in which students are first learning about equivalent fractions.

Find two fractions between $\frac{8}{13}$ and $\frac{10}{17}$ by using two different approaches, neither of which involves using decimals and explain how you know your answer is between the two fractions.

Use good number sense (without using decimals) to place the fractions $\frac{9}{16}, \frac{4}{11}, \frac{4}{13}, \frac{5}{8}, \frac{5}{13}$ in order from smallest to largest and provide a justification that does not involve any use of a calculator.

Explain a process that can be used to write any fraction as the sum of distinct unit fractions and demonstrate the process by decomposing $\frac{6}{7}$ and $\frac{5}{21}$ as the sum of distinct unit fractions.

Ratio and Proportional Relationships, Grades 6 – 7

The work of teaching Ratio and Proportional Relationships

Ratios and proportional relationships are used to describe how quantities are related and how quantities vary together. Their study bridges multiplication and division of the elementary grades with linear functions in the upper middle grades and high school.

Teachers may introduce ratios with juice or paint mixtures and walking distances and times, for example. When two juices or paints are mixed, there are certain quantities that yield mixtures that have identical flavors or colors. When people walk, there are distances and times that correspond with the same pace of walking. Such situations motivate the concepts of ratio and rate.

Teachers teach students to represent ratios in tables, on double number lines, and with tape diagrams and to use ratio language (e.g., "A for every B," "A parts to B parts") in discussing ratios. With the aid of these representations, students learn to reason about quantities that are in equivalent ratios and they solve ratio problems flexibly, and in increasingly sophisticated and abbreviated ways.

A central idea in the study of ratio and proportional relationships is the notion of a (unit) rate, which tells the amount of one quantity per 1 unit of another quantity. Teachers emphasize this rate language and they guide students toward problem-solving methods that rely on unit rates. As students work with proportional relationships, in which two quantities vary together, teachers help students see that a constant of proportionality, c, in an equation of the form $y=cx$ and the slope of the graph are unit rates.

Especially important is that teachers guide students to understand the distinction between additive relationships and proportional relationships, which are commonly confused. Teachers must draw students' attention to wording such as "for every," "for each," and "per," which indicate a proportional relationship. They may also highlight that "for each" and "per" refer implicitly to 1 unit of a quantity. For example, 3/2 meters per second means 3/2 meters for every 1 second.

Ratios and proportional relationships are widely applicable. Teachers will teach students to use them in geometry, when working with scale drawings, and in statistics, when using samples to make inferences about populations. They will also teach multi-step problems in which ratios are used, as in situations of percent increase or decrease.

Key understandings to support this work

- Know the meanings of ratio and rate and use ratio and rate language.
- Explain how to use tables, double number lines, and tape diagrams to represent and reason about ratios and equivalent ratios and to solve problems.
- Find unit rates in tables and on double number lines and use unit rates to describe situations and solve problems.
- Represent proportional relationships with tables, graphs, and equations and explain how to relate the different representations.
- Know how to determine when quantities described in a situation or with a table or graph are or are not in a proportional relationship.
- Recognize that a common error is for students to make an additive comparison in cases where such a comparison does not apply.
- Know ways of reasoning about and solving simple percent problems as well as multi-step percent problems, such as those involving percent increase and percent decrease.

Illustrative examples

Will a mixture of 1 cup blue paint with 3 cups yellow paint be the same color as a mixture of 4 cups blue paint and 6 cups yellow paint? Why might sixth graders say that these two paint mixtures will be the same color? Make two ratio tables, one for each paint mixture, and use these tables to explain in several ways how the mixtures compare.

To make "grapple juice," we mix grape juice and apple juice in a ratio of 2 to 3. Describe ways to interpret each of the fractions 2/3, 3/2, 2/5, and 3/5 in terms of grapple juice (*other than* as ratios). [e.g., There is 2/3 of a cup of grape juice in the mixture for every 1 cup of apple juice. There is 2/3 times as much grape juice in the mixture as apple juice.] Show how these fractions occur in tables, equations, and graphs relating quantities of juice.

If it takes 4 people 6 hours to mow a field, how long will it take 2 people to mow a field of the same size? (Assume all the people work at the same steady rate.) Can we solve this problem by setting up and solving a proportion, 4 people/6 hours = 2 people/H hours? Why or why not? How long would it take 3 people to mow a field of the same size? Is that amount of time half way between the amounts of time for 4 people and 2 people or not?

Some pants are on sale for 20% off. The new, reduced price is $30. Why can't you find the original price by calculating 20% of $30 and then adding that to $30? How can you reason correctly to find the original price?

Statistics and Probability, Grades 6 – 8

The work of teaching statistics and probability

Teachers need to design activities that help student see statistical reasoning as a four-step investigative process that involves:

- formulating questions that can be answered with data;
- designing and using a plan to collect relevant data;
- analyzing the data with appropriate methods;
- interpreting results and drawing valid conclusions from the data that relate to the questions posed.

Teachers guide students to use random sampling to collect data and learn to differentiate between the variability in a sample and the variability inherent in a sample statistic when samples are repeatedly selected from the same population.

Teachers help students to understand random sampling by first building their understanding of elementary probability. The probability of a chance event as approximated by a long-run relative frequency is introduced using coins, number cubes, cards, spinners, bead bags. Once the connection between relative frequency and theoretical probability is clear, probability experiments are simulated with technology. It is critical that teachers help students understand how the product rule for counting outcomes for chance events is used in situations such as tossing three coins or rolling two number cubes.

Teachers introduce students to the analysis of bivariate measurement data graphed on a scatterplot and help them describe shape (a cloud of points on a plane), center (a line drawn through the cloud that captures the essence of its shape) and spread (how far the data points stray from this central line). Teachers also introduce students to numerical descriptions of center and spread such as median, quartiles, the interquartile range, and mean absolute deviation.

Key understandings to support this work

- Understand and know how to guide students in the 4-step investigative statistical process.
- Recognize that approximating the probability of a chance event by observing its long-run relative frequency may result in strings and patterns in the short run. Understand that a basic tenet of statistical reasoning is that random sampling allows results from a sample to be generalized to a much larger body of data, namely, the population from which the sample was selected.
- Distinguish between variability in a sample and variability in a sample statistic when samples are repeatedly selected from the same population.
- Know how to find and explain various measures of shape, center, and spread of univariate and bivariate data (including mean, median, quartiles, mean absolute deviation). Identify clusters, gaps, and unusual data points for bivariate data.

- Identify skewness when looking at a graph of data and explain what it means in the context of the problem.
- Choose the best measure of center for a given situation, and explain why it is best.
- Know how to find theoretical probability in uniform and non-uniform probability models, and be able to explain to students why theoretical and empirical probabilities differ for a given situation.
- Know the product rule for finding outcomes of two or three independent events (such as rolling 2 dice or flipping 3 coins) and be able to help students understand why this is a multiplicative (rather than additive) situation.
- Know when and how to use simulations to help students make sense of statistical ideas.
- Interpret the slope of the line fitting a scatter plot as a rate of change.
- Describe the association between two categorical variables.

Illustrative examples

1. In the last ten games of the 2006 season, the Gwinnett Women's Soccer team had an average score of 2 goals per game. Create four different dot plots with data values for the ten games. Remember that the mean number of goals for the ten games is 2. Label your plots a, b, c, and d. The plots should satisfy the following conditions:

Plot a: Only one game had 2 goals.
Plot b: Exactly two games had 4 goals.
Plot c: One game had an amazing 7 goals!
Plot d: The median of the data set is 3.

For the next soccer game played, we might use the mean of 2 to predict how many goals a team will score given that plot a, b, c, or, d describes the previous ten games. For which distribution do you think the prediction of 2 is most likely to be closest to the actual number of goals scored? Give a statistical justification for your answer.

2. In 1972, 48 male bank supervisors were each given the same personnel file and asked to judge whether the person should be promoted to a branch manager job that was described as "routine" or whether the person's file should be held and other applicants interviewed. The files were identical except that half of the supervisors had files showing the person was male while the other half had files showing the person was female. Of the 48 files reviewed, 35 were promoted. (Reference. B. Rosen and T. Jerdee (1974), "Influence of sex role stereotypes on personnel decisions," *J. Applied Psychology*, 59:9-14.)

Create three tables for the data, one showing no discrimination, one showing strong evidence of discrimination, and one where discrimination is not obvious without further investigation.

What percentage of males and females were recommended for promotion? Without exploring the data any further, do you think there was discrimination?

Conduct a simulation of this situation, and record the results in a dot plot. Using these data, estimate the likelihood that 21 or more males out of 35 will be selected if the selection process is random. Look at the dot plot and describe the shape, center, and variability of the distribution. Is the behavior of this distribution what you might expect? Based on the simulation, does there appear to be evidence to support the claim of discrimination?

Algebra, Grade 8 and High School

18 September 2011

The work of teaching algebra*

Teaching algebra intertwines teaching symbol manipulation skills with inculcating an understanding of the principles behind them. This starts with the introduction of symbols themselves, where it is easy for students to lose sight of the basic fact that the symbol stands for a number. Helping students keep a firm grasp on this fact enables them to see symbol manipulation as the result of properties of number operations.

For example, students have learned in earlier grades to explain $3 \times 29 = 3 \times 20 + 3 \times 9$ in terms of the distributive property. Now teachers help them see the distributive property for algebraic expressions as an expansion of that for whole numbers, so that a statement like $3(x + y) = 3x + 3y$ is viewed as a consequence of the distributive property, rather than as a mysterious new rule called "moving the 3 inside the parentheses," which refers more to the appearance of the expression on the page than it does to the meaning it expresses. Teachers understand how an area model showing $3 \times 29 = 3 \times 20 + 3 \times 9$ can be connected to one for $3(x + y) = 3x + 3y$ and they know that students have been introduced to the former in earlier grades.

Students have trouble making the transition from situations in which symbols are used to represent specific unknown numbers, for example the unknown quantity in a word problem, to situations where they are used to make universal statements in some domain (Kieran, *What do Students Struggle with When First Introduced to Algebra Symbols?*, NCTM, 2007). Surrounding expressions and equations with complete sentences helps make this distinction, e.g. "We are looking for a number p for which $p + (p + 5) = 47$" or "$2n$ is greater than n for any positive number n."

Precise language about expressions and equations, and later functions, helps students avoid common confusions about these objects. For example, when given an expression that is to be manipulated in some way, students might instead set it equal to zero. An important part of the work of teaching is continually bringing students back the roots of algebra in work with quantities and relationships between quantities. Graphs, tables, and appropriate contexts that engage students are the tools of the trade in this work.

*The main focus in this draft is on the early stages of algebra in late middle and early high school.

Key understandings to support this work

1. That symbols stand for numbers; expressions and equations do not stand by themselves, but live in sentences that ask questions or make statements about numbers.

2. That the rules for manipulating expressions are consequences of the properties of operations with numbers.

3. That the process of solving equations is a process of logical deduction, in which the equation is assumed to be true, and step by step consequences of that assumption are derived until the solutions are made manifest.

4. That equations in two variables represent relationships between two quantities; they can be represented by graphs and tables; finding the value of one variable that corresponds to a value of the other variable involves solving an equation in one variable.

5. How to represent symbolically problems external to mathematics, manipulate the symbolic representation purposefully, and thus solve the problem.

Illustrative examples

Manipulating expressions Students are considering a formula for the contribution of three test grades t_1, t_2, and t_3 to their final grade, which is

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right).$$

They are trying to figure out how much a 10 point increase in t_3 would change the final grade. One students says "you can just move the 3 over, so this is the same as

$$\frac{0.6}{3}\left(t_1+t_2+t_3\right) = 0.2t_1 + 0.2t_2 + 0.2t_3.\text{"}$$

Why can you "move the 3 over"?

Solving equations A student writes the following solution up on the board:

$$\begin{array}{rcl} x^2-3x-4 & = & 0 \\ x^2-3x & = & 4 \\ x(x-3) & = & 4 \\ x=2, & & x-3=2 \\ x=2, & & x=5 \end{array}$$

Another student says "you made a mistake in the second line, you shouldn't have moved the 4 over." Did the first student make a mistake and where is it?

Solving equations graphically A student is trying to solve the equation $x^2 = 2x + 3$. Her mother suggestions graphing the equations $y = x^2$ and $y = 2x + 3$ and finding the x-coordinates of the points of intersection. The students says "why are those the solutions?" Give an answer which explains the mathematical reasoning in terms understandable by a high school student who has taken a CCSS-based course in middle grades.

Functions, Grade 8 and High School

13 September 2011

The work of teaching functions

Functions are general purpose tools that pervade all of mathematics, and hence teachers are faced with helping students build and use different kinds of functions in every course and in every mathematical area, from algebra to geometry to probability to calculus.

State standards and published curricula have concentrated on helping students interpret and use standard types of functions defined on the real numbers—linear, quadratic, exponential, and trigonometric. This involved using function notation and purposely transforming formulas that define functions to reveal certain information—extrema, end behavior, and so on.

The Common Core State Standards (CCSS) keep attention on these skills but adds explicit standards that require students to *build functions of their own* to model some situation or to express some perceived regularity. Teachers know that, for most students (and for most adults, as well), creating functions for a purpose is much harder than using "stock" functions. For example, many students who can compute the cost of a text messaging plan for a fixed number of messages find it very difficult to create an algebraic expression that defines a function that will do the calculation for *any* number of messages. Helping students develop the skill of expressing general relationships with functions is intimately tied up with the mathematical practice that CCSS calls expressing regularity in repeated reasoning.

Student-created functions are often not conceived in standard algebraic forms—they are often expressed as incomplete recurrences or as a mix of formal algebra and verbal descriptions. The work of teaching functions requires one to find ways to help students refine and make precise what they are trying to express about the regularity they see in a given situation. Once again, one of the CCSS standards for practice comes into play: the habit of using precise language to express ideas is useful when trying to shoe-horn an insight into a mathematical formalism.

Another aspect of this work is to help students become at home with the many ways to represent functions. The most common of these representations in high school is the Cartesian graph. There are all kinds of issues to deal with in this representation. One of the most thorny is to understand the effect of a linear change of variable on the corresponding graph: a common misconception is that the graph of $y = f(x - 3)$ is obtained from the graph of $y = f(x)$ by a translation 3 units to the left. Overcoming this misconception takes considerable teaching skill.

In many applications, functions play a two-fold role: they express relationships between mathematical objects, and they are mathematical objects in their own right. The standard example of this is in calculus where, for example, ordinary functions become inputs to the operation of taking the derivative, which may be viewed as a function in its own right, whose inputs and outputs are themselves functions. But the idea shows up earlier—as soon as one talks about equality or composition of two functions (as in geometry, for example, when transformations are composed). It's a difficult task to help students understand this reification of functions.

Key understandings to support this work

There are two kinds of background supports that help teachers carry out this work: epistemological and mathematical.

The theories behind the abstracting regularity habit are useful when thinking about how the function concept develops in learners. This approach has its roots in the genetic epistemology of Piaget. Several researchers have have identified landmarks on the continuum of the levels of abstraction students exhibit in their use of functions, calling them the *action*, *process*, and *object* concepts of function. Briefly,

- Students who have an *action* concept of function see a function as a set of isolated calculations. The output of the function $x \mapsto 3x + 2$ is calculated by performing the multiplication, writing down the answer, and then adding 2 to that. Students who have this point of view make extensive use of the "=" button on their calculator, obtaining several partial results on the way to doing a calculation.

- As students repeatedly perform a sequence of calculations, they begin to chunk the individual steps together into coherent and self contained sequences or networks of calculations. They perform chain calculations on their calculators (using the = key only when absolutely necessary)

- As processes are further interiorized and students further suppress the details of the actual calculations that take place, students can begin to *encapsulate* processes into *objects*, manipulating them as data (this is very much in the spirit of how young children develop the concept of an integer by encapsulating the counting process).

These levels of abstraction are markers on a continuum rather than discrete stages. The action-process-object dimension is not the only one that can be used to analyze the function concept. Another is to look at how functions are used in mathematics and at what people do with them in actual mathematical contexts:

- **Function as continuous change**. Classical analysis and physics were invented to study situations that vary continuously. Although continuity makes sense in more general situations, in school mathematics this point of view makes implicit use of the topology of the real line.
- **Function as algorithm**. Algorithms, in the sense of a sequence of explicit instructions for transforming one object into another, play an important role in many parts of algebra and number theory. Many highly theoretical investigations in algebra begin with a careful analysis of patterns that emerge from algorithmic calculations.
- **Function as mapping**. In combinatorics, it is quite common to set up an abstract correspondence between two sets in order to analyze cardinality questions. The functions under consideration are just pairings, and they are usually represented by arrow diagrams or by sets of ordered pairs.

Teacher preparation and professional development programs often don't make these ideas explicit, but, with minor modifications, they could. For example, in number theory, Eulcid's algorithm can be viewed as a recursively defined function of two variables. Another example: in Galois theory, automorphisms of a field are both functions and elements of a group.

Illustrative examples

Function as continuous change; domain; range. Little congruent squares are cut out of the corners of a 5×7 rectangle, and the sides are folded up to make an open box. The volume of the box is a continuous function of the side-length of the cutout, and before they write down an explicit expression for the volume of the box, teachers can ask students some qualitative questions about the variation: what are the limits on the domain? Must there be a maximum volume? A minimum? Where might these extreme values occur?

Function as algorithm; domain; equal functions. Given an input-output table (say, with inputs 0, 1, 2, 3 ,4) that can be matched by a linear function f defined by $f(n) = 5n + 2$, many students see a pattern that can be modeled by a recursively defined algorithm: The output at n is obtained from the output at $n-1$ by adding 5. A complete model is

$$g(n) = \begin{cases} 2 & n = 0 \\ g(n-1) + 5 & n > 0 \end{cases}$$

Teachers can use these two models as a jumping off point for several productive discussions: Does $f = g$? Does $f = g$ on all non-negative integers? How can you be sure?

Function as mapping; function as object. What's the probability that, in a room of 25 people, two have the same birthday? There are many ways to think about this, but one is to count functions. A room-birthday sample can be represented by a mapping from the set $\{1, 2, \ldots, 25\}$ to the set $\{1, 2, \ldots, 365\}$. Combinatorial reasoning shows that there are 365^{25} such pairings. A double birthday occurs in the pairings that are not one-to-one. It's not easy to count these, but students can count the functions that *are* one-to-one: there are

$$365 \cdot (365 - 1) \cdot (365 - 2) \cdot \cdots \cdot (365 - 24)$$

of these. Hence the number of non-one-to-one functions is 365^{25} minus this, and the desired probability is

$$\begin{aligned}
&\frac{365^{25} - 365 \cdot (365 - 1) \cdot (365 - 2) \cdots (365 - 24)}{365^{25}} \\
&\qquad = 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{24}{365}\right)
\end{aligned}$$

Analysis of the structure of this expression, generalizations to rooms of other sizes, and numerical approximations can push this example quite far over several years in high school.

Geometry, High School

11 September 2011

The work of teaching geometry

During middle school students develop an intuitive understanding of geometric definitions and theorems, through hands on experience with physical or virtual geometric objects. They begin to reason mathematically, seeing, for example, how the angle-sum of a triangle can be derived from facts about transversals an parallel lines. Students enter high school with varying conceptions of definition and proof, and varying levels of preparedness to understand proofs. Although reasoning and proof are important in all areas of mathematics, the study of geometry provides a particularly important opportunity for students to make the transition from empirical to deductive reasoning. It is important to focus on understanding the ideas and the logical relationships between them rather than the form or structure of the proof:

> . . . it is not the deductive scheme that commands most attention. It is, in fact, the mathematical ideas, whose relationships are illuminated by the proof in a new way, which appeal for understanding, and it is the intuitive bridging of the gaps in logic that forms the essential component of that understanding. When a mathematician evaluates an idea, it is significance that is sought, the purpose of the idea and its implications, not the formal adequacy of the logic in which it is couched. . . . what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinized and revised. While these skills may involve a degree of formalization, the emphasis must be clearly placed on the clarity of the ideas.
>
> —G Hanna, *More than formal proof*, For the Learning of Mathematics 9 (1989), no. 1, 20-23.

Clear ideas are based on precise definitions, and therefore another important part of the work of teaching geometry is helping students understand the role of definitions. This involves creating a classroom environment in which definitions are analyzed and discussed, not simply received on authority.

Dynamic geometry software environments provide ways to illustrate the meaning of interesting theorems, e.g. the concurrency theorems for triangles, and reveal the element of surprise in such theorems, with the potential for triggering in the student an appreciation of the need to arrive at conviction by means other than repeated empirical observations or acceptance of authority.

Key understandings to support this work

1. Understand that the angle- and distance- preserving properties of rotations, reflections, rotations and dilations may be taken as an intermediate platform of axioms on which to build the more advanced theorems of Euclidean geometry, without requiring a long march from more primitive axioms through seemingly obvious statements.

2. Understand that the intuitive notions of congruence and similarity in terms of size and shape may be given a precise form using rotations, reflections, translations, and dilations.

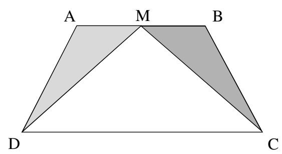
3. Understand that the standard triangle congruence and similarity criteria can be derived using the definition of congruence and similarity in terms of transformations. Familiarity with multiple ways of proving the basic theorems of Euclidean geometry.

4. Understand the way measures such as length, area, and volume scale under similarity transformations, and the application of this to modeling probems involving scale drawings.

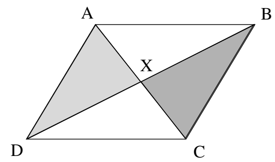
5. Understand that Cartesian coordinates are based on Euclidean geometry, particularly the theorems about transversals intersecting pairs of parallel lines, and that coordinates in turn provide an algebraic way of proving the basic theorems of Euclidean geometry.

Illustrative examples

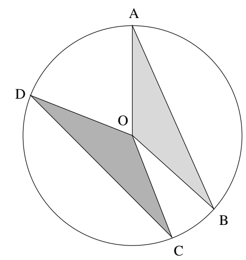
Experience with deductive reasoning Situations where students are asked to judge for themselves whether a given statement is true or not, and whether enough information has been given to judge, give them experience with the practices of mathematics rather than the form. For example, students could be asked to decide whether there is enough information to prove that two triangles are congruent, rather than to prove that they are congruent:



$ABCD$ is a trapezoid with $\overline{AB} || \overline{CD}$, and M is the midpoint of side $\overline{AB}$



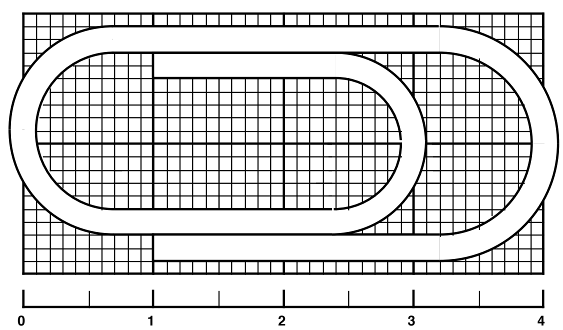
$ABCD$ is a parallelogram



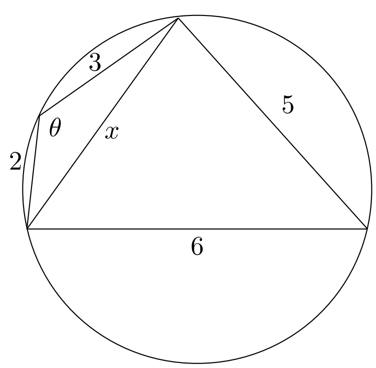
The circle is centered at point O, and the arc AB is congruent to the arc CD

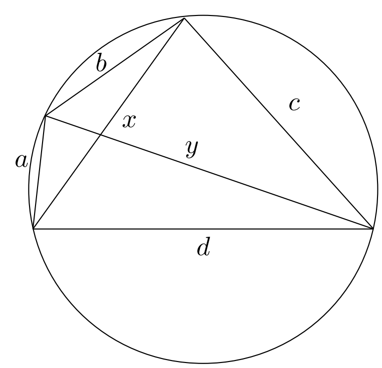
Modeling with geometry Geometry provides students with opportunities for modeling and looking for structure, as illustrated by the following example.

> This paper clip is just over 4 cm long. How many paper clips like this may be made from a straight piece of wire 10 metres long?



Connection between geometry and algebra The interplay between geometry and algebra provides students with opportunities to generalize repeated calculations and see structure in expressions and geometric figures.





For example, students who have studied trigonometry can use the law of cosines to express the square of the diagonal in the cyclic quadrilateral on the left in terms of the sides in two different ways:

$$\begin{aligned} x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\ &= 25 + 36 + 2 \cdot 30 \cos\theta. \end{aligned}$$

By eliminating $\cos\theta$ they can find the value of x.

Whereas students might stop there, teachers in professional development might conduct a further exploration with the figure on the right, generalizing the calculation to an arbitrary cyclic quadrilateral

$$\begin{aligned} x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\ &= \frac{(ac + bd)(ad + bc)}{ab + cd}. \end{aligned}$$

The surprising factorization of the numerator is a sophisticated illustration of looking for structure, where regrouping the squared factor in each term reveals an underlying structure of the expanded form of a product of two binomials. A similar formula for y^2 results in a remarkable simplification, namely Ptolemy's Theorem that the product of the two diagonals in a cyclic quadrilateral is the sum of the products of opposite sides:

$$xy = ac + bd.$$

Acknowledgments: the examples in this section were provided by Cody Patterson, The Shell Center, and Dick Askey.

Statistics and Probability, High School

There are two unique, challenging aspects of teacher preparation in the area of statistics and probability. First, the substantially increased status of statistics in the Common Core State Standards means that most future high school mathematics teachers need to master material they never saw in their own high school education. Second, a large part of the new CCSS statistics component involves empirical, qualitative issues related to experimental design, data collection and interpretation. Realistic data analysis only makes sense in a context, e.g., one cannot talk meaningfully about stock market data without knowing something about economic events that influenced the market. This is a very different type of knowledge being taught to students than that in the standard mathematical framework of concepts, skills and problem-solving techniques in other areas of high school mathematics. Furthermore, most college statistics courses that math majors encounter devote little time to experimental design, data collection and interpretation. Thus, major changes are needed in the pre-service and in-service statistical education of high school mathematics teachers.

The Common Core State Standards for high school statistics and probability reflect the greater role for collecting and interpreting data:

A. Interpreting Categorical and Quantitative Data
- Summarize, represent, and interpret data on a single count or measurement variable
- Summarize, represent, and interpret data on two categorical and quantitative variables
- Interpret linear models

B. Making Inferences and Justifying Conclusions
- Understand and evaluate random processes underlying statistical experiments
- Make inferences, justify conclusions in sample surveys, experiments and observational studies

C. Conditional Probability and the Rules of Probability
- Understand independence and conditional probability and use them to interpret data
- Compute probabilities of compound events in a uniform probability model

D. Using Probability to Make Decisions
- Calculate expected values and use them to solve problems
- Use probability to evaluate outcomes of decisions

Statistics. Through pre-service and in-service education, high school mathematics teachers need to acquire the knowledge to offer good statistics and probability instruction that meets the above standards. They also should know the material in the Introduction to Statistics college course and AP Statistics course, to prepare their students for this popular college course.

Basic techniques of representing data in Standard A above, such as box plots and stem-and-leaf plots, are now frequently covered in high school and college statistics instruction, but the topics in Standard B are rarely addressed at all. For example, one important item in Standard B is, “recognize the purposes of, and differences among, sample surveys, experiments, and observational studies; explain how randomization relates to each.” The philosophy driving this part of the statistics standards is that before students learn standard statistics inference procedures such as a t-test, they should first understand the range of issues and assumptions about where data come from and why it is variable. Variability can come just from inherent variability in the differences of random samples taken from the same population (this is the focus of traditional mathematical statistics) but it can also arise from variability induced by the experimenter, such as error-prone measurements. This in turn leads to questions about how best to collect the data and how best to analyze it once collected.

As noted earlier, this is very different instruction than is normally associated with mathematics standards. Moreover, many CCSS statistics standards have almost nothing in common with the standard upper-level probability-statistics survey course for future teachers and other mathematics majors (or even worse with the all-probability first course in a two-semester probability-statistics sequence). *The message from statistics educators is that the current probability-statistics course in the mathematics major falls well short of what future high school mathematics teachers need.*

The American Statistical Association (ASA) and NCTM have produced helpful materials to facilitate the design of a more appropriate course for future teachers: ASA's Guidance for Assessment and Instruction in Statistics Education, and NCTM's Navigation Books in Data Analysis and Probability and Essential Understandings for High School Volume in probability and statistics. A few institutions offer such courses and similar in-service courses; see the CBMS MET website for information about some of these courses.

Currently, the general-audience Introduction to Statistics course is closer to what teachers need than the upper-level math major course. However, the typical introduction to statistics course gives minimal attention to the new issues about designing studies to produce good data for answering well-formed questions. There are some recent textbooks for this intro course that do look at these issues[1] and would make such intro courses much more valuable for future teachers (and other students in such courses).

Given the importance of statistics in most people's daily life, given future teachers' minimal exposure in high school currently to the CCSS statistics topics, and given the holes that remain after the typical introductory college statistics course (huge holes in the case of an upper-division probability-statistics course), there is a strong argument that future teachers merit their own upper stat-prob course, using one of the introductory textbooks mentioned above. The following are the sort of questions that would be discussed.

- What exactly is wrong with saying, "I accept the null hypothesis"?
- Does a high correlation indicate that a linear model is appropriate?
- Why do different calculators and statistical software sometimes give different values for the median and quartiles?

[1] Examples are the books by Agresti and Franklin, by Peck, Olsen and Devore, by Watkins Scheaffer and Cobb, and by Starnes, Yates and Moore.

- Describe three things you could say to a high school student who asks why we use the standard deviation to measure spread.
- How would you explain why we divide by $n - 1$ in the denominator of the standard deviation?
- What happens to the mean and standard deviation if you transform data by adding the same constant to each value? Prove your answer.
- Why is the coverage for a standard 95% confidence interval for a proportion less than 95% and what alternatives are available?
- What's the difference between the idea of independent events that we saw in the probability chapter and the idea of independence in this chi-square chapter?

In the ideal world, this would be a two-semester course. Note that this course would also be valuable to many math majors. In the absence of such a one- or –two-semester course for teachers, it would be valuable for future teachers to take, along with the intro course, a data analysis (statistical methods) course, either a general-audience sequel to the intro course or a stand-alone, upper-level data analysis course for math majors.

Professional Development: From the preceding discussion, it is clear that most high school mathematics teachers will need extensive professional development in statistics. In the immediate future, they will typically graduate with one upper-level probability-statistics course that covers at best a modest amount of the CCSS statistics standards. The American Statistical Association (ASA) and NCTM publication mentioned above can be used for the in-service statistical education of teachers. The footnoted textbooks for introductory statistics courses are also useful. The needed material is not technically advanced. However, because many of the statistics topics are more science than mathematics, e.g., experimentation, data exploration, hypothesis generation, it is valuable for teachers to receive some instruction from a statistician in a graduate course or some sort of mini-course. See the CBMS MET about such mini-courses and graduate courses (which could be co-listed as an undergraduate course for pre-service teachers).

Probability. The probability component of this area gets some coverage in current high school curricula and is reasonably covered in the upper-level probability-statistics course that future high school mathematics teachers are normally required to take in their mathematics major programs. Key topics are sample spaces, simple distributions, probability of compound events, independence, conditional probability and expectations. Note that the given greater amount of statistics that needs to be covered in a probability-statistics course, combinatorial probability problems should be minimized and left to a discrete mathematics course.

Teaching probability well is a skill that requires years of continual learning through experience. Probability is famous for its tricky logic. An example is, in an equiprobable sample space, conditioning questions can make some outcomes more likely than others—for example, if a family is known to have two children, how does knowing that one child is a girl affect the probability that the other child is a girl. Another challenge is that there are many different ways to approach some problems. Throughout their careers, teachers will have to respond to a student's suggested solution strategy that the teacher has not seen before, possibly the strategy is correct, possibly seriously flawed, and sometimes wrong but correctible in an interesting way.

Professional Development: As noted above, statistics educators recommend eliminating the required upper-level probability-oriented course that future teachers take along with other mathematics majors and replacing it with a lower-level introduction to statistics or better a separate course for teachers. This will leave future teachers with a deficit in their probability knowledge. The professional development materials of the American Statistical Association and NCTM cited in the statistics section have the needed probability discussions. As with statistics, these materials can be used for: individual teachers to study on their own; to support study groups among teachers; and for use in professional development workshops. Given the subject's subtleties, these materials can be helpful for teachers who did have a probability course in college.

AP Statistics. The AP Statistics test was started in 1997 and now has close to 150,000 students taking it annually in over 6,000 high schools. Given the shortcomings described here in the typical pre-service statistics education of high school mathematics teachers, it is clear that extensive additional preparation in statistics is required to teach AP Statistics. Several graduate courses in statistics are desirable (chosen in individual consultation with faculty in a graduate statistics program). The minimum preparation would be a good lower-level introductory statistics course, based on the sort of textbooks mentioned above, followed by either a second undergraduate statistics course or a graduate statistics course designed for teachers (see the MET Professional Development website for details about such a course).

Explanatory note. This is a draft of recommendations that update those in the *Mathematical Education of Teachers* report published in 2001, hereafter called MET I. That report may be downloaded at http://www.cbmsweb.org/MET_Document/index.htm.

This draft makes reference to examples posted on the CBMS web site. Such examples are not currently posted, but will be posted when MET II is published.

Draft MET II Recommendations

1. Prospective teachers need mathematics courses that develop a good understanding of the mathematics they will teach. To produce well-started beginning teachers, coursework should include the opportunity to study in depth and from a teacher's perspective most of the mathematics they will initially teach. The mathematical knowledge needed by teachers at all levels is substantial yet quite different from that required by students pursuing other mathematics-related professions. Prospective teachers need to understand the fundamental principles that underlie school mathematics, so that they can teach it to diverse groups of students as a coherent, reasoned activity and communicate an appreciation of the elegance and power of the subject.

2. Coursework that allows time to engage in reasoning, explaining, and making sense of the mathematics that prospective teachers will initially teach is needed to produce well-started beginning teachers. Although the quality of mathematical preparation is more important than the quantity, we offer the following recommendations for the amount of mathematics coursework for prospective teachers.

- *(i) Prospective elementary grade teachers should be required to take at least 9 semester-hours on fundamental ideas of elementary school mathematics.*

- *(ii) Prospective middle grades teachers of mathematics should be required to take at least 21 semester hours of mathematics that includes at least 12 semester-hours on fundamental ideas of school mathematics appropriate for middle grades teachers.*

- *(iii) Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that includes three courses with a primary focus on high school mathematics from an advanced viewpoint.*

3. *All courses designed for prospective teachers should develop the habits of mind of a mathematical thinker: reasoning and explaining; modeling and using tools; seeing structure and generalizing. Courses should also develop and demonstrate the flexible, interactive styles of teaching that will enable teachers to develop these habits of mind in their students.* A long-standing goal of mathematics education is to develop in students not only knowledge of content but the ability to work in ways characteristic of the discipline. Such ability has been described in various ways in the last 10 years: the National Council of Teachers of Mathematics process standards, the *Adding It Up* strands of mathematical proficiency, and the CCSS standards for mathematical practice. In order to develop these

abilities in their students, teachers must experience them in their own mathematical education, through, for example, immersion experiences, research projects, or seminars devoted to doing mathematics.

4. *Teacher education must be recognized as an important part of mathematics departments' mission at institutions that educate teachers. More mathematics faculty should consider becoming deeply involved in K–12 mathematics education.* Mathematics departments need to devote commensurate resources to designing and offering courses for teachers. At a minimum, these courses should be supervised by a faculty member (full-time or part-time) with expertise in teacher education.

5. *Mathematics teaching, including the mathematical education of teachers, can be greatly strengthened by the growth of a professional community that includes mathematicians as one of many constituencies committed to working together to improve mathematics education.*

This recommendation updates recommendations 6, 7 and 8 in MET I. It remains important to encourage partnerships between mathematics faculty and mathematics education faculty, between faculty in two-year and four-year institutions and between mathematics faculty and school mathematics teachers. But more is needed to sustain efforts to strengthen the mathematical education of teachers, both future teachers and those who currently teach in K–12 US schools and efforts to improve mathematics teaching at all levels, from early childhood learning to our colleges and universities. Certainly, we already have many initiatives, communities, and professional organizations focused on some aspect of this work. Needed are more intentional efforts to bridge current communities in ways that are built upon mutual respect and the recognition that there are opportunities for professional growth for university faculty in mathematics and mathematics education as well as the mathematics teachers and supervisors in the K–12 community. The CBMS web site includes examples of collaborative work that enhances the mathematical education of teachers include mathematicians and K–12 teachers who co-teach professional development courses for teachers, integrative approaches to teaching future teachers, and partnerships between universities and two-year colleges. Also needed are more opportunities to make public the work of teaching, including learning communities, math circles, conferences, and publications, from newsletters to scholarly articles. Because the potential for contributions are great and research mathematicians are underrepresented in the community described by Recommendation 5, it is especially important for research mathematicians to be members of this community.

6. *Efforts to improve standards for school mathematics instruction as well as for teacher preparation accreditation and teacher certification will be strengthened by the full-fledged participation of the academic mathematics community.* Mathematicians have recently become more involved in this arena as witnessed by their major role in writing the Common Core State Standards. One growing area of mutual interest is alternative routes for state licensing of mathematics teachers, where better standards for mathematical knowledge are needed.

7. *Throughout their careers, teachers need opportunities for continued professional growth that include opportunities to learn mathematics on the job, through informal teacher-driven initiatives, and as part of graduate education.*

The goal of initial certification should be to create well-started beginners. Initial certification programs do not, and probably cannot, allocate sufficient time to learning mathematics so as to ensure that even the best prepared future teachers have the knowledge of mathematics, of teaching, and of students that is possessed by outstanding master teachers. A well designed professional development program offers teachers opportunities for professional growth appropriate for their experience as they make the transition from new teacher to mid-career professional, to master teacher.

Reflective practitioners will, of course, learn from the work of teaching itself, but if their careers are to allow continued professional growth, they will need regular opportunities to learn and do mathematics and to increase their pedagogical content knowledge. This should include both school- and district-sponsored content-based professional development and university-based professional development offered either as "short courses" or graduate coursework. Highly motivated teachers will also seek out opportunities for teacher driven professional experiences such as teachers' math circles.

Continued opportunities to learn mathematics are particularly important at the secondary level. A well prepared new teacher should be ready to teach the mathematics of grades 7 and 8, as well as algebra and geometry. But, continued graduate education in mathematics is needed before a high school teacher is ready to teach, and teach well, the subjects typically found in grades 11 and 12, including precalculus, calculus, discrete mathematics, matrix algebra, and statistics.

8. *To ensure high quality mathematics teaching and learning in K–12 schools, collegiate mathematics faculty, in collaboration with mathematics education faculty, must become more involved in professional development programs in mathematics.*

State departments of education and local school districts recognize the need for continuing education and implement policies requiring either inservice work or graduate education. Unfortunately, in far too many colleges and universities, mathematics departments have few, if any, graduate education opportunities designed to meet the professional needs of K–12 mathematics teachers. Instead, this important work is left to faculty in the College of Education. In the absence of opportunities to learn mathematics, teachers often take graduate work leading to administrative certification, or other graduate work that, while valuable, does little to strengthen their knowledge of mathematics.

There are notable exceptions that can serve as models for mathematics departments interested in supporting and serving this important part of the mathematics community. Several of the Math Science Partnerships sponsored by National Science Foundation (NSF) include significant professional development programs for K–12 mathematics teachers. The Master Teaching Fellows track of the Robert Noyce Teacher Scholarship Program is an example of how universities can support professional development for master teachers who then work to benefit mathematics teaching in their district, often by providing professional development opportunities for fellow teachers. Math for America supports both early career teachers and master teachers through a program that centers on solid knowledge of mathematics to strengthen mathematics teaching in secondary mathematics classrooms. Both the American Institute of Mathematics and the Mathematical Sciences Research Institute have made a commitment to encourage mathematicians and mathematics teachers to participate together in teachers' math circles. Examples of these and other professional development programs for mathematics teachers can be found on the CBMS web site.

9. *The mathematics community should support raising the standards of professionalism for teachers.* Strategies for raising these standards include: (i) ensuring that teachers have adequate knowledge of mathematics at certification; (ii) developing a career ladder that keeps outstanding teachers in the profession; and (iii) providing professional development opportunities for teachers to grow from early career teachers to mid-career teachers to master teachers.

Society will be better served by focusing efforts to raise standards at the time of entry into the teaching profession rather than looking to weed out ineffective practicing teachers. Advancement in a career ladder provides another framework to establish high standards. In return, schools need to provide high-quality, content-based professional development to support teachers' efforts to deepen their mastery of mathematics and to apply this mastery to provide high-quality instruction. Models for creating high standards of professionalism in teaching can be found in other professions such as health care, law, and engineering.

Building a greater sense of community among all mathematics educators—school teachers and collegiate mathematics faculty members—(as discussed in Recommendation 5) will contribute to higher professional standards for teaching.

10. *In grades 5–8, mathematics should be taught by teachers whose preparation and knowledge base meets the standards outlined in this document for middle grades teachers. The mathematical preparation of an elementary teacher is not sufficient*.

This recommendation seeks to clarify Recommendation 11 in MET I. Historically, grades 5 and 6 teachers have been prepared as elementary teachers and certification requirements in many states permitted teachers with an elementary preparation to teach 7^{th} and 8^{th} grade mathematics. This is not adequate preparation to teach challenging mathematics in grades 5–8. The need for mathematics specialists starting in grade 5 is especially important for a challenging curriculum such as that consistent with the CCSS. As topics such as algebra and functions receive increased attention and work with fractions and decimals becomes more complex, teaching mathematics well requires subject matter expertise beyond that of even a well prepared elementary teacher. It is also important, especially in grades 7 and 8, for middle grades mathematics teachers to know the high school mathematics curriculum well and to understand how their instruction lays the foundation for high school mathematics.

Ideally, all states will acquire different certification requirements for elementary and middle grades, so that elementary certification ensures preparation for teaching grades K–4 and middle level certification programs are developed and required to teach grades 5–8 mathematics. In the interim, states without a stand-alone middle school teacher certification program can assist in educating future grades 5–6 teachers by requiring candidates to pursue a strong specialist program (focusing major attention on a key content area such as mathematics) as part of their current elementary education program. Grades 5–6 teachers with strong preparation to teach mathematics will prove an important resource for their elementary schools, both as classroom teachers and as mathematics specialists who can assist K–4 teachers in strengthening their mathematical knowledge and delivering a strong mathematics program to their students.