# EXERCISES

## 1. Colourings of integers

**Question 1.** For each equation below try to prove or disprove that for any $k \geq 1$ there is a threshold, $n_0 = n_0(k)$ such that if $n \geq n_0$ then any $k$-colouring of the first $n$ integers contains three numbers $x, y, z \in [n]$ from the same colour class giving solution to the equation

(1) $x^2 - y = z$
(2) $x + y = 3z$
(3) $xy = z^2$
(4) an interesting equation of your choice.
(5) In this question we colour all real numbers with $k$ colours. Is it true that for any colouring there are three numbers $x, y, z \in \mathbb{R}$ from the same colour class giving solution to the equation $\frac{z-y}{y-x} = \frac{z-x}{z-y}$ ?

**Question 2.** Let $a, m, k, s$ and $a_0 \dots a_{m-1}$ be positive integers. Then the set

$$\left\{ a + \sum_{i<m} \delta_i a_i | \delta_i \in \{k, s\} \right\}$$

is the affine $m_{k,s}$-cube generated by $a, a_0, \dots a_{m-1}$. Similar to Hilbert's cube lemma, prove the following: Let $m, k, s$ and $r$ be positive integers. Then for every $r$-colouring of the positive integers there exists an affine $m_{k,s}$-cube which is monochromatic.

**Question 3.*** Let $a, m, k, s, t$ and $a_0 \dots a_{m-1}$ be positive integers. Then the set

$$\left\{ a + \sum_{i<m} \delta_i a_i | \delta_i \in \{k, s, t\} \right\}$$

is the affine $m_{k,s,t}$-cube generated by $a, a_0, \dots a_{m-1}$. Similar to Hilbert's cube lemma, prove the following: Let $m, k, s, t$ and $r$ be positive integers. Then for every $r$-colouring of the positive integers there exists an affine $m_{k,s,t}$-cube which is monochromatic.

**Question 4.** Let's colour the first $n$ natural numbers with $\ell$ colours independently at random, assigning a colour with probability $1/\ell$ to every number. What is the probability that there is a monochromatic $m_{k,s,t}$-cube? (Let's assume first that all entries in the sum are distinct)

**Conjecture 1** $*^{**}$[ A special case of Hindman's Conjecture] For any colouring of the natural numbers (with finitely many colours) there are two distinct natural numbers, $n$ and $m$ such that all four numbers, $n, m, n+m, nm$, received the same colour.

## 2. Cartesian products

**Question 1.** Let $A, B \subset \mathbb{C}$ such that $|A| = |B| = n$. Prove that the number of collinear triples in $A \times B$ is $O(\log n \cdot n^2)$.

**Question 2.** Let $A_1, \ldots, A_d \subset \mathbb{R}$ such that $|A_1| = \ldots = |A_d| = n$. Give an upper bound on the number of collinear triples in $A_1 \times \ldots \times A_d$ in terms of $n$ and $d$.

**Question 3.*** Let $A, B \subset \mathbb{R}$ such that $|A| = |B| = n$. Give an upper bound on the number of four-tuples in $A \times B$ where the four points are co-circular, they lie on the same circle.

**Question 4.*** Let $S$ consists of the points of the integer grid $[n] \times [n]$,

$$S = \{(a, b) | a \leq n, b \leq n, a, b \in \mathbb{N}\}.$$

Give an upper bound, $B(n, r)$, on the number of triples in $S$ where the three points are co-$r$-circular, they lie on the same circle with radius $r$.

**Question 5.** Given a set of $n$ points in the plane, denoted by $P$. What is the maximum number of six-tuples $p, q, r, p', q', r' \in P$ such that the triangle $p, q, r$ is similar to $p', q', r'$?

## 3. Convexity and sums

**Question 1.** Let $A \subset \mathbb{N}$ such that $|A| = n$. Prove that $A(A + A) = \Omega(n^2)$.

**Question 2.** Let $A \subset \mathbb{R}$ a convex set. Prove that $A + (A - A) = \Omega(n^2)$.

**Question 3.** Let $A \subset \mathbb{R}$ where $A = \{a_1 < a_2 < \ldots < a_n\}$. If the set of consecutive differences $(a_i - a_{i-1})$ is denoted by $D$, then give a lower bound on $|A + A|$ in terms of $n$ and $|D|$.

**Question 4.*** Let $A \subset \mathbb{R}$ a convex set. Prove that $|A - A| = \Omega(n^{3/2+c})$ for some $c > 0$.

**Question 5.** Let $G(A, B)$ a bipartite graph with $e$ edges and $|A| = n, |B| = m$ vertices. Give a bound on the crossing number of $G(A, B)$ in terms of $n, m$ and $e$, which is better than the

$$\Omega\left(\frac{e^3}{(n+m)^2}\right)$$

bound for $n \ll m$.

## 4. Spectra

**Question 1.** In the following question perform all steps as required. You can use a calculator or a computer if needed.

**a,** Your task is to count the number of edges between the top four vertices and the bottom four vertices of the graph of a 3 dimensional cube, C (fig. 1). There are clearly 4 edges, but you should use linear algebra. Multiply the adjacency matrix with the indicator vectors of the two vertex sets. Don't actually multiply the matrix, use the eigenvectors instead. So, find an orthonormal basis of eigenvectors and the projections (coordinates) of the indicator vector. Then calculate the inner products of the vectors to get the desired result.

**b,** Repeat the counting with a $d$-dimensional hypercube.

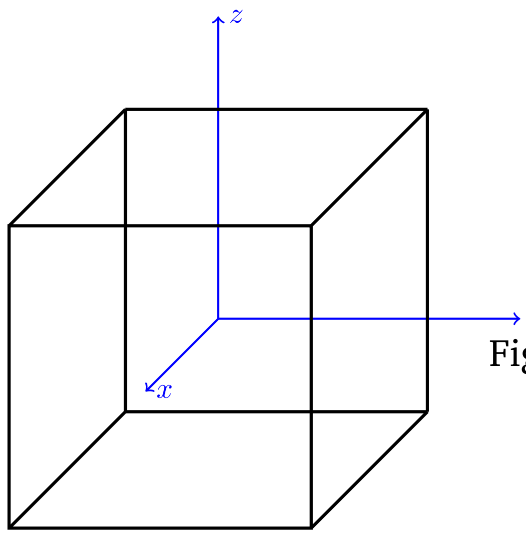
$C =$ 

Figure 1

**Question 2.** Let us define a graph $G$ with vertex set given by the elements of the special linear group of $2 \times 2$ matrices with determinant 1, over the field $\mathbb{F}_p$, $SL(2, \mathbb{F}_p)$. In $G$ two vertices, $A, B$, are connected by an edge iff $\det(A - B) = 0$. Find the eigenvalues of $G$. (Find the largest and second largest eigenvalues first)

**Question 3.*** Try to prove a sum-product type inequality for $SL(2, \mathbb{F}_p)$, similar to the one we proved for $\mathbb{F}_p$ using the sum-product graph.

**Question 4.**** Try to prove a sum-product type inequality for $SL(n, \mathbb{F}_p)$, similar to the one we proved for $\mathbb{F}_p$ using the sum-product graph.

## 5. Regularity

**Question 1.** Let $G_{4n}$ be a four-partite graph on vertex sets $V_1, V_2, V_3, V_4$ such that

- $|V_1| = |V_2| = |V_3| = |V_4| = n$
- there are no edges in $V_i$-s ($1 \leq i \leq 4$)

- all $G(V_i, V_j)$ are $\varepsilon$-regular ($1 \leq i < j \leq 4$, $\varepsilon > 0$)
- all $G(V_i, V_j)$ are at most $\delta$-dense ($1 \leq i < j \leq 4$, $\delta \geq 0$)

What is the maximum number of $K_4$-s in $G$? Check if your bound is sharp up to some $\varepsilon$ error.

**Question 2.**** If a 3-uniform hypergraph on $n$ vertices, $H_n^3$, contains no subgraph on 14 vertices, $F_{14}^3$, such that the number of edges is at least 10, $e(F_{14}^3) \geq 10$, then it is sparse, $e(H_n^3) = o(n^2)$.

**Question 3.*** For any colouring of the natural numbers (with finitely many colours) there are three distinct natural numbers, $n, m$ and $k$ such that all four numbers,

$$n^{n+k}, n^{m+k}, m^{n+k}, m^{m+k},$$

received the same colour.

**Conjecture**$**^{**}$ If a 3-uniform hypergraph on $n$ vertices, $H_n^3$, contains no subgraph on 14 vertices, $F_{14}^3$, such that the number of edges is at least 11, $e(F_{14}^3) \geq 11$, then it is sparse, $e(H_n^3) = o(n^2)$. (This is a particular case of the Brown-Erdős-Sós conjecture)

## 6. Polynomial method

**Question 1.** Given a symmetric polynomial $P(X, Y)$ of degree $d$, and two finite sets $A, B \subset \mathbb{R}$, $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_m\}$. We define an $n \times m$ matrix $M$ such that $M = \{\nu_{i,j}\}_{i,j} = \{P(a_i, b_j)\}$. What is the maximum rank of $M$? Prove that your bound is sharp. (Did you use that $P(X, Y)$ is symmetric?)

**Question 2.** Let $F$ be a finite field with $q$ elements. Let $S \subset F^n$ be a set such that for each point $p \in F^n$ there is a circle $C$ such that $p \in C$ and $|S \cap C| \geq p/4$. Give a lower bound on the size of $S$.

**Question 3.** Let $S$ consists of the points of the integer grid $[n] \times [n]$,

$$S = \{(a, b) | a \leq n, b \leq n, a, b \in \mathbb{N}\}.$$

The $n^2$ points of $S$ can be covered by a polynomial of degree at most $n$, by taking the product of the $n$ vertical (or horizontal) lines intersecting $S$. Is there a polynomial, $F(x, y)$, of degree $n - 1$ covering the $n^2$ points? (i.e. every point is a solution of the $F(x, y) = 0$ equation.) What happens if we remove the diagonal, $(a, a)$, points? What is the smallest degree polynomial covering the $n^2 - n$ points?