CHAPTER 6

High School Teachers

What mathematics should prospective high school teachers study to prepare for their careers? What kinds of coursework and programs will prepare high school teachers for teaching mathematics? What sorts of professional development experiences will develop and sustain high quality mathematics teaching in high school? How can mathematicians make valuable contributions to these endeavors? These questions are the topics of this chapter. Coursework in mathematical pedagogy is assumed to be part of a preparation program, but is not discussed in detail.

In *Elementary Mathematics from an Advanced Standpoint*, Felix Klein described what he called the double discontinuity experienced by prospective high school teachers:

> The young university student [was] confronted with problems that did not suggest ... the things with which he had been concerned at school. When, after finishing his course of study, he became a teacher ... he was scarcely able to discern any connection between his task and his university mathematics.[1]

The double discontinuity consists of the jolt experienced by the high school student moving from high school to university mathematics, followed by the second jolt moving from the mathematics major to teaching high school. This discontinuity still exists today. Many high school students, even those who are successful in their mathematics courses, graduate with a view of mathematics as a static body of knowledge and skills, full of special-purpose tools and methods that are used to solve small classes of problems. Missing is the overall coherence and parsimony of the discipline and the beautiful simplicity of a subject in which a small number of ideas can be used to build intricate and textured edifices of interconnected results.

As noted in the first MET report, analyses by Ed Begle in the 1970s and David Monk in the 1990s suggest that the set of upper-division courses typical of a mathematics major have minimal impact on the quality of a teacher's instruction, as measured by student performance.[2] This is not to say that subject matter of those courses is not valuable for teachers. However, it may be that the choice of topics and the way they are developed is not helpful. For example, teachers might emerge from a course on Galois theory without having seen its connection with the quadratic formula. In this regard, Hung-Hsi Wu makes the following recommendation for the preparation of high school teachers:

Note that the MET II web resources at `www.cbmsweb.org` give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

[1]Translation of the third edition, Macmillan, 1932, p. 1.

[2]This line of research and its limitations are discussed in more detail in Chapter 2.

> In contrast with the normal courses that are relentlessly "forward-looking" (i.e., the far-better-things-to-come in graduate courses), considerable time should be devoted to "looking back."[3]

Thus, one theme of this chapter is that the mathematical topics in courses for prospective high school teachers and in professional development for practicing teachers should be tailored to the work of teaching, examining connections between middle grades and high school mathematics as well as those between high school and college.

A second theme concerns not the topics studied but the practice of mathematics. The National Academy report *Adding It Up* defines five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. High school teachers have the responsibility of building on their students' mathematical experiences in previous grades to give them a sturdy proficiency composed of all five strands interwoven. The Common Core State Standards for Mathematical Practice describe how this proficiency might look in various mathematical situations and specific examples are given in the Progressions for the CCSS.

To achieve this result, teachers need opportunities for the full range of mathematical experience themselves: struggling with hard problems, discovering their own solutions, reasoning mathematically, modeling with mathematics, and developing mathematical habits of mind. Thus, in addition to describing topics, this chapter describes varieties of mathematical experience for teacher preparation and professional development.

Some topics and experiences will occur during preparation, others will occur in professional development over the course of teachers' careers. This chapter describes:

- Essentials in the mathematical preparation of high school teachers.
- Important additional mathematics content that can be learned in undergraduate electives or in professional development programs for practicing teachers.
- Essential mathematical experiences for practicing teachers.

With regard to topics, in all three categories, teachers might take standard courses for mathematics majors. This chapter describes ways in which such courses can be adjusted to better connect with the mathematics of high school. Here "the mathematics of high school" does not mean simply the syllabus of high school mathematics, the list of topics in a typical high school text. Rather it is the structure of mathematical ideas from which that syllabus is derived.

Much of this structure is absent from current courses that prospective teachers often take, either in high school or in college. For example, the method of completing the square may or may not be present, as a pure technique, in a high school algebra course. However, there is more to know about completing the square than how to carry out the technique: it reduces every single-variable quadratic equation

[3] "On the Education of Mathematics Majors" in *Contemporary Issues in Mathematics Education*, Mathematical Sciences Research Institute, 1999, p. 13.

to an equation of the form $x^2 = k$, and thereby leads to a general formula for the solution of a single-variable quadratic equation; it generalizes to a method of eliminating the next to highest order term in higher order equations; it allows one to translate the graph of every quadratic function so that its vertex is at the origin, and thereby allows one to show that all such graphs are similar; and it provides an important step in simplifying quadratic equations in two variables, leading to a classification of the graphs of such equations. The treatment of completing the square in high school is often the merest decoration on this body of knowledge, and a university mathematics major might hear no more of the matter. A prospective teacher who sees some of these connections is better prepared to teach completing the square in a manner consistent with the Standards for Mathematical Practice in the Common Core.

This report recommends that the mathematics courses taken by prospective high school teachers include at least a three-course calculus sequence, an introductory statistics course, an introductory linear algebra course, and 18 additional semester-hours of advanced mathematics, including 9 semester-hours explicitly focused on high school mathematics from an advanced standpoint. It is desirable to have a further 9 semester-hours of mathematics; the appendix for this chapter gives suggestions for a short and a long mathematics course sequence exemplifying these recommendations. A full program would also include all education courses required for certification, which are not described in this report. It is recommended that the methods courses required for certification focus on instructional strategies for high school mathematics rather than generic instructional methods.

Whatever the length of the program, the recommendations described here, particularly the 9 semester-hours of coursework designed for prospective teachers, are ambitious and will take years to achieve. They are, however, what is needed. Institutions that serve only a few prospective teachers per year may be unable to offer many courses with its prospective teachers as the sole audience. These programs may need to consider innovative solutions, similar to the regional centers recommended by the National Task Force on Teacher Education in Physics (see the web resources for examples).[4] Furthermore, courses designed for prospective high school teachers can also serve the needs of other mathematics majors, of prospective middle school teachers if the department has a dedicated program for them, and, if offered at convenient times and locations, of practicing teachers seeking further education.

Essentials in Mathematical Preparation

A primary goal of a mathematics major program is the development of mathematical reasoning skills. This may seem like a truism to higher education mathematics faculty, to whom reasoning is second nature. But precisely because it is second nature, it is often not made explicit in undergraduate mathematics courses. A mathematician may use reasoning by continuity to come to a conjecture, or delay the numerical evaluation of a calculation in order to see its structure and create

[4] See Recommendation 13 of *National Task Force on Teacher Education in Physics: Report Synopsis*, American Association of Physics Teachers, the American Physical Society, & the American Institute of Physics, 2010.

a general formula, but what college students see is often the end result of this thinking, with no idea about how it was conceived.

Reasoning from known results and definitions supports retention of knowledge in a mathematical domain by giving it structure and connecting new knowledge to prior knowledge. This kind of reasoning is useful in most careers, even if domain knowledge is forgotten. It is especially important for teachers, because a careful look at the mathematics that is taught in high school reveals that it is often developed as a collection of unrelated facts that are not always justified or precisely formulated.[5] Hence, many incoming undergraduates are not used to seeing the discipline as a coherent body of connected results derived from a parsimonious collection of assumptions and definitions. One necessary ingredient to breaking this cycle is the next generation of teachers, who must have a coherent view of the structure of mathematics in order to develop reasoning skills in their students. Thus, this report recommends that when courses include prospective teachers, instructors pay careful attention to building and guiding mathematical reasoning—generalizing, finding common structures in theorems and proofs, seeing how a subject develops through a sequence of theorems, and forming connections between seemingly unrelated concepts. At the heart of mathematical reasoning is asking the right questions. As George Pólya liked to tell his classes, "It is easy to teach students the right answers; the challenge is to teach students to ask the right questions."

To teach mathematical reasoning requires a classroom where learners are active participants in developing the mathematics and are constantly required to reflect on their reasoning. Definitions and theorems should be well motivated so that they are seen as helpful, powerful tools that make it easier to organize and understand mathematical ideas. Such classrooms will also benefit prospective teachers by serving as models for their own future classrooms. A corollary of this approach is that to emphasize mathematical reasoning, upper-division mathematics courses for teachers may need to spend more time on only a part of the traditional syllabus. The end of such a course could survey further material. Learning mathematical reasoning is more important than covering every possible topic.

Finally, learning mathematical reasoning and actively participating in class will be easier when the learning builds on existing knowledge of high school mathematics. For example, undergraduates have more experience to draw upon in an algebra class when discussing polynomial rings than non-commutative groups. Of course, building theories directly connected to high school mathematics can also strengthen and deepen prospective teachers' knowledge of what they will teach.

We begin with some suggestions for the courses in the short sequence outlined in the chapter appendix. There are two special suggestions that cut across all the rest—experiences that should be integrated across the entire spectrum of undergraduate mathematics:

> *Experience with reasoning and proof.* Reasoning is essential for all mathematical professions, especially for teaching. Making sense of mathematics makes it easier to understand, easier to teach, and intellectually satisfying for all students, including high school students who have no intention of going into technical fields. And proof is essential to all of mathematics. Accordingly, reasoning and proof, while it is often the focus of a particular

[5] For examples, see Wu, "Phoenix Rising," *American Educator*, 2011.

course, should be present, not always at the same level of rigor, in most undergraduate courses. For example, the Intermediate Value Theorem can be motivated and made plausible in a first calculus course; a more rigorous proof can be developed in a course in real analysis.

Experience with technology. Teachers should become familiar with various software programs and technology platforms, learning how to use them to analyze data, to reduce computational overhead, to build computational models of mathematical objects, and to perform mathematical experiments. The experiences should include dynamic geometry environments, computer algebra systems, and statistical software, used both to apply what students know and as tools to help them understand new mathematical ideas—in college, and in high school. Not only can the proper use of technology make complex ideas tractable, it can also help one understand subtle mathematical concepts. At the same time, technology used in a superficial way, without connection to mathematical reasoning, can take up precious course time without advancing learning.

Courses Taken by a Variety of Undergraduate Majors

Single- and multi-variable calculus. The standard three-semester calculus sequence can help prospective teachers bring together many of the ideas in high school mathematics. They can derive results that may have been taken for granted in high school, such as the formulas for the volume of a cone and sphere, and they have an opportunity to master the ideas of algebra, clearing up common confusion among expressions, equations, and functions.

Calculus also opens up the arena of applied mathematics, deepening prospective teachers' understanding of mathematical modeling. Many calculus courses include a brief treatment of differential equations, providing prospective teachers an opportunity to see where a subject they teach in high school is heading in college.

Multi-variable calculus provides the same opportunity, opening up the subject into more serious applications to science and engineering than are available in single-variable calculus. Multi-variable calculus also provides essential background in analytic geometry. A careful treatment of geometry in $\mathbf{R}^2$ and $\mathbf{R}^3$ using dot product to extend notions of length and angle, and developing equations of lines and planes, is extremely useful background for high school teaching.

Introduction to linear algebra. After calculus, linear algebra is the most powerful, comprehensive theory that teachers will encounter. It is an excellent place to begin proving theorems because of the computational nature of many of its proofs, and provides an opportunity for teachers to experience the mathematical practice of abstracting a mathematical idea from many examples. A concrete course anchored in specific examples and contemporary applications is more likely to serve the needs of prospective teachers than a course in the theory of abstract vector spaces. Important examples include $\mathbf{R}^n$ and the vector space of polynomials on which differentiation and integration act as linear operators; contemporary applications such as regression, computer visualization, and web search engines.

The Common Core State Standards include operations on vectors and matrices and the use of matrices to solve systems of equations. Matrices and matrix algebra represent an important generalization of numbers and number algebra, providing an opportunity to reflect on the properties of operations as general rules for algebraic manipulation. The representation of complex numbers by matrices is a particularly relevant instance of this for high school teachers. In addition to this algebraic aspect of matrices, the geometric interpretation of matrices as transformations of the plane and three-space is also useful for prospective teachers, providing, for example, a connection between solving equations and finding inverse functions. Linear equations and functions are prominent in secondary school mathematics, and geometric interpretations of them in higher dimensions can deepen teachers' understanding of these notions. For example, the classification by dimension of solution sets of systems of linear equations gives perspective on the one-dimensional case, making the cases of no or infinitely many solutions to linear equations in one variable seem less exceptional.

Statistics and probability. The Common Core State Standards include interpretation of data, an informal treatment of inference, basic probability (including conditional probability), and, in the + standards,[6] the use of probability to make decisions. In preparation for teaching this, teachers should see real-world data sets, understand what makes a data set good or bad for answering the question at hand, appreciate the omnipresence of variability, and see the quantification and explanation of variability via statistical models that incorporate variability.

For this purpose, the standard statistics course that serves future engineers and science majors in many institutions may not be appropriate. A modern version, given as one course or a two-course sequence, centers around statistical concepts and real-world case studies, and makes use of technology in an active learning environment. It would contain the following topics: formulation of statistical questions; exploration and display of univariate data sets and comparisons among multiple univariate data sets; exploration and display of bivariate categorical data (two-way tables, association) and bivariate measurement data (scatter plots, association, simple linear regression, correlation); introduction to the use of randomization and simulation in data production and inferential reasoning; inference for means and proportions and differences of means or proportions, including notions of p-value and margin of error; and introduction to probability from a relative frequency perspective, including additive and multiplicative rules, conditional probability and independence. If given as a two-course sequence, it would include additionally the topics described on page 66.

Courses Intended for All Mathematics Majors

This section describes ways in which courses commonly occurring in the mathematics major can be geared to the needs of prospective teachers.

Introduction to proofs. In order to be able to recognize, foster, and correct their students' efforts at mathematical reasoning and proof, prospective high school teachers should analyze and construct proofs themselves, from simple derivations to proofs

[6]The CCSS standards for high school include standards marked with a +, indicating standards that are beyond the college- and career-ready threshold.

of major theorems. Also, they need to see how reasoning and proof occur in high school mathematics outside of their traditional home in axiomatic Euclidean geometry. Important examples include proof of the quadratic formula and derivation of the formula for the volume of a cone from an informal limiting argument that starts from the volume of a pyramid. Moreover, teachers must know that proof and deduction are used not only to convince but also to solve problems and gain insights. In particular, teachers need to see why solving equations is a matter of logical deduction and be able to describe the deductive nature of each step in solving an equation.

Prospective teachers can gain experience with reasoning and proof in a number of different courses, including a dedicated introduction to proofs course for mathematics majors, Linear Algebra, Abstract Algebra, Geometry, or a course on high school mathematics from an advanced standpoint. In the last course, polynomial algebra, geometry, number theory, and complex numbers are good venues for learning about proofs.

Abstract algebra. An advanced standpoint reveals much of high school mathematics as the algebra of rings and fields. Abstract algebra for prospective high school teachers should therefore emphasize rings and fields over groups. These structures underlie the base-ten arithmetic of integers and decimals, and operations with polynomials and rational functions. This course is an opportunity for prospective teachers to gain an understanding of how the properties of operations determine the permissible manipulations of algebraic expressions and to appreciate the distinction between these properties and "rules" that are merely conventions about notation (for example, the order of operations). Attention to the concepts of identity and inverse help prospective teachers see the concepts of multiplicative inverse, additive inverse, inverse matrix, and inverse function as examples of the same idea. Particularly important is the isomorphism between the additive group of the real numbers and the multiplicative group of the positive real numbers given by the exponential and logarithm functions, and the fact that the laws of exponents and the laws of logarithms are just two isomorphic collections of statements.

The study of rings also provides an opportunity for teachers to see that nonnegative integers represented in base ten can be viewed as "polynomials in 10," and to consider ways in which polynomials might be considered as numbers in base x, as well as ways in which these analogies have shortcomings. Whole-number arithmetic can be viewed as a restriction of operations on polynomials to "polynomials in 10."

The division algorithm and the Euclidean algorithm for polynomials and integers, the Remainder Theorem, and the Factor Theorem are important for teachers, because these theorems underlie the algebra that they will teach.

It is also valuable for prospective teachers to see the historical development of methods for representing and performing numerical and symbolic calculations, and of formal structures for number systems.

Another example of a connection between abstract algebra and high school mathematics is the connection between $\mathbf{C}$ and $\mathbf{R}[x]$. It would be quite useful for prospective teachers to see how $\mathbf{C}$ can be "built" as a quotient of $\mathbf{R}[x]$ and, more generally, how splitting fields for polynomials can be gotten in this way. The quadratic formula, Cardano's method, and the algorithm for solving quartics by radicals can all be developed from a structural perspective as a preview to Galois theory, bringing some coherence to the bag of tricks for factoring and completing the square that are traditional in high school algebra. Indeed, this coherence is a major goal of the CCSS high school standards.

Many of the examples given in the description of the number theory course on page 61 can also serve as ingredients for an abstract algebra course geared to prospective high school teachers.

The short sequence in the chapter appendix contains an additional elective for all mathematics majors. Here are some suggestions for material that can be included in such courses. Much of this material is also suitable for the recommended three courses designed specifically for prospective teachers.

The real number system and real analysis. It is an often unstated assumption of high school mathematics that the real numbers exist and satisfy the same properties of operations as the rational numbers. Teachers need to know how to prove what is unstated in high school in order to avoid false simplifications and to be able to answer questions from students seeking further understanding. Thus, a construction of the real numbers, a proof that they satisfy the properties of operations (the CCSS term for the field axioms), and a proof that they satisfy the Completeness Axiom are necessary for teachers. A definition of continuity for a function of a real variable and a proof of the Intermediate Value Theorem provide the underpinnings of the graphical methods for solving equations that are taught to high school students. Thus, they are needed ingredients in teachers' backgrounds. A treatment of the real numbers can also include a treatment of their representation as infinite decimals, including an understanding of decimal expansions as an address system on the number line and an analysis of the periods of decimal expansions of rational numbers using modular arithmetic.

These topics provide opportunities to make use of original historical sources, which can motivate the theory and make it seem less disconnected from school mathematics.

Modeling. In many departments, there is a modeling course for mathematics majors. The Common Core State Standards include an emphasis on modeling in high school, and prospective teachers should have experience modeling rich real-world problems. This includes some aspects of quantitative literacy: the ability to construct and analyze statistical models; the ability to construct and analyze expressions, equations, and functions that serve a given purpose, derived either from a real-world context or from a mathematical problem, and to express them in different ways when the purpose changes; and the ability to understand the limitations of mathematical and statistical models and modify them when necessary.

Differential equations. It would be useful for calculus teachers to see where the subject is going. The traditional course for engineers, heavy on analytical techniques, is not the best choice for teachers. Rather they would benefit from a course including quantitative and qualitative methods, and experience constructing and interpreting classical differential equations arising in science, possibly including partial differential equations. This is also an excellent course in which to include some historical material.

Group theory. A number of applications of group theory are useful for teachers. One is the application of groups to understand the permutations of the roots of an equation that leave the coefficients fixed.[7] A concrete introduction to Galois theory helps place the quadratic formula into a larger theoretical perspective. A second application is connected to the CCSS approach to geometry through rigid motions and dilations. Study of groups and group structures of transformations, and even the isometries of polygons and polyhedra, is very useful background for high school teachers as they integrate this transformational approach into their geometry teaching. A third application is the affine group of the line, that is, the group of transformations of the form $f(x) = ax + b$ for real numbers a and b, which underlies much of the work in high school algebra on transformations of graphs of functions, among other things.[8]

Number theory. Modular arithmetic lends itself to extensions of numerous topics appropriate for STEM-intending high school students, from the analysis of periods of decimal expansions of rational numbers to an understanding of public key cryptography.

A comparison of arithmetic in $\mathbf{Z}$ and $\mathbf{Z}/n\mathbf{Z}$ helps teachers understand the importance of the lack of zero divisors when teaching the "factor to solve" techniques for quadratic and higher-degree equations. It also provides examples of two different polynomials that define the same function, reinforcing the distinction between polynomials and polynomial functions.

A detailed examination of the parallel between $\mathbf{Z}$ and $\mathbf{Q}[x]$, showing, for example, the close connection between the Chinese Remainder Theorem and Lagrange interpolation, would help teachers tie together two of the main algebraic structures in pre-college mathematics.

Although high school mathematics curricula often mention irrational numbers, there are not many cases where numbers are proven irrational. The proof of unique factorization often included in a number theory course allows a proof that any fractional power of any rational number is irrational, unless it is obviously rational.

Many of the examples given in the description of the abstract algebra course can also serve as ingredients for a number theory course geared to prospective high school teachers.

History of mathematics. The history of mathematics can either be woven into existing mathematics courses or be presented in a mathematics course of its own. In both instances, it is important that the history be accurate; instructors who have no contact with historians need to be aware that findings from historical research may contradict popular accounts.

The history of mathematics can be used to raise some general issues about mathematics, such as the role of axiomatic systems, the nature of proof, and perhaps most importantly, mathematics as a living and evolving subject.[9]

History can illustrate the significance of notation. In medieval Europe, computations were made with counters or an abacus, and recorded with Roman numerals. By the sixteenth century, arithmetic was frequently done using Hindu-Arabic

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[7]From a modern viewpoint, this is an application, but the notion of group arose in this context. See Grattan-Guinness's discussion of "irresolving the quintic" in *The Rainbow of Mathematics: A History of the Mathematical Sciences*, Norton, 1997.

[8]See, e.g., Howe, "The Secret Life of the $ax + b$ Group" in the web resources.

[9]These and other ideas are listed in Kleiner's "The Teaching of Abstract Algebra: An Historical Perspective" in *Learn From the Masters!*, MAA, 1995.

numerals—suggesting base-ten notation's affordances for written computation.[10] An innovation that occurred just before 1600 was the representation of quantities by using letters systematically rather than as abbreviations.[11] The power of the symbolic algebra that developed is suggested by the enormous mathematical and scientific progress of the following century, and the cumbersome nature of the preceding "rhetorical algebra."[12]

It is particularly useful for prospective high school teachers to work with primary sources. Working with primary sources gives practice in listening to "wrong" ideas. Primary documents show how hard some ideas have been, for example, the difficulties that Victorian mathematicians had with negative and complex numbers helps prospective teachers appreciate how hard these ideas can be for students who encounter them for the first time. Finally, primary documents exhibit older techniques, and so give an appreciation of how mathematics was done and how mathematical ideas could have developed.

The array of undergraduate electives in mathematics around the country is vast; the list above gives some examples, showing how existing electives can be used to help prepare majors for the profession of high school teaching.

Courses Designed Primarily for Prospective Teachers

As part of the mathematics major for prospective teachers, Recommendation 1 in Chapter 3 calls for three courses with a primary focus on high school mathematics from an advanced viewpoint.

Many of the electives described in the preceding section can, with some modification, meet the goals of this recommendation. Indeed, at many universities, enrollments in teacher preparation programs are too low to create special courses (and many majors don't decide to become teachers until late in their undergraduate program). Dual purpose electives can meet this recommendation if they meet the dual criterion of developing content expertise and reasoning skills described at the beginning of this chapter.

Instructors who design and teach these courses find them mathematically satisfying. Arnold Ross held up "thinking deeply about simple things" as a beacon in our discipline that shines brightly when we consider the mathematics of high school—bringing out its essential coherence and fitting it into the larger landscape of mathematics.

Specialized courses for prospective teachers need not follow a particular theme or format. Practicing high school teachers with insight about the mathematics that is useful to them in their profession can help in the design. Organizing principles that have been useful are:

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[10]See Smith & Karpinski, *Hindu-Arabic Numerals*, Ginn and Company, 1911, pp. 136–137.

[11]This distinction is illustrated by x^2 and x vs. sq and rt (or square and root).

[12]For details of previous and subsequent notations, see Cajori, *A History of Mathematical Notations*, Dover, 1993. A similarity between base-ten notation and symbolic algebra is that they are "action notations" in which computations can occur, rather than "display notations" that only record results. See Kaput, "Democratizing Access to Calculus," *Mathematical Thinking and Problem Solving*, Erlbaum, 1994, p. 101.

Treat high school mathematics from an advanced standpoint. Courses following this principle should emphasize the inherent coherence of the mathematics of high school, the structure of mathematical ideas from which the high school syllabus is derived.

Take up a particular mathematical terrain related to high school mathematics and develop it in depth. For example, a course might develop the mathematics necessary to prove the fundamental theorem of algebra or the impossibility of the classical straight-edge and compass constructions.

Develop mathematics that is useful in teachers' professional lives. Courses following this principle might take up classical ideas that are not normally included in a mathematics major but are of special use for teachers, such as the classical theory of equations or three-dimensional Euclidean geometry.

Here are some examples of courses and course sequences that illustrate these principles:

A treatment of the Pythagorean Theorem and Pythagorean triples that leads to the unit circle, trigonometry and Euclidean formulas for areas of triangles, leading in turn to polygonal approximations of the unit circle and π (one-semester course).[13]

The theory of rational numbers based on the number line, the Euclidean algorithm, complex numbers and the Fundamental Theorem of Algebra, roots and factorizations of polynomials, Euclidean geometry (including congruence and similarity), geometric transformations, axiomatic systems, basic trigonometry, equation, functions, graphs (three-semester sequence).[14]

Classical number systems, starting with the natural numbers and progressing through the integral, rational, real, and complex number systems; mathematical systems, including operations within fields and the foundations of the real number system; modern Euclidean and non-Euclidean geometry, including topics in plane and solid geometry, the axioms of Euclidean, projective, and non-Euclidean geometry; and an introduction to group theory: permutation groups, cyclic groups, theory of finite groups, group homomorphisms and isomorphisms, and abelian groups (five-quarter sequence).[15]

Here are some ideas for the ingredients of specialized courses for teachers:

Geometry and transformations. The approach to geometry in the Common Core State Standards replaces the initial phases of axiomatic Euclidean geometry. In the latter, the triangle congruence and similarity criteria are derived from axioms. The Common Core, on the other hand, uses a treatment based on translations, rotations, reflections, and dilations, whose basic angle and distance preserving properties are taken as axiomatic. It is essential that teachers see a detailed exposition of this development.

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[13] Rotman's *Journey Through Mathematics* has been used for such a course.

[14] This description is based on the University of California at Berkeley courses 151, 152, 153.

[15] This description is based on the University of California, Santa Barbara courses 101A-B, 102A-B, 103.

The Pythagorean Theorem is a fundamental topic in school geometry, and students should see a proof of the theorem and its converse. It can also provide an excursion into some number theory topics such as generating Pythagorean triples, the Congruent Number Problem, and Fermat's Last Theorem, the last two being examples of how questions arising in high school can lead to the frontiers of current research.

An understanding of the role played by the parallel postulate in Euclidean geometry is essential for geometry teachers. Knowing where the postulate is hiding underneath the major theorems in plane geometry, from angle sums in polygons to area formulas, helps teachers build a coherent and logical story for their students. It can also be helpful to know that there are geometries in which the parallel postulate does not hold.

There are many classical results in geometry that may fall through the cracks in undergraduate preparation. Everything from Heron's (and Brahmagupta's) formula to Ptolemy's theorem to the theory of cyclic quadrilaterals can, in the hands of a well-prepared teacher, enrich high school geometry.

Analytic geometry. Many connections between high school topics and the content of undergraduate mathematics can be highlighted in a course in analytic geometry. For example, teachers would benefit from analytic and vector proofs of standard geometric theorems from high school (showing that a line in the coordinate plane is characterized by constant slope is highlighted in the Common Core State Standards[16]). The complete analysis of the graph of a quadratic equation in two variables gives teachers tools that they can use in their high school classes, and it is also a concrete example of the use of the theory of eigenvalues and eigenvectors that they may have studied in linear algebra. Applications of conic sections to everything from quadratic forms to optics shows the power of some of these classical methods.

Complex numbers and trigonometry. Complex numbers can fall into the chasm between high school and college, with high school teachers assuming they will be taught in college and college instructors assuming they have been taught in high school. And the trigonometry that students learn in high school often ends up being a jumble of facts and techniques, with little texture and coherence. Treating complex numbers and trigonometry together can prepare teachers with a solid foundation in both areas, and it can help them make sense of some seemingly disconnected terrains in upper-level high school mathematics.

An understanding of the historical evolution of the complex number system—that complex numbers evolved from a mysterious tool used to solve cubic equations with real coefficients and roots to one of the most useful structures in mathematics—is extremely valuable for high school teachers. For example, $\mathbf{C}$ allows one to connect algebra and geometry in ways that are very hard to see otherwise; de Moivre's Theorem establishes a tight connection between the algebraic structure of roots of the equation and the geometry of the regular n-gon. And there's an equally tight connection between complex numbers and trigonometry—many trigonometric identities can be viewed as algebraic identities in $\mathbf{C}$.

[16]Part of an eighth grade standard is: "Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane."

Teachers would also benefit from seeing applications of complex numbers, ranging from how they underlie electromagnetic communication such as cell phones to how they can be used to generate Pythagorean triples.

In another direction, prospective teachers could bring some needed parsimony and coherence to the high school trigonometry that they will teach if they understand that a small number of basic ideas—the invariance of the unit circle under rotation and the Pythagorean Theorem, for example—can be used to generate all of the results in elementary trigonometry, from the formulas for the sine and cosine of sums and differences to the relationships among co-functions.

Research experience. In all mathematical professions, experts are able to view the discipline from several perspectives. Expert high school teachers should know mathematics in at least four overlapping ways:

> As *scholars.* They should have a solid grounding in classical mathematics, including major results and applications, the history of ideas, and connections to pre-college mathematics.
>
> As *educators.* They should understand the habits of mind that underlie different branches—arithmetic, algebra, geometry, analysis, modeling, and statistics—and how these develop in learners.
>
> As *mathematicians.* They should have experienced a sustained immersion in mathematics that includes performing experiments and grappling with problems, building abstractions as a result of reflection on the experiments, and developing theories that bring coherence to the abstractions.
>
> As *teachers.* They should be expert in uses of mathematics that are specific to the profession, e.g., finding simple ways to make mathematics tractable for beginners; the craft of task design, the ability to see underlying themes and connections in school mathematics, and the mining of student ideas.

Much of the education that prospective high school teachers get as undergraduates focuses on knowing mathematics as a scholar (in mathematics departments) and knowing mathematics as an educator (in education departments). Few undergraduates get a chance to develop knowledge of mathematics as a mathematician or as a teacher. Knowing mathematics as a mathematician is important for prospective high school teachers (in fact, for any mathematics major). It is possible that the discontinuity between how mathematicians and teachers view the whole enterprise of mathematics—what is important, what is convention, what constitutes expertise, and even what it means to understand the subject—is because the typical mathematics major does not provide an intense immersion experience in mathematics. Teachers who have engaged in a research-like experience for a sustained period of time frequently report that it greatly affects what they teach, how they teach, what they deem important, and even their ability to make sense of standard mathematics courses.[17] The research experiences available in many departments and summer

[17]For example, see the reports of Focus on Mathematics (a Math Science Partnership). Comments from teachers include: "Study groups have made 'asking the next question' a much

programs are recommended for prospective mathematics teachers. (See the web resources for examples.)

Important Additional Mathematics

It is impossible to learn all the mathematics one will use in any mathematical profession, including teaching, in four years of college. Therefore teachers will need opportunities to learn further topics throughout their careers. This section describes important additional mathematics that can be the content of undergraduate electives, graduate courses for prospective and practicing teachers, or professional development programs for practicing teachers.

The involvement of the mathematical community in career-long, content-based professional development programs for practicing teachers provides an opportunity for mathematicians and statisticians to have a profound effect on the content and direction of high school mathematics. And it provides teachers with years of opportunities to learn more mathematics and statistics that is especially useful in their profession and to be partners with mathematicians and statisticians in a desperately needed effort to improve professional development experiences. Such in-service should be offered at times of the day and year that allow teachers to participate, such as during the summer, evenings, or weekends.

All of the topics listed in the previous section as possible ingredients for specialized courses for prospective teachers are also fair game for constructing in-service courses. Here are some additional ideas:

Further statistics. For teachers who plan to teach statistics, including high school courses that address the more advanced parts of the statistics standards in the CCSS or AP courses, a second course is recommended. Suggested topics include: regression analysis, including exponential and quadratic models; transformations of data (logs, powers); categorical data analysis, including logistic regression and chi-square tests; introduction to study design (surveys, experiments, and observational studies); randomization procedures for data production and inference; and introduction to one-way analysis of variance.

Discrete mathematics and computer science. Many states are beginning to require a fourth year of high school mathematics. Not all students will be inclined or able to satisfy this requirement with precalculus, calculus, or statistics. States are developing additional courses that build on the modeling and + standards in the CCSS. Teachers of modeling courses will benefit from courses that include topics such as the basics of graph theory; finite difference equations, iteration and recursion; the Binomial Theorem and its use in algebra and probability; and computer programming.

Further geometry. Geometry teachers could profitably study geometric limit problems of the sort studied in ancient Greece, for example the method used to determine the area of a disk. Other possible topics include geometric optimization (finding shortest paths, for example); equi-decomposibility, area, and volume; non-Euclidean geometries; axiomatic approaches to geometry; and a brief introduction

more intriguing mathematical exploration than I previously had imagined or realized I could access," *Focus on Mathematics Summative Evaluation Report 2009,* p. 29.

to computational geometry. The latter can draw heavily on high school geometry; for example, questions such as: Given the three-dimensional coordinates of an observer and the corners of a tetrahedron, which faces of the tetrahedron can the observer see?

Further algebra. There are applications of abstract algebra that are especially useful for high school teachers. These include straight-edge and compass constructions, solvability of equations by radicals, and applications of cyclotomy and roots of unity in geometry. Rational points on conics and norms from quadratic fields can be applied to the problem of creating problems for students that have integer solutions. An introduction to algebraic geometry can help teachers bring some coherence to the analytic geometry they teach in precalculus, helping them make deeper connections between the algebra of polynomials and the graphs of polynomial curves.

Further history of mathematics. Many topics in the history of mathematics are closely related to high school mathematics, for example, history of statistics, history of trigonometry, and history of (premodern) algebra. It is important to make sure that the materials used for courses on these topics include a significant amount of mathematical content.

Further study of the mathematics of high school. Teachers should study the mathematics of high school in their undergraduate programs, as suggested above. Further coursework that focuses on the mathematics they are teaching and how it fits into the broader landscape of mathematics is valuable. Especially important in such coursework is a goal of bringing mathematical coherence to high school mathematics, showing how a few general-purpose ideas and methods can be used across the entire high school spectrum of topics, replacing much of the special-purpose paraphernalia that clutters many high school programs, such as the various mnemonics for formulas in trigonometry. Such courses can fit in graduate degree programs for teachers offered by mathematics departments alone or in conjunction with education departments.

Other advanced topics. The terrain is vast—much mathematics and statistics is missed in undergraduate programs simply because of lack of time, and much of this can bring new insights into high school topics. Applications of the arithmetic-geometric mean inequality to optimization problems, the use of measure theory to connect area and probability, the irrationality or transcendence of the classical constants from algebra and geometry, the famous impossibility theorems (squaring the circle, trisecting the angle), Gödel's Incompleteness Theorems, properties of iterated geometric constructions, Hilbert's axioms for area and volume, and so many other areas can help teachers learn mathematics that is useful in their work and at the same time important in the field.

Essential Experiences for Practicing Teachers

All teachers need continuing opportunities to deepen and strengthen their mathematical knowledge for teaching. Many teachers prepared before the era of the CCSS will need opportunities to study content that they have not previously taught, particularly in the areas of statistics and probability.

In addition to learning more mathematical topics, teachers need experiences that renew and strengthen their interest in and love for mathematics, help them represent mathematics as a living discipline to their students by exemplifying mathematical practices, figure out how to pose tasks to students that highlight the essential ideas under consideration, to listen to and understand students' ideas, and to respond to those ideas and point out flaws in students' arguments. Being able to place themselves in the position of mathematics learners can help them think about their students' perspectives. These needs create opportunities for mathematics departments to participate in the creation of important professional development experiences for high school mathematics teachers.

The research experiences described above for prospective teachers can also be important for practicing teachers. Here are some additional ideas:

Math teachers' circles and study groups. Math teachers' circles, in which teachers and mathematicians work together on interesting mathematics, provide ongoing opportunities for teachers to develop their mathematical habits of mind while deepening their understanding of mathematical connections and their appreciation of mathematics as a creative, open subject. Unlike more structured courses, math teachers' circles are informal sessions that meet regularly and can include the same participants for multiple years. A substantial benefit of such programs is that they address the isolation of both high school teachers and practicing mathematicians: they establish communities of mathematical practice in which teachers and mathematicians can learn about each others' profession, culture, and work.

Immersion experiences. For all the reasons discussed earlier under "research experience" (p. 65), teaching mathematics is greatly enhanced when teachers work themselves as mathematicians and statisticians. For practicing teachers, an immersion experience (usually over a summer) in which one works on a small, low-threshold, high-ceiling cluster of ideas for a sustained period of time has profession-specific benefits. For example, it helps teachers understand the nature of doing mathematics and statistics, it reminds them that frustration, confusion, and struggle are all natural parts of being a learner, it helps them connect ideas that seem on the surface to be quite different, it shows the value of refining ill-formed ideas through the use of precise language, and it keeps alive the passion for mathematics that was ignited in undergraduate school. Mathematicians and statisticians are the ideal resources to help design and implement such immersion programs.

Lesson study. In lesson study, teachers work in small teams including fellow teachers, mathematicians, mathematics educators, and administrators. The teams carefully and collaboratively craft lesson plans designed to meet both content goals and general learning or affective goals for students—such as working together to solve problems or being excited to learn about nature. One or more members of the team teaches the lesson while the other team members observe the lesson implementation.

The team then debriefs the lesson and makes revisions, sometimes teaching the revised lesson to another group of students. Mathematicians can play an important role as part of a lesson study team in helping to think flexibly about the mathematical goals of the lesson, tasks to include, mathematical issues to address following observation of the lesson in action, and mathematical issues to consider when revising the lesson. Working as part of a team, mathematicians, along with others, can bring expertise to bear on the interdisciplinary work of teaching. Hosting a high school lesson study group on teaching topics in trigonometry or precalculus can have an added benefit for instructors of related courses at universities or two- and four-year colleges.

Chapter 6 Appendix: Sample Undergraduate Mathematics Sequences

Short sequence (33 semester-hours).

I Courses taken by undergraduates in a variety of majors (15+ semester-hours)
- Single- and Multi-variable Calculus (9+ semester-hours)
- Introduction to Linear Algebra (3 semester-hours)
- Introduction to Statistics (3 semester-hours)

II Courses intended for all mathematics majors (9 semester-hours)
- Introduction to Proofs (3 semester-hours)
- Abstract Algebra (approach emphasizing rings and polynomials) (3 semester-hours)
- A third course for all mathematics majors (e.g., Differential Equations) (3 semester-hours)

III Courses designed primarily for prospective teachers (9 semester-hours).

Long sequence (42 semester-hours).

I Courses taken by undergraduates in a variety of majors (21 semester-hours)
- Single- and Multi-variable Calculus (9+ semester-hours)
- Introduction to Linear Algebra (3 semester-hours)
- Introduction to Computer Programming (3 semester-hours)
- Introduction to Statistics I, II (6 semester-hours)

II Courses intended for all mathematics majors (12 semester-hours)
- Introduction to Proofs (3 semester-hours)
- Advanced Calculus (3 semester-hours)
- Abstract Algebra (approach emphasizing rings and polynomials) (3 semester-hours)
- Geometry or Mathematical Modeling (3 semester-hours)

III Courses designed primarily for prospective teachers (9 semester-hours).