Chapter 1—6 footnotes with hyperlinks

The footnotes from Chapters 1–6 are listed below and hyperlinked (when possible) to the references cited.

Many of the documents cited are freely available. National Research Council reports such as *Adding It Up* can be read on-line. They can be downloaded without charge as can documents from the Conference Board of the Mathematical Sciences and the Council of Chief State School Officers. In some cases, cited portions of documents can be seen via Google Books.

Mathematics education research journal articles are likely to require a subscription. At many academic institutions, these journals will be accessible via institutional subscription. Attempts to access a JSTOR link without such a subscription will get the response “Cannot download the information you requested.”

**Chapter 1**

1. An overview of the CCSS structure appears as Appendix B of this report.

2. The full text of these standards appears as Appendix C.

3. Between 2004 and 2008, the Park City Mathematics Study Group (a group of research mathematicians) conducted discussions of school mathematics, including extended discussions with NCTM representatives. *Principles and Standards* and *Adding It Up* (published in 2000 and 2001) summarize findings from previous decades of research in mathematics education.

4. Such connections are outlined in the *Progressions for the CCSS* (see the web resources for this report).

5. Examples are given by Ma, *Knowing and Teaching Elementary Mathematics*, Erlbaum, 1999: change in number, p. 74; change in manipulative and problem context, p. 5.

6. For a summary (p. 400) and further examples of teaching tasks, see Ball et al., “Content Knowledge for Teaching,” *Journal of Teacher Education*, 2008; also Senk et al., “Knowledge of Future Primary Teachers for Teaching Mathematics: An International Comparative Study,” *ZDM*, 2012, p. 310.

7. See, e.g., the findings of the *Teacher Education and Development Study in Mathematics* (TEDS-M).

8. These are intertwined and occur on a variety of levels. For example, the institutional arrangement of having teachers share a room affords the professional practice of discussing mathematics. An institutionalized career hierarchy based on teaching shapes the professional activities of Chinese master teachers and “super rank” teachers described in *The Teacher Development Continuum*.

9. Chapter 2 discusses this claim further, but note the findings of Effects of Teacher Professional Development on Gains in Student Achievement, Council of Chief State School Officers, 2009. Most successful professional development programs continued for 6 months or more, and the mean contact time with teachers was 91 hours.

10. For example, the Mathematics Common Core Coalition (comprised of professional societies and assessment consortia) addresses educators, teachers, teacher leaders, supervisors, administrators, governors and their staffs, other policy-makers, and parents.

11. The CBMS surveys (conducted every five years) consistently document large proportions of undergraduates enrolled in remedial mathematics courses (see, e.g., Table S.2 of the 2005 report).

12. The 2005 CBMS survey suggests that many mathematics departments do not have courses especially designed for elementary teachers (see Table SP.6). In 2010, Masingila et al. surveyed 1,926 U.S. higher education institutions that prepared elementary teachers. Of those who responded (43%), about half (54%) reported that requirements included two mathematics courses designed for teachers. See “Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey,” Journal of Mathematics Teacher Education, 2012, Table 2. A more detailed picture for three states is presented by McCrory & Cannata, “Mathematics Classes for Future Elementary Teachers: Data from Mathematics Departments,” Notices of the American Mathematical Society, 2011.

13. Chapter 2 gives an overview of teaching–learning paths.

14. In Masingila et al.’s survey less than half of respondents reported giving training or support to instructors of mathematics courses for elementary teachers.

15. For example, when surveyed in 2000, 86% of K–4 teachers reported studying mathematics for less than 35 hours over a period of three years, an average of less than 12 hours per year. See Horizon Research’s 2000 National Survey of Science and Mathematics Education. More recent studies show large increases in elementary student mathematics achievement when their teachers receive content-based professional development. Student scores of teachers who do not receive such professional development do not show these gains (see the sections on curriculum-specific professional development in Chapter 2 and on mathematics specialists in Chapter 4). Thus, unsatisfactory student performance may suggest a greater need for content-based professional development.
16. The Association for Middle Level Education (AMLE) position statement notes, “in some states, virtually anyone with any kind of degree or licensure is permitted to teach young adolescents.” According to the AMLE website, 28 states and the District of Columbia offer separate licenses for middle grades generalists. Separate licenses, however, do not necessarily imply the existence of separate preparation programs or different mathematics requirements. The 2005 CBMS survey found that 56% of mathematics departments at four-year institutions had the same mathematics requirements for K–8 certification in early and later grades (see Table SP.5). See also the discussion of opportunity to learn for U.S. prospective lower secondary teachers in Tatto & Senk, “The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics,” Journal of Mathematics Teacher Education, 2011, p. 127.

Chapter 2

1. These were: Einleitung zur Rechen-Kunst (Introduction to the Art of Reckoning), St Petersburg (vol. 1, 1738, vol. 2, 1740); The Elements of Arithmetic, London, 1830.

2. Hodgson points out that “one could even see the ICMI as having been formed on the very assumption that university mathematicians should have an influence on school mathematics.” See The Teaching and Learning of Mathematics at University Level, Kluwer, 2001, p. 503.


4. Teaching Teachers Mathematics (Mathematical Sciences Research Institute, 2009) gives an overview of past and recent counterexamples.

5. In 2010, Masingila et al. surveyed 1,926 U.S. higher education institutions that prepared elementary teachers. Of those who responded (43%), less than half reported giving training or support for instructors of mathematics courses for elementary teachers. However, the authors write that “there appears to be interest in training and support as a number of survey respondents contacted us to ask where they could find resources for teaching these courses.” See “Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey,” Journal of Mathematics Teacher Education, 2012.
6. Quoted from Schoenfeld, “Learning to Think Mathematically” in *Handbook for Research on Mathematics Teaching and Learning*, 1992, p. 359. Note that these beliefs may not be explicitly stated as survey or interview responses, but displayed as classroom behaviors, e.g., giving up if a problem is not quickly solved. This discussion is not meant to exclude the possibility of exceptional mathematical talent, but focuses on the idea that K–12 mathematics can be learned in its absence.


8. Note that such beliefs may vary according to domain, e.g., one may believe in a “math gene,” but favor continued practice in order to improve sports performance.


12. This is a slight reformulation of Lampert, 1990 as quoted by Schoenfeld, “Learning to Think Mathematically” in *Handbook for Research on Mathematics Teaching and Learning*, 1992, p. 359. The surrounding text discusses research on school experiences that shape such beliefs.


16. For example, middle grades and high school teachers who participated in an MSP based on an immersion approach (involving intensive sessions of doing mathematics) reported changes in beliefs that affected their teaching, e.g., communicating that it is “OK” to struggle. See ,” Focus on Mathematics Summative Evaluation Report 2009, p. 73. Gains in student test scores are shown on p. 93 (high school) and p. 96 (middle grades).

17. For a snapshot from one such collaboration, see Teaching Teachers Mathematics, Mathematical Sciences Research Institute, 2009, p. 34; for descriptions of three Math Science Partnerships, see pp. 32–41.

18. Test quality can be a major limitation for this measure. An analysis of state mathematics tests found low levels of cognitive demand, e.g., questions that asked for recall or performance of simple algorithms, rather than complex reasoning over an extended period. See Hyde et al., “Gender Similarities Characterize Math Performance,” Science, 2008, 494–495.

19. See Preparing Teachers: Building Evidence for Sound Policy, National Research Council, 2010, p. 112. See also, Telese, “Middle School Mathematics Teachers' Professional Development and Student Achievement.” Journal of Educational Research, 2012. Telese’s measure of student achievement was the Grade 8 National Assessment of Educational Progress, which includes items with a high level of cognitive demand. It found number of mathematics courses to be a strong predictor, but like many such studies, it did not have an experimental or quasi-experimental design.


37. CAEP was formed by the merger of the National Council for the Accreditation of Teacher Education (NCATE) and the Teacher Education Accreditation Council (TEAC). Two of the MET II writers are engaged in the development of the CAEP standards.


39. CBMS 2005 Survey, Table SP.3.


41. In addition to the forthcoming CAEP standards, note the 2012 report *Supporting Implementation of the Common Core State Standards for Mathematics: Recommendations for Professional Development*, Friday Institute for Educational Innovation at the North Carolina State University College of Education.

42. For an overview of MSP outcomes, including increases in student achievement, see *National Impact Report: Math and Science Partnership Program*, National Science Foundation, 2010, pp. 6, 10–12.

Chapter 3

1. The recommendations for teacher preparation in this report are formulated in terms of courses and semester-hours, but this is not meant to exclude other ways of awarding credit or organizing teacher education. For example, collegiate institutions that do not follow a semester system with most courses earning 3 credit-hours will need to adapt these recommendations accordingly.

2. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons. Math teachers’ circles focus primarily on giving teachers an experience to be learners and doers of mathematics. See the web resources for further information and examples.

4. In the 2005 CBMS survey, special courses for K–8 teachers were offered by 11% of Ph.D.-granting and 33% of M.A.-granting statistics departments. Less than 0.5% of statistics departments surveyed reported that special sections of regular courses were designated for K–8 teachers. See Table SP.3.


Chapter 4

Note that the MET II web resources at [www.cbmsweb.org](http://www.cbmsweb.org) give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

1. As noted in Chapter 3, “Although elementary certification in most states is still a K–6 and, in some states, a K–8 certification, state education departments and accreditation associations are urged to require all grade 5–8 teachers of mathematics to satisfy the 24-hour requirement recommended by this report.” Chapters 4 and 5 allow for a period of transition.

2. For example, “It is widely assumed—some would claim common sense—that teachers must know the mathematical content they teach” (*Foundations for Success: Reports of Task Groups of the National Mathematics Advisory Panel*, 2008, p. 5-6). “Aspiring elementary teachers must begin to acquire a deep conceptual knowledge of the mathematics that they will one day need to teach, moving well beyond mere procedural understanding” (*No Common Denominator*, 2008, National Council on Teacher Quality). “Mathematics courses for future teachers should develop ‘deep understanding’ of mathematics, particularly of the mathematics taught in schools at their chosen grade level” (*Curriculum Foundations Project*, 2001, Mathematical Association of America). See also *Preparing Teachers: Building Sound Evidence for Sound Policy*, 2010, National Research Council, p. 123.

3. An international comparison of prospective elementary teachers found that 48% of the U.S. teachers did not score above “Anchor point 2.” Teachers with this score often had trouble reasoning about factors, multiples, and percentages. See Tasto & Senk, “‘The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics,’ *Journal of Mathematics Teacher Education*, 2011, pp. 129–130. *Preparing Teachers* discusses concern about the adequacy of current teacher
preparation in mathematics, especially for elementary teachers. See Chapter 6, especially p. 124.

4. See the National Research Council report *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (2009) and the *Counting and Cardinality Progression*.

5. See the CCSS, pp. 88–89; or the *Operations and Algebraic Thinking Progression* for details and examples of situation and solution equations.

6. For examples of how teachers may construe the base-ten system, see Thanheiser, “*Pre-service Elementary School Teachers’ Conceptions of Multidigit Whole Numbers*.” *Journal for Research in Mathematics Education*, 2009.


8. For instance, a study of prospective elementary and secondary teachers found that many either did not know that division by 0 was undefined or were unable to explain why it was undefined. On average, the secondary teachers had taken over 9 college-level mathematics courses. Ball, “*Prospective Elementary and Secondary Teachers’ Understanding of Division*.” *Journal for Research in Mathematics Education*, 1990.

9. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons.

10. See this chapter’s section on mathematics specialists for more discussion about their roles in professional development for teachers.


16. In general, a math specialist’s roles and responsibilities are not analogous to those of a reading specialist.

17. Examples include the Vermont Mathematics Initiative (a Math Science Partnership), see Teaching Teachers Mathematics, Mathematical Sciences Research Institute, 2009, pp. 36–38. A 3-year randomized study in Virginia found that specialists’ coaching of teachers had a significant positive effect on student achievement in grades 3–5. The specialists studied completed a mathematics program designed by the Virginia Mathematics and Science Coalition (also a Math Science Partnership) and the findings should not be generalized to specialists with less expertise. See Campbell & Malkus, “The Impact of Elementary Mathematics Coaches on Student Achievement,” Elementary School Journal, 2011.


19. These examples come from Lee & Ginsburg, “Early Childhood Teachers’ Misconceptions about Mathematics Education for Young Children in the United States,” Australasian Journal of Early Childhood, 2009. This article summarizes research in this area and discusses possible sources of such beliefs.


Chapter 5

Note that the MET II web resources at www.cbmsweb.org give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.

1. As noted in Chapter 3, “Although elementary certification in most states is still a K–6 and, in some states, a K–8 certification, state education departments and accreditation associations are urged to require all grade 5–8 teachers of mathematics to satisfy the 24-hour requirement recommended by this report.” Chapters 4 and 5 allow for a period of transition.
2. See the listing at the Association for Middle Level Education web site.


4. See the Ratio and Proportion Progression for further details, including examples of double number lines and tape diagrams, and discussion of unit rates. In the CCSS, “fractions” refers to non-negative rational numbers in grades 3–5. Note that distinctions made in the CCSS between fractions, ratios, and rates may be unfamiliar to teachers.

5. For descriptions of multiplication and division problem types, see the CCSS, p. 89 or the Operations and Algebraic Thinking Progression.

6. “I don’t think it’s equal because I think that would be confusing to kids to say that 99 cents can be rounded up to a dollar” and other examples of conceptions that teachers may hold about this equation are given in Yopp et al., “Why It is Important for In-service Elementary Mathematics Teachers to Understand the Equality .999\ldots = 1,” Journal of Mathematical Behavior, 2008. Note that undergraduates may use decimal notation in ways that suggest notions of nonstandard analysis, see Ely, “Nonstandard Student Conceptions About Infinitesimals,” Journal for Research in Mathematics Education, 2010.

7. It is important for middle grades teachers to have an elementary teacher’s perspective on this content because they may need to provide support and instruction for students who have not yet achieved proficiency.

8. Note that the CCSS use the term “number line diagram” instead of “number line.”

9. Examples of these representations occur in the Progressions for the CCSS.


11. Lesson study is a process in which teachers jointly plan, observe, analyze, and refine actual classroom lessons.
12. Math teachers’ circles and immersion experiences focus primarily on giving teachers an experience to be learners and doers of mathematics. See Chapter 6 and the web resources for further information and examples.


Chapter 6

Note that the MET II web resources at www.cbmsweb.org give URLs for the CCSS, the Progressions for the CCSS, and other relevant information.


2. This line of research and its limitations are discussed in more detail in Chapter 2.


5. For examples, see Wu, “Phoenix Rising,” American Educator, 2011.

6. The CCSS standards for high school include standards marked with a +, indicating standards that are beyond the college- and career-ready threshold.

7. From a modern viewpoint, this is an application, but the notion of group arose in this context. See Grattan-Guinness’s discussion of “irresolving the quintic” in The Rainbow of Mathematics: A History of the Mathematical Sciences, Norton, 1997.

8. See, e.g., Howe, “The Secret Life of the ax + b Group” in the web resources.
9. These and other ideas are listed in Kleiner’s “The Teaching of Abstract Algebra: An Historical Perspective” in Learn From the Masters!, MAA, 1995.


11. This distinction is illustrated by $x^2$ and $x$ vs. sq and rt (or square and root).

12. For details of previous and subsequent notations, see Cajori, A History of Mathematical Notations, Dover, 1993. A similarity between base-ten notation and symbolic algebra is that they are “action notations” in which computations can occur, rather than “display notations” that only record results. See Kaput, “Democratizing Access to Calculus,” Mathematical Thinking and Problem Solving, Erlbaum, 1994, p. 101.

13. This description is based on the University of California at Berkeley courses 151, 152, 153.

14. This description is based on the University of California, Santa Barbara courses 101A-B, 102A-B, 103.

15. Rotman’s Journey Through Mathematics has been used for such a course.

16. Part of an eighth grade standard is: “Use similar triangles to explain why the slope $m$ is the same between any two distinct points on a non-vertical line in the coordinate plane.”

17. For example, see the reports of Focus on Mathematics (a Math Science Partnership). Comments from teachers include: “Study groups have made ‘asking the next question’ a much more intriguing mathematical exploration than I previously had imagined or realized I could access,” Focus on Mathematics Summative Evaluation Report 2009, p. 29.