NSF–CBMS CONFERENCE ON THE CAHN–HILLIARD EQUATION: RECENT ADVANCES AND APPLICATIONS PRELIMINARY READING

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1. The Cahn–Hilliard equation

The Cahn–Hilliard system

$$\frac{\partial u}{\partial t} = \kappa \Delta \mu, \ \kappa > 0, \ \mu = -\alpha \Delta u + f(u), \ \alpha > 0, \tag{1.1}$$

usually rewritten, equivalently, as the fourth-order in space parabolic equation

$$\frac{\partial u}{\partial t} + \alpha \kappa \Delta^2 u - \kappa \Delta f(u) = 0 \tag{1.2}$$

which is precisely the equation known as the Cahn–Hilliard equation, was proposed by J.W. Cahn[1] and J.E. Hilliard in 1958 (see [63]). These equations play an essential role in materials science as they describe important qualitative features of two-phase systems related with phase separation processes, assuming isotropy and a constant temperature. This can be observed, e.g., when a binary alloy (e.g., Aluminium/Zinc, see [390], or Iron/Chromium, see [309, 310, 311]) is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nuclides in the material) or a total nucleation, known as spinodal decomposition: the initially homogeneous material quickly becomes inhomogeneous, resulting in a very finely dispersed microstructure. In a second stage, which is called coarsening and occurs at a slower time scale, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength, hardness, fracture, toughness and ductility. We refer the reader to, e.g., [61, 63, 275, 280, 302, 303, 345, 347] for more details.

Here, u is the order parameter (we will consider a rescaled density of atoms or concentration of one of the material's components which takes values between -1

[1]John Werner Cahn (January 9, 1928-March 14, 2016) played a major role in materials science.

and 1, -1 and 1 corresponding to the pure states; the density of the second component is $-u$, meaning that the total density is a conserved quantity[2]) and μ is the chemical potential (more precisely, the difference of chemical potentials between the two components). Furthermore, f is the derivative of a double-well potential F. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(s) = \frac{\theta_c}{2}(1-s^2) + \frac{\theta}{2}[(1-s)\ln(\frac{1-s}{2}) + (1+s)\ln(\frac{1+s}{2})],\ s \in (-1,1), \tag{1.3}$$
$$0 < \theta < \theta_c,$$

i.e.,

$$f(s) = -\theta_c s + \frac{\theta}{2}\ln\frac{1+s}{1-s}, \tag{1.4}$$

although such a function is very often approximated by regular ones, typically, $F(s) = \frac{1}{4}(s^2-1)^2$, i.e., $f(s) = s^3 - s$; more generally, one can take $F(s) = \frac{1}{4}(s^2-\beta^2)^2$, $\beta \in \mathbb{R}$. The logarithmic terms in (1.3) correspond to the entropy of mixing and θ and θ_c are proportional to the absolute temperature (assumed constant during the process) and a critical temperature, respectively; the condition $\theta < \theta_c$ ensures that F has indeed a double-well form and that phase separation can occur. Also note that the polynomial approximation is reasonable when the quench is shallow, i.e., when the absolute temperature is close to the critical one. Finally, κ is the mobility and α is related to the surface tension at the interface.

From a phenomenological point of view, the Cahn–Hilliard system can be derived as follows.

One considers the following (total) free energy, called Ginzburg–Landau free energy:

$$\Psi_\Omega(u, \nabla u) = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))\,dx, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, is the domain occupied by the material. The gradient term in (1.5) was proposed in [63] in order to model the surface energy of the interface (i.e., capillarity; note that such gradients go back to J.D. van der Waals, see [395]); F is also called homogeneous free energy.

One then has the mass balance

[2]If u_A and u_B denote the densities of the two components, then, before rescaling, one has $u_A + u_B = 1$. Replacing u by $2u - 1$, one obtains, after rescaling, $u_A + u_B = 0$.

$$\frac{\partial u}{\partial t} = -\mathrm{div} h, \tag{1.6}$$

where h is the mass flux which is related to the chemical potential μ by the following (postulated) constitutive equation which resembles the Fick's law:

$$h = -\kappa \nabla \mu. \tag{1.7}$$

The usual definition of the chemical potential is that it is the derivative of the free energy with respect to the order parameter. Here, such a definition is incompatible with the presence of ∇u in the free energy. Instead, μ is defined as a variational derivative of the free energy with respect to u, which yields (assuming proper boundary conditions)

$$\mu = -\alpha \Delta u + f(u); \tag{1.8}$$

hence the Cahn–Hilliard system. This variational derivative can be (formally) seen by writing that, for a small variation,

$$\delta \Psi_\Omega = \int_\Omega (\alpha \nabla u \cdot \nabla \delta u + f(u) \delta u)\, dx,$$

where $\cdot$ denotes the usual Euclidean scalar product. Assuming compatible boundary conditions and integrating by parts, this yields

$$\delta \Psi_\Omega = \int_\Omega (-\alpha \Delta u + f(u)) \delta u \; dx,$$

from which the definition follows.

The Cahn–Hilliard system, in a bounded and regular domain Ω, usually is endowed with Neumann boundary conditions, namely,

$$\frac{\partial \mu}{\partial \nu} = 0, \text{ on } \Gamma, \tag{1.9}$$

meaning that there is no mass flux at the boundary (note that $h.\nu = -\kappa \frac{\partial \mu}{\partial \nu}$), and

$$\frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma, \tag{1.10}$$

which is a natural variational boundary condition (by natural, we mean that it allows to write down a convenient variational/weak formulation in view of the mathematical analysis of the problem; this boundary condition also yields that the interface is orthogonal to the boundary). Here, $\Gamma = \partial\Omega$ and ν is the unit outer normal to the

boundary. In particular, it follows from the first boundary condition that we have the conservation of mass, i.e., of the spatial average of the order parameter, obtained by (formally) integrating the first equation of (1.1) over Ω,

$$\langle u(t)\rangle \equiv \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega u(t,x)\,dx = \langle u(0)\rangle,\ \forall t \geq 0. \tag{1.11}$$

If we have in mind the fourth-order in space Cahn–Hilliard equation, we can rewrite these boundary conditions, equivalently, as

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0,\ \text{on } \Gamma. \tag{1.12}$$

We can also consider periodic boundary conditions (in which case $\Omega = \Pi_{i=1}^n (0, L_i)$, $L_i > 0$, $i = 1, \cdots, n$); in that case, we still have the conservation of mass. Note that we generally do not consider Dirichlet boundary conditions, due precisely to the fact that they do not yield the conservation of mass, although such boundary conditions certainly simplify the mathematical analysis.

Now, the question of how the phase separation process (i.e., the spinodal decomposition) is influenced by the presence of walls has gained much attention (see [164, 165, 265] and the references therein). This problem has mainly been studied for polymer mixtures (although it should also be important for other systems, such as binary metallic alloys): from a technological point of view, binary polymer mixtures are particularly interesting, since the occurring structures during the phase separation process may be frozen by a rapid quench into the glassy state; microstructures at surfaces on very small length scales can be produced in this way.

We also recall that the usual variational boundary condition $\frac{\partial u}{\partial \nu} = 0$ on the boundary yields that the interface is orthogonal to the boundary, meaning that the contact line, when the interface between the two components meets the walls, is static, which is not reasonable in many situations. This is the case, e.g., for mixtures of two immiscible fluids: in that case, the contact angle should be dynamic, due to the movements of the fluids. This can also be the case in the context of binary alloys, whence the need to define dynamic boundary conditions for the Cahn–Hilliard equation.

In that case, we again write that there is no mass flux at the boundary (i.e., that (1.9) still holds). Then, in order to obtain the second boundary condition, following the phenomenological derivation of the Cahn–Hilliard system, we consider, in addition to the usual Ginzburg–Landau free energy and assuming that the interactions with the walls are short-ranged, a surface free energy of the form

$$\Psi_\Gamma(u, \nabla_\Gamma u) = \int_\Gamma \left(\frac{\alpha_\Gamma}{2}|\nabla_\Gamma u|^2 + G(u)\right) d\sigma,\ \alpha_\Gamma > 0, \tag{1.13}$$

where ∇_Γ is the surface gradient and G is a surface potential. Thus, the total free energy of the system reads

$$\Psi = \Psi_\Omega + \Psi_\Gamma. \tag{1.14}$$

Writing finally that the system tends to minimize the excess surface energy, we are led to postulate the following boundary condition:

$$\frac{1}{d}\frac{\partial u}{\partial t} - \alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma, \tag{1.15}$$

i.e., there is a relaxation dynamics on the boundary. This boundary condition usually is referred to as dynamic boundary condition, in the sense that the kinetics, i.e., $\frac{\partial u}{\partial t}$, appears explicitly. Here, Δ_Γ is the Laplace–Beltrami operator, $g = G'$ and $d > 0$ is some relaxation parameter. Furthermore, in the original derivation, one has $G(s) = \frac{1}{2}a_\Gamma s^2 - b_\Gamma s$, where $a_\Gamma > 0$ accounts for a modification of the effective interaction between the components at the walls and b_Γ characterizes the possible preferential attraction (or repulsion) of one of the components by the walls (when b_Γ vanishes, there is no preferential attraction). We can note that it follows from the boundary conditions that, formally,

$$\frac{d\Psi}{dt} = -\frac{1}{\kappa}\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} - \frac{1}{d}\|\frac{\partial u}{\partial t}\|^2_{L^2(\Gamma)} \leq 0,$$

where $\|\cdot\|_X$ denotes the norm on the Banach space X; in the case of the classical Neumann boundary conditions, one has

$$\frac{d\Psi_\Omega}{dt} = -\frac{1}{\kappa}\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} \leq 0.$$

We also refer the reader to [30, 166] for other physical derivations of the dynamic boundary condition, obtained by taking the continuum limit of lattice models within a direct mean-field approximation and by applying a density functional theory, respectively, to [362] for the derivation of dynamic boundary conditions in the context of two-phase fluid flows and to [372, 377] for an approach based on concentrated capacity.

Actually, it would seem more reasonable, in the case of nonpermeable walls, to write the conservation of mass both in the bulk Ω and on the boundary Γ, i.e.,

$$\frac{d}{dt}\Big(\int_\Omega u\,dx + \int_\Gamma u\,d\sigma\Big) = 0.$$

Indeed, due to the interactions with the walls, one should expect some mass on the boundary. We assume that the first equation of (1.1) still holds. Then, writing that

$$\mu = \partial_u \Psi,$$

where ∂ is the variational derivative mentioned above (note that, in the original derivation, one has $\mu = \partial_u \Psi_\Omega$), we obtain the second equation of (1.1), together with the boundary condition

$$\mu = -\alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu}, \text{ on } \Gamma.$$

We now note that, owing to the first equation of (1.1), the above mass conservation reads

$$\int_\Gamma (\frac{\partial u}{\partial t} + \kappa \frac{\partial \mu}{\partial \nu})\, d\sigma = 0.$$

A class of boundary conditions which ensure this mass conservation reads

$$\frac{\partial u}{\partial t} + \beta_\Gamma \Delta_\Gamma u + \kappa \frac{\partial \mu}{\partial \nu} = 0, \text{ on } \Gamma, \ \beta_\Gamma \geq 0.$$

We can thus see that, when $\beta_\Gamma > 0$, we also have a Cahn–Hilliard type system on the boundary. Note that, when $\beta_\Gamma = 0$, it follows from the above that

$$\frac{d\Psi}{dt} = -\kappa \|\nabla \mu\|^2_{L^2(\Omega)^n} \leq 0.$$

Similar dynamic boundary conditions, in the case of semipermeable walls, are considered in [174, 175, 185]. Furthermore, in [294], based on an energetic variational approach and Onsager's principle of maximum energy dissipation, one recovers these dynamic boundary conditions, together with the no mass flux consition (1.9); in that case, one has mass conservation in the bulk and on the boundary, separately.

The Cahn–Hilliard system/equation is now quite well understood, at least from a mathematical point of view. In particular, one has a rather complete picture as far as the existence, the uniqueness and the regularity of solutions and the asymptotic behavior of the associated dynamical system are concerned. We refer the reader to (among a huge literature), e.g., [5, 35, 59, 83, 89, 93, 96, 109, 122, 131, 141, 143, 147, 149, 152, 174, 175, 185, 198, 206, 207, 212, 264, 278, 287, 294, 297, 300, 324, 333, 334, 337, 341, 342, 343, 344, 345, 347, 360, 363, 370, 388, 412, 424]. As far as the asymptotic behavior of the system is concerned, one has, in particular, the existence of finite-dimensional attractors. Such sets give information on the global/ all possible dynamics of the system. Furthermore, the finite dimensionality means, very roughly speaking, that, even though the initial phase space is infinite-dimensional, the limit dynamics can be described by a finite number of parameters. We refer the interested

reader to, e.g., [19, 97, 134, 335, 388] for more details and discussions on this. One also has the convergence of single trajectories to steady states.

We can note that we assumed so far that the mobility κ is a strictly positive constant. Actually, κ is often expected to depend on the order parameter and to degenerate at the singular points of f in the case of a logarithmic nonlinear term (see [62, 147, 148, 191, 417]; see also [421] for a discussion in the context of immiscible binary fluids). Note however that this essentially restricts the diffusion process to the interfacial region and is observed, typically, in physical situations in which the movements of atoms are confined to this region (see [364]). In that case, the first equation of (1.1) reads

$$\frac{\partial u}{\partial t} = \operatorname{div}(\kappa(u)\nabla\mu),$$

where, typically, $\kappa(s) = 1 - s^2$. In particular, the existence of solutions to the Cahn–Hilliard equation with degenerate mobilities and logarithmic nonlinearities is proved in [147]; note that, up to now, only existence of weak solutions is known, nothing else. The asymptotic behavior, and, more precisely, the existence of attractors, of the Cahn–Hilliard equation with nonconstant and nondegenerating mobilities is studied in [378, 379].

We can also note that the gradient term in the Ginzburg–Landau free energy (1.5) accounts for the fact that the interactions between the material's components are assumed to be short-ranged. Actually, this term is obtained by approximation of a nonlocal term which also accounts for long-ranged interactions (see [63]). The Cahn–Hilliard equation, with a nonlocal term, was derived rigorously by G. Giacomin and J.L. Lebowitz in [204, 205], based on stochastic arguments, by considering a lattice gas with long range Kac potentials (i.e., the interaction energy between two particles at x and y (x, $y \in \mathcal{Z}^n$) is given by $\gamma^n \mathcal{K}(\gamma|x-y|)$, $\gamma > 0$ being sent to 0 and $\mathcal{K}$ being a smooth function). In that case, the (total) free energy reads

$$\Psi_\Omega(u) = \int_{\mathbb{T}^n} [f(u(x)) + u(x) \int_{\mathbb{T}^n} \mathcal{K}(|x-y|)(1-u(y))\,dy]\,dx, \tag{1.16}$$

where $\mathbb{T}^n$ is the n-dimensional torus. Furthermore, rewriting the total free energy in the form

$$\Psi_\Omega(u) = \int_{\mathbb{T}^n} [f(u(x)) + k_1(x)u(x)(1-u(x)) + \frac{1}{2}\int_{\mathbb{T}^n} \mathcal{K}(|x-y|)|u(x)-u(y)|^2\,dy]\,dx,$$

where $k_1(x) = \int_{\mathbb{T}^n} \mathcal{K}(|x-y|)\,dy$, one can, by expanding the last term and keeping only some terms in the expansion, recover the Ginzburg–Landau free energy (this is reasonable when the scale on which the free energy varies is large compared with

γ^{-1}; the macroscopic evolution is observed here on the spatial scale γ^{-1} and time scale γ^{-2}); see also [307]. Such models were studied, e.g., in [3, 27, 173, 176, 177, 178, 181, 254] (see also [78, 133, 230, 231, 232, 415] for the numerical analysis and simulations).

Now, it is interesting to note that the Cahn–Hilliard equation and some of its variants are also relevant in other phenomena than phase separation in binary alloys. We can mention, for instance, dealloying (this can be observed in corrosion processes, see [154]), population dynamics (see [100]), tumor growth (see [18, 266]), bacterial films (see [273]), thin films (see [349, 391]), chemistry (see [396]), image processing (see [28, 29, 72, 79, 132]) and even in astronomy, with the rings of Saturn (see [394]), and ecology (for instance, the clustering of mussels can be perfectly well described by the Cahn–Hilliard equation, see [296] (see also Youtube,

https://www.youtube.com/watch?v=u-mEjfBaYks

and

https://www.youtube.com/watch?v=OYcXZ7Ho4o8,

for real mussels clustering and simulations, respectively); of course, in that case the time scale is much larger, typically, weeks or months).

In particular, several such phenomena can be modeled by the following generalized Cahn–Hilliard equation:

$$\frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) + g(x,u) = 0,\ \alpha,\ \kappa > 0 \tag{1.17}$$

(here, α and κ do not necessarily have the same physical meaning as in the original Cahn–Hilliard equation). The above general equation contains, in particular, the following models:

(i) Mixed Allen–Cahn/Cahn–Hilliard system. In that case, we consider the following system of equations:

$$\frac{\partial u}{\partial t} = \varepsilon^2 D\Delta\mu - \mu,\ D,\ \varepsilon > 0,\ \mu = -\Delta u + \frac{f(u)}{\varepsilon^2},$$

which can be rewritten, equivalently, as

$$\frac{\partial u}{\partial t} + \varepsilon^2 D\Delta^2 u - \Delta(Df(u) + u) + \frac{f(u)}{\varepsilon^2} = 0$$

and is indeed of the form above. In particular, without the term $\varepsilon^2 D\Delta\mu$ in the first equation, we have the Allen–Cahn equation and, without the term $-\mu$, we have the Cahn–Hilliard equation. These equations were proposed in order to account for microscopic mechanisms such as surface diffusion and adsorption/desorption (see [258, 260, 261, 308]) and were studied in [250, 251, 252, 253, 259].

(ii) Cahn–Hilliard–Oono[3] equation (see [317, 348, 397]). In that case,

$$g(x,s) = g(s) = \beta s, \ \beta > 0.$$

This function was proposed in [348] in order to account for long-ranged (i.e., nonlocal) interactions in phase separation, but also to simplify numerical simulations, due to the fact that we do not have to account for the conservation of mass, although it seems that this equation has never been considered in simulations.

Actually, it can be surprising that nonlocal interactions can be described by such a simple linear term. This can be seen by noting that one considers here the free energy

$$\Psi_\Omega = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u) + \int_\Omega u(y)g(y,x)u(x)\ dy)\ dx, \tag{1.18}$$

where the function g describes the long-ranged interactions. In particular, in Oono's model and in three space dimensions, one takes

$$g(y,x) = \frac{4\pi\beta}{|y-x|}. \tag{1.19}$$

Note that the long-ranged interactions are repulsive when $u(y)$ and $u(x)$ have opposite signs and thus favor the formation of interfaces (see [397] and the references therein). Writing finally, as in the derivation of the classical Cahn-Hilliard equation,

$$\frac{\partial u}{\partial t} = \kappa\Delta\partial_u\Psi_\Omega, \tag{1.20}$$

we find the Cahn–Hilliard–Oono equation, noting that $-\frac{1}{|y-x|}$ is the Green function associated with the Laplace operator (see [397] and the references therein for more details).

A variant of this model, proposed in [95] to model microphase separation of diblock copolymers, consists in taking

$$g(x,s) = g(s) = \beta(s - \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega u_0(x)\,dx), \ \beta > 0,$$

where u_0 is the initial condition. In that case, we have the conservation of mass and efficient simulations, based on multigrid solvers, were performed in [17]. This variant of the Cahn–Hilliard–Oono equation can also be coupled with the incompressible

[3] A better name would be Cahn–Hilliard–Oono–Puri equation; we will however keep the customary one.

Navier–Stokes equations to model a chemically reacting binary fluid (see [246, 247]; see also [48] for the mathematical analysis).

(iii) Proliferation term. In that case,

$$g(x,s) = g(s) = \lambda s(s-1),\ \lambda > 0.$$

This function was proposed in [266] in view of biological applications and, more precisely, to model wound healing and tumor growth (in one space dimension; in that case, one can think of a propagation front) and the clustering of malignant brain tumor cells (in two space dimensions); see also [396] for other quadratic functions with chemical applications and [18] for other polynomials with biological applications.

(iv) Fidelity term. In that case,

$$g(x,s) = \lambda_0 \chi_{\Omega\setminus D}(x)(s-h(x)),\ \lambda_0 > 0,\ D \subset \Omega,\ h \in L^2(\Omega),$$

where χ denotes the indicator function, and we consider the following equation:

$$\frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon}\Delta f(u) + g(x,u) = 0,\ \varepsilon > 0.$$

Written in this way, ε corresponds to the interface thickness. This function g was proposed in [28, 29] in view of applications to binary image inpainting (i.e., black and white images). Here, h is a given (damaged) image and D is the inpainting (i.e., damaged) region. Furthermore, the fidelity term $g(x,u)$ is added in order to keep the solution u close to the image outside the inpainting region. The idea in this model is to solve the equation up to steady state to obtain an inpainted (i.e., restored) version $u(x)$ of $h(x)$.

The generalized equation (1.17) was studied in [318, 322] (see also [157]) under very general assumptions on the additional term g, when endowed with Dirichlet boundary conditions. In that case, one essentially recovers the results (well-posedness, regularity and existence of finite-dimensional attractors) known for the original Cahn–Hilliard equation. The case of Neumann boundary conditions is much more involved, due to the fact that one no longer has the conservation of mass, i.e., of the spatial average of the order parameter, when compared with the original Cahn–Hilliard equation with Neumann boundary conditions (see [79, 80, 90, 157, 158, 197]).

Another variant of the Cahn–Hilliard equation is concerned with higher-order Cahn–Hilliard models. More precisely, G. Caginalp and E. Esenturk recently proposed in [60] (see also [76]) higher-order phase-field models in order to account for anisotropic interfaces (see also [274, 387, 407] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified free energy, in which we omit the temperature:

$$\Psi_{\mathrm{HOGL}} = \int_\Omega \Big(\frac{1}{2}\sum_{i=1}^{M}\sum_{|k|=i} a_k |\mathcal{D}^k u|^2 + F(u)\Big)\,dx,\ \ M \in \mathbb{N}, \tag{1.21}$$

where, for $k = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$,

$$|k| = k_1 + k_2 + k_3$$

and, for $k \neq (0,0,0)$,

$$\mathcal{D}^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)} v = v$). The corresponding higher-order Cahn–Hilliard equation then reads

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{M} (-1)^i \sum_{|k|=i} a_k \mathcal{D}^{2k} u - \Delta f(u) = 0. \tag{1.22}$$

For $M = 1$ (anisotropic Cahn–Hilliard equation), we have an equation of the form

$$\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = 0$$

and, for $M = 2$ (sixth-order anisotropic Cahn–Hilliard equation), we have an equation of the form

$$\frac{\partial u}{\partial t} - \Delta \sum_{i,j=1}^{3} a_{ij} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} + \Delta \sum_{i=1}^{3} b_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = 0.$$

We studied in [86, 312] the corresponding higher-order isotropic model, namely,

$$\frac{\partial u}{\partial t} - \Delta P(-\Delta) u - \Delta f(u) = 0, \tag{1.23}$$

where

$$P(s) = \sum_{i=1}^{M} a_i s^i,\ \ a_k > 0,\ \ M \geq 1,\ \ s \in \mathbb{R},$$

and, in [87], the anisotropic higher-order model (1.22) (there, numerical simulations were also performed to illustrate the effects of the higher-order terms and of the

anisotropy). Furthermore, these models contain sixth-order Cahn–Hilliard models. We can note that there is currently a strong interest in the study of sixth-order Cahn–Hilliard equations. Such equations arise in situations such as strong anisotropy effects being taken into account in phase separation processes (see [393]), atomistic models of crystal growth (see [28, 29, 153, 186]), the description of growing crystalline surfaces with small slopes which undergo faceting (see [377]), oil-water-surfactant mixtures (see [215, 216]) and mixtures of polymer molecules (see [121]). We refer the reader to [74, 223, 227, 228, 229, 243, 276, 277, 304, 305, 317, 319, 320, 321, 323, 350, 351, 356, 357, 398, 399, 411] for the mathematical and numerical analysis of such models.

We can also note that the variant (1.17) can be relevant in the context of higher-order models (we can mention, for instance, anisotropic effects in tumor growth). We refer the reader to [88] for the analysis and numerical simulations of such models.

We finally mention several other important generalizations and variants of the Cahn–Hilliard equation.

A first one consists in studying systems of Cahn–Hilliard equations to describe phase separation in multicomponent alloys (see [56, 96, 112, 148, 149, 155, 191, 192, 193, 330]). Note that the Cahn–Hilliard equation can be rewritten, equivalently, as a system of two (Cahn–Hilliard) equations. Let us indeed denote by A and B the two components and consider, with obvious notation, the free energy

$$\Psi_\Omega(u_A, \nabla u_A, u_B, \nabla u_B) = \frac{1}{2}\int_\Omega (\frac{\alpha}{2}|\nabla u_A|^2 + \frac{\alpha}{2}|\nabla u_B|^2 + F(u_A) + F(u_B))\, dx.$$

Then, the Cahn–Hilliard system (1.1) is equivalent, again with obvious notation and noting that f is an odd function in both cases of interest, to

$$\begin{gathered}\frac{\partial u_A}{\partial t} = \kappa\Delta\mu_A,\ \ \mu_A = \frac{1}{2}(-\alpha\Delta u_A + f(u_A))\,(= \partial_{u_A}\Psi_\Omega),\\ \frac{\partial u_B}{\partial t} = \kappa\Delta\mu_B,\ \ \mu_B = \frac{1}{2}(-\alpha\Delta u_B + f(u_B))\,(= \partial_{u_B}\Psi_\Omega),\\ u_A + u_B = 0,\ \ \mu_A + \mu_B = 0.\end{gathered}$$

Furthermore, we can see that

$$\mu_A - \mu_B = -\alpha\Delta u_A + f(u_A).$$

We also mention the stochastic Cahn–Hilliard equation (also called the Cahn–Hilliard–Cook equation) which takes into account thermal fluctuations (see [33, 34, 37, 38, 68, 71, 116, 118, 123, 124, 142, 172, 217, 218, 233, 376]).

Then, an important generalization of the Cahn–Hilliard equation is the viscous Cahn–Hilliard equation,

$$-\epsilon\Delta\frac{\partial u}{\partial t}+\frac{\partial u}{\partial t}+\alpha\kappa\Delta^2 u-\kappa\Delta f(u)=0,\ \epsilon>0,$$

proposed by A. Novick–Cohen in [344] to account for viscosity effects in the phase separation of polymer/polymer systems (see also [20, 70, 98, 151]). The viscous Cahn–Hilliard equation can also be seen as a particular case of the generalizations proposed by M. Gurtin in [235] (which, in particular, also account for anisotropy) and which are based on a microforce balance, i.e., a new balance law for interactions at a microscopic level (see [41, 42, 44, 85, 139, 140, 211, 221, 314, 315, 316, 325, 326, 331, 336, 367, 368, 369, 408] for the mathematical analysis); we also refer the reader to yet another approach proposed by P. Podio–Guidugli in [359] and studied in, e.g., [102, 103, 104, 105, 110].

Another important generalization of the Cahn–Hilliard equation is the hyperbolic relaxation of the equation,

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\alpha\kappa\Delta^2 u-\kappa\Delta f(u)=0,\ \epsilon>0,$$

proposed in [187, 188, 189, 190, 281] to model the early stages of spinodal decomposition in certain glasses (see also [43, 200, 201, 224, 225, 226, 375] for the mathematical analysis and [373, 374] for the hyperbolic relaxation of the Cahn–Hilliard–Oono equation in the whole space). Actually, the hyperbolic relaxation of the equation is a particular case of more general memory relaxations (for an exponentially decreasing memory kernel) which were studied, e.g., in [111, 113, 115, 202, 203] (see also [361]).

We also mention the convective Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t}+\alpha\kappa\Delta^2 u+u\cdot\nabla u-\kappa\Delta f(u)=0,$$

which describes the dynamics of driven systems such as faceting of growing thermodynamically unstable crystal surfaces (see [135, 136, 137, 213, 291, 405] for the mathematical analysis).

It is important to note that, in realistic physical systems, quenches are usually carried out over a finite period of time, so that phase separation can begin before the final quenching is reached. It is thus important to consider nonisothermal Cahn–Hilliard models. Such models were derived and studied in [12, 13, 183, 184, 329, 382].

The Cahn–Hilliard equation can be coupled with the Allen–Cahn equation which describes the ordering of atoms during the phase separation process (see [11]). This problem was studied, e.g., in [32, 120, 293, 327, 328, 346, 422].

It can also be coupled with the equations for elasticity or viscoelasticity, to account for mechanical effects (see, e.g., [16, 31, 39, 40, 69, 128, 192, 193, 194, 314, 315, 352, 353, 354, 355, 366]).

We also mention the coupling of the Cahn–Hilliard equation with the Navier–Stokes equations in the context of two-phase (multiphase) flows (see, e.g., [2, 4, 48, 49, 50, 51, 52, 54, 67, 64, 84, 91, 99, 167, 168, 179, 180, 182, 210, 236, 242, 255, 267, 270, 272, 279, 292, 299, 332, 419, 423]) and some related models such as the Cahn–Hilliard–Hele–Shaw and Cahn–Hilliard–Brinkman equations (see, e.g., [47, 114, 125, 126, 127, 162, 209, 240, 402, 404, 409, 418]). Related models can also be used to model tumor growth (see, e.g., [101, 106, 107, 108, 119, 169, 195, 196, 257, 298]).

We finally refer the reader to, e.g., [1, 9, 10, 14, 15, 20, 21, 22, 23, 24, 25, 26, 36, 45, 51, 52, 53, 54, 55, 57, 58, 65, 66, 73, 75, 77, 92, 94, 117, 129, 130, 144, 145, 146, 150, 156, 159, 160, 161, 163, 169, 170, 171, 199, 214, 219, 220, 222, 234, 237, 238, 239, 241, 248, 249, 256, 262, 263, 265, 267, 268, 269, 270, 271, 272, 282, 283, 284, 285, 286, 288, 289, 295, 301, 306, 313, 338, 339, 364, 371, 380, 381, 385, 386, 392, 400, 401, 403, 406, 410, 413, 414, 416, 425, 426] for the numerical analysis and simulations of the Cahn–Hilliard equation (and several of its generalizations). Note that, as suggested in [146], it is in general preferable to build numerical schemes for the Cahn–Hilliard system (1.1) rather than the equivalent fourth-order in space Cahn–Hilliard equation. This has indeed the advantage of splitting the fourth-order equation into a system of two second-order ones which is easier to deal with.

Notation. We denote by $\|\cdot\|$ the usual norms on $L^2(\Omega)$ and $L^2(\Omega)^n$ (with associated scalar product $((\cdot,\cdot))$). More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X.

In what follows, Ω is a bounded and regular (as regular as needed) domain of $\mathbb{R}^n$, $n = 1$, 2 or 3.

2. Linear operators [388]

We consider the spaces $L^2(\Omega)$ and $H^1(\Omega)$ which, endowed with their usual scalar products and associated norms, are Hilbert spaces.

Of course, $(v, v) \mapsto ((\nabla u, \nabla v))$ is not a scalar product in $H^1(\Omega)$, as it is not coercive.

To overcome this, we set

$$\langle u \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u \, dx, \ u \in L^1(\Omega),$$
$$\langle u \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle u, 1 \rangle, \ u \in H^{-1}(\Omega),$$

where $H^{-1}(\Omega)$ is the topological dual of $H^1(\Omega)$, $H^{-1}(\Omega) = H^1(\Omega)'$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$.

We then set

$$H = \dot{L}^2(\Omega) = \{u \in L^2(\Omega),\ \langle u \rangle = 0\},$$
$$V = \dot{H}^1(\Omega) = H^1(\Omega) \cap H.$$

These spaces are also Hilbert spaces, when endowed with the induced scalar products. Furthermore, $((\cdot,\cdot))_V = ((\nabla\cdot, \nabla\cdot))$ is a scalar product on V, with associated norm $\|\cdot\|_V$, which is equivalent to the usual H^1-scalar product (owing to the Poincaré-Wirtinger inequality; see below).

Let $V' = \{u \in H^{-1}(\Omega),\ \langle u \rangle = 0\}$ be the topological dual of V. Then, we know from Riesz' representation theorem that, $\forall l \in V'$, there exists a unique $u \in V$ such that $((u,v))_V = \langle l, v\rangle$, $\forall v \in V$, where $\langle\cdot,\cdot\rangle$ also denotes the duality pairing between V' and V.

Identifying H with its topological dual H', we have the Hilbert triplet $V \subset H \equiv H' \subset V'$, with dense, continuous and compact injections. Furthermore, if u and v are in H and V, respectively, then $\langle u, v\rangle = ((u,v))_H$.

We can then define the linear operator $A : V \to V'$ by

$$\langle Au, v\rangle = ((u,v))_V,\ \forall u,\ v \in V.$$

This operator is an isomorphism from V onto V'.

We now set

$$D(A) = A^{-1}(H) = \{u \in V,\ Au \in H\} = \{u \in \dot{H}^1(\Omega),\ -\Delta u \in \dot{L}^2(\Omega)\},$$

called domain of A. Note indeed that, if $((u,v))_V = ((f,v))$, $\forall v \in V$ and for $f \in H$, then

$$((\nabla u, \nabla v)) = ((f,v)),\ \forall v \in H^1(\Omega)$$

(it suffices to replace v by $v - \langle v\rangle$). Taking then $v \in \mathcal{D}(\Omega) \equiv \mathcal{C}_c^\infty(\Omega)$, it is easy to see that $-\Delta u = f$ in the sense of distributions and thus in $L^2(\Omega)$.

Next, since $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, the trace $\frac{\partial u}{\partial \nu}$ can be defined in $H^{-\frac{1}{2}}(\Gamma)$ and a generalized form of Green's formula is valid for every $v \in H^1(\Omega)$ (see [389]; see also [388], Chapter II, Example 2.5), yielding

$$-((\Delta u, v)) = -\langle \frac{\partial u}{\partial \nu}, v\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} + ((\nabla u, \nabla v)),\ \forall\ v \in H^1(\Omega).$$

We thus deduce that

$$\langle \frac{\partial u}{\partial \nu}, v\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0,\ \forall\ v \in H^1(\Omega),$$

so that

$$\frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma$$

(in $H^{-\frac{1}{2}}(\Gamma)$). Having this, it follows from classical elliptic regularity results (see [6, 7, 8]) that $u \in H^2(\Omega)$. Finally,

$$D(A) = \{u \in H^2(\Omega) \cap V, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\}$$

and $Au = f$, $u \in D(A)$ and $f \in H$, is equivalent to

$$-\Delta u = f, \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma.$$

Remark 2.1. More generally, if $f \in H^m(\Omega)$, $m \geq 0$, then $u \in H^{m+2}(\Omega)$.

2.1. **Spectral properties of the operator A.** First note that A is selfadjoint (since $((\cdot,\cdot))_V$ is symmetric). Furthermore, since $V \subset H$ is compact, then $A^{-1} : H \to H$ is compact (and selfadjoint). Indeed, $A^{-1} : H \to D(A)$ is continuous, so that $A^{-1} : H \to V$ is also continuous (note that it follows from the regularity mentioned above that the injection $D(A) \subset V$ is continuous) and we conclude owing to the compact injection $V \subset H$.

We thus conclude that A^{-1} is compact, selfadjoint and positive (as an operator in H). Therefore, there exists an orthonormal basis (w_j), $j \in \mathbb{N}$, of H formed of eigenvectors of A:

$$A^{-1}w_j = \mu_j w_j, \ \mu_j \to 0 \text{ as } j \to +\infty, \ \mu_j > 0.$$

Since $w_j = \frac{1}{\mu_j} A^{-1} w_j \in D(A)$, then

$$Aw_j = \lambda_j w_j, \ \lambda_j = \frac{1}{\mu_j},$$

and w_j, λ_j are eigenvectors/eigenvalues of A, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to +\infty$ as $j \to +\infty$ (note that $\langle Aw_j, w_j \rangle = \lambda_j \|w_j\|^2 > 0$). Furthermore, the w_j's are orthogonal in V for $((\cdot,\cdot))_V$. Indeed, if $j \neq k$, then

$$((w_j, w_k))_V = \langle Aw_j, w_k \rangle = \lambda_j((w_j, w_k)) = 0.$$

However, this family is not orthonormal, since

$$\langle Aw_j, w_j \rangle = ((w_j, w_j))_V = \lambda_j \|w_j\|^2 = \lambda_j.$$

We can now compute the norms of u in terms or the spectral elements of A. Indeed:

- If $u \in H$, $u = \sum_{j=1}^{\infty} u_j w_j$, then $\|u\|^2 = \sum_{j=1}^{\infty} |u_j|^2$.

• If $u \in V$, then $\|u\|_V^2 = ((\sum_{j=1}^\infty u_j w_j, \sum_{j=1}^\infty u_j w_j))_V = \sum_{j=1}^\infty |u_j|^2((w_j, w_j))_V = \sum_{j=1}^\infty \lambda_j |u_j|^2$. We also have

$$V = \{u \in H,\ u = \sum_{j=1}^\infty u_j w_j,\ \sum_{j=1}^\infty \lambda_j |u_j|^2 < +\infty\}.$$

• If $u \in D(A)$, then $Au = A(\sum_{j=1}^\infty u_j w_j) = \sum_{j=1}^\infty u_j A w_j = \sum_{j=1}^\infty \lambda_j u_j w_j$ (the sequence converges) and $\|u\|_{D(A)}^2 = \|Au\|^2 = \sum_{j=1}^\infty \lambda_j^2 |u_j|^2$ (graph norm). We also have

$$D(A) = \{u \in H,\ u = \sum_{j=1}^\infty u_j w_j,\ \sum_{j=1}^\infty \lambda_j^2 |u_j|^2 < +\infty\}.$$

Furthermore, on $D(A)$, the graph norm $\|A \cdot\|$ is equivalent to the usual H^2-norm, owing to the elliptic regularity results mentionned above.

2.2. **The spaces $D(A^\alpha)$ and the operators A^α.** We first consider the case $\alpha \geq 0$.

By convention, $D(A^0) = H$ and $A^0 = I$ (identity operator). Then, for $\alpha > 0$, we set (see also above)

$$D(A^\alpha) = \{u \in H,\ u = \sum_{j=1}^\infty u_j w_j,\ \sum_{j=1}^\infty \lambda_j^{2\alpha} |u_j|^2 < +\infty\}$$

which we endow with the graph norm

$$\|u\|_{D(A^\alpha)} = (\sum_{j=1}^\infty \lambda_j^{2\alpha} |u_j|^2)^{\frac{1}{2}}$$

and the associated scalar product

$$((u, v))_{D(A^\alpha)} = \sum_{j=1}^\infty \lambda_j^{2\alpha} u_j v_j,\ u = \sum_{j=1}^\infty u_j w_j,\ v = \sum_{j=1}^\infty v_j w_j.$$

Endowed with this scalar product, $D(A^\alpha)$ is a Hilbert space.

The operator $A^\alpha : D(A^\alpha) \to H$ is then defined by

$$A^\alpha u = \sum_{j=1}^\infty \lambda_j^\alpha u_j w_j,\ u = \sum_{j=1}^\infty u_j w_j.$$

Example 2.2. (i) We recover the above convention and definition for $\alpha = 0$ and $\alpha = 1$.

(ii) For $\alpha = \frac{1}{2}$, $D(A^{\frac{1}{2}}) = V$.

(iii) For $\alpha = 2$,

$$D(A^2) = \{u \in H,\ u = \sum_{j=1}^{\infty} u_j w_j,\ \sum_{j=1}^{\infty} \lambda_j^4 |u_j|^2 < +\infty\} = \{u \in D(A),\ Au \in D(A)\}.$$

It thus follows for the elliptic regularity results of [6, 7, 8] that

$$D(A^2) = \{u \in H^4(\Omega) \cap V,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

Note that we recover the two (Neumann) boundary conditions associated with the Cahn–Hilliard equation. Indeed, $Au = f$, $u \in D(A^2)$ and $f \in H$ is equivalent to

$$\Delta^2 u = f,\ \text{in } \Omega,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0,\ \text{on } \Gamma.$$

Furthermore, the graph norm $\|A^2 \cdot\|$ is equivalent to the usual H^4-norm on $D(A^2)$.

Remark 2.3. If $\alpha > \alpha' \geq 0$, then $D(A^\alpha) \subset D(A^{\alpha'})$ with continuous, dense and compact injection.

We now turn to the case $\alpha < 0$.

We consider, on H, the continuous norm $(\sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2)^{\frac{1}{2}}$, $u = \sum_{j=1}^{\infty} u_j w_j$. Then, by definition, $D(A^\alpha)$ is the closure of H for this norm.

We thus have $H \subset D(A^\alpha)$ with continuous injection (this injection is indeed continuous, since $\sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2 \leq \lambda_1^{2\alpha} \sum_{j=1}^{\infty} |u_j|^2$).

Remark 2.4. a) If $\alpha > \alpha'$, $\alpha, \alpha' \in \mathbb{R}$, then $D(A^\alpha) \subset D(A^{\alpha'})$ with continuous, dense and compact injection. Furthermore, if $\alpha > 0$, then $D(A^\alpha)' = D(A^{-\alpha})$ (topological dual).

b) We could have also defined $D(A^\alpha)$, $\alpha < 0$, as the topological dual of $D(A^{-\alpha})$.

The operator A^α, $\alpha < 0$, is then defined as above, $A^\alpha : D(A^\alpha) \to H$.

Example 2.5. (i) For $\alpha = -1$, $D(A^{-1}) = D(A)'$, as mentioned above.

(ii) For $\alpha = -\frac{1}{2}$, $D(A^{-\frac{1}{2}}) = D(A^{\frac{1}{2}})' = V'$ and the graph norm, $\|A^{-\frac{1}{2}} \cdot\|$, is equivalent to the usual H^{-1}-norm on $D(A^{-\frac{1}{2}})$.

Remark 2.6. The norm

$$u \mapsto (\|A^{-\frac{1}{2}}(u - \langle u \rangle)\|^2 + \langle u \rangle^2)^{\frac{1}{2}}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual H^{-1}-norm.

We can note that $D(A) \subset D(A^{\frac{1}{2}}) \equiv V \subset D(A^0) \equiv H \subset D(A^{-\frac{1}{2}}) \equiv V'$, with continuous, dense and compact injections. Furthermore,

$$\|u\|_V^2 = \sum_{j=1}^{+\infty} \lambda j |u_j|^2 \geq \lambda_1 \sum_{j=1}^{+\infty} |u_j|^2 = \lambda_1 \|u\|^2,$$

which yields the Poincaré type inequality

$$\|u\| \leq \frac{1}{\sqrt{\lambda_1}} \|u\|_V, \ \forall u \in V.$$

Note that we can obtain similar estimates on the norms of all injections.

Remark 2.7. We deduce from the above the Poincaré–Wirtinger inequality

$$\|u - \langle u \rangle\| \leq \frac{1}{\sqrt{\lambda_1}} \|u\|_V, \ \forall u \in H^1(\Omega). \tag{2.1}$$

2.3. **The linear equation $\frac{du}{dt} + A^2 u = f(t)$.** The linear equation associated with the Cahn–Hilliard equation reads

$$\frac{\partial u}{\partial t} + \Delta^2 u = f(x,t),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma.$$

Having in mind the operator A defined in the previous subsection, we can rewrite this boundary value problem in the following functional form:

$$\frac{du}{dt} + A^2 u = f(t).$$

In particular, if $f \in L^2(0,T;H)$, $u \in L^2(0,T;D(A^2))$ and $\frac{du}{dt} \in L^2(0,T;H)$, $T > 0$ given, this equation makes sense in $L^2(0,T;H)$. Furthermore, it is associated with the variational formulation

Find $u : [0,T] \to D(A)$ such that

$$\frac{d}{dt}((u,v)) + ((Au, Av)) = ((f(t), v)), \ \forall v \in D(A),$$

in the sense of distributions.

We have the following result (existence and uniqueness of weak solutions).

Theorem 2.8. *We assume that $f \in L^2(0,T;D(A^{-1}))$ and $u_0 \in H$, $T > 0$ given. Then, the linear initial value problem*

$$\frac{du}{dt} + A^2 u = f(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1})), \tag{2.2}$$

$$u(0) = u_0 \text{ in } H, \tag{2.3}$$

possesses a unique solution u *such that* $u \in \mathcal{C}([0,T];H) \cap L^2(0,T;D(A))$ *and* $\frac{du}{dt} \in L^2(0,T;D(A^{-1}))$.

Remark 2.9. Note that, since $D(A) \subset H \subset D(A^{-1}) \equiv D(A)'$ with continuous injections, then it follows from Lions' theorem (see [290]; see also the next subsection below) that, if $u \in L^2(0,T;D(A))$ and $\frac{du}{dt} \in L^2(0,T;D(A^{-1}))$, then $u \in \mathcal{C}([0,T];H)$. Therefore, $u(0)$ makes sense in H.

Proof. **Uniqueness:**

Let u_1 and u_2 be solutions to

$$\frac{du_1}{dt} + A^2 u_1 = f_1(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1})),$$
$$u_1(0) = u_{0,1} \text{ in } H,$$

and

$$\frac{du_2}{dt} + A^2 u_2 = f_2(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1})),$$
$$u_2(0) = u_{0,2} \text{ in } H.$$

Then, $u = u_1 - u_2$, $f = f_1 - f_2$ and $u_0 = u_{1,0} - u_{2,0}$ satisfy

$$\frac{du}{dt} + A^2 u = f(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1})), \tag{2.4}$$

$$u(0) = u_0 \text{ in } H. \tag{2.5}$$

The above equation is associated with the variational formulation

$$\frac{d}{dt}((u,v)) + ((Au,Av)) = \langle f(t), v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in D(A), \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product in $D(A^{-1})$.

Taking $v = u$ in (2.6), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|Au\|^2 = \langle f(t), u\rangle,$$

which yields, noting that

$$|\langle f(t),u\rangle| \le \|f(t)\|_{D(A^{-1})}\|Au\| \le \frac{1}{2}\|f(t)\|^2_{D(A^{-1})} + \frac{1}{2}\|Au\|^2$$

the differential inequality

$$\frac{d}{dt}\|u\|^2 + \|Au\|^2 \le \|f(t)\|^2_{D(A^{-1})},$$

so that, integrating over $[0,t]$, $0 \le t \le T$,

$$\|u(t)\|^2 \le \|u_0\|^2 + \|f\|^2_{L^2(0,T;D(A^{-1}))}.$$

This gives the continuous dependence with respect to the data (i.e., $(u_0,f) \mapsto u(t)$ is continuous from $H \times L^2(0,T;D(A^{-1}))$ onto H), as well as the uniqueness (taking $f_1 = f_2$ and $u_{1,0} = u_{2,0}$).

Existence:

The proof of existence is based on a standard Galerkin scheme.

We consider, for $m \in \mathbb{N}$ given, the following approximated problem:

Find $u_m = \sum_{i=1}^m u_i w_i$, $u_i = u_i(t)$, $i = 1, \cdots, m$, such that

$$\frac{d}{dt}((u_m,v)) + ((Au_m,Av)) = \langle f(t),v\rangle, \ \forall v \in W_m \equiv \mathrm{Span}(w_1,\cdots,w_m), \tag{2.7}$$

$$u_m(0) = u_{0,m}, \tag{2.8}$$

where $u_{0,m}$ is the projection (in H) of u_0 onto W_m,

$$u_{0,m} = \sum_{i=1}^m ((u_0,w_i))w_i.$$

Here, the w_j's are eigenfunctions of A as constructed above. Note that the u_i's do not depend on m, since the problem is linear. Furthermore, (2.7) is equivalent to (taking $v = w_i$, $i = 1, \cdots, m$, in (2.7))

$$\frac{du_i}{dt} + \lambda_i^2 u_i = \langle f(t),w_i\rangle, \ i = 1, \cdots, m,$$

so that

$$u_i(t) = e^{-\lambda_i^2 t}u_{i,0} + e^{-\lambda_i^2 t}\int_0^t e^{\lambda_i^2 s}\langle f(s),w_i\rangle\, ds, \ i = 1, \cdots, m,$$

where $u_{i,0} = ((u_0,w_i))$.

We thus have the existence of the maximal solution u_m to (2.7); actually, this solution is global and defined on $[0,T]$.

We now take $v = w_i$ in (2.7) and multiply the resulting equation by u_i to find

$$u_i \frac{du_i}{dt} + \lambda_i^2 |u_i| = \langle f(t), u_i w_i \rangle.$$

Summing over $i = 1, \cdots, m$ and noting that $u_i \frac{du_i}{dt} = \frac{1}{2}\frac{d}{dt}|u_i|^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \|Au_m\|^2 = \langle f(t), u_m \rangle,$$

which yields, proceeding as above,

$$\frac{d}{dt}\|u_m\|^2 + \|Au_m\|^2 \leq \|f(t)\|^2_{D(A^{-1})}.$$

Therefore, u_m is bounded, independently of m, in $L^2(0,T;D(A))$ and $L^\infty(0,T;H)$. Hence, up to a subsequence which we do not relabel, there exists $u \in L^\infty(0,T;H) \cap L^2(0,T;D(A))$ such that $u_m \to u$ in $L^\infty(0,T;H)$ weak star and in $L^2(0,T;D(A))$ weak. This allows to pass to the limit in (2.7) and finish the proof of existence (the regularity on $\frac{du}{dt}$ immediately follows from the equation and the regularity on u and f).

□

We also have the following result (existence and uniqueness of strong solutions).

Theorem 2.10. *We assume that $u_0 \in D(A)$ and $f \in L^2(0,T;H)$. Then, the solution u to (2.2) given in Theorem 2.8 satisfies $u \in \mathcal{C}([0,T];D(A)) \cap L^2(0,T;D(A^2))$ and $\frac{du}{dt} \in L^2(0,T;H)$ and is a strong solution to (2.2) (i.e., (2.2) holds in $L^2(0,T;H)$).*

Remark 2.11. Note that $D(A^2) \subset D(A) \subset H$ with continuous injections. Therefore, $u \in L^2(0,T;D(A^2))$ and $\frac{du}{dt} \in L^2(0,T;H)$ imply that $u \in \mathcal{C}([0,T];D(A))$. Therefore, $u(0)$ makes sense in $D(A)$.

Proof. Owing to the regularity on f, (2.6) and (2.7) now read

$$\frac{d}{dt}((u,v)) + ((Au,Av)) = ((f(t),v)) \text{ in } \mathcal{D}'(0,T),\ \forall v \in D(A), \tag{2.9}$$

and

$$\frac{d}{dt}((u_m,v)) + ((Au_m,Av)) = ((f(t),v)),\ \forall v \in W_m, \tag{2.10}$$

respectively.

We take $v = w_i$ in (2.10) and multiply the resulting equation by $\lambda_i^2 u_i$. This yields

$$\lambda_i^2 u_i \frac{du_i}{dt} + \lambda_i^4 |u_i| = ((f(t), \lambda_i^2 u_i w_i)).$$

Summing over $i = 1, \cdots, m$, we have

$$\frac{1}{2}\frac{d}{dt}\|Au_m\|^2 + \|A^2u_m\|^2 = ((f(t), Au_m)),$$

Noting that

$$|((f(t), Au_m))| \leq \|f(t)\|\|Au_m\| \leq \frac{1}{c_0}\|f(t)\|\|A^2u_m\|,$$

where c_0 is such that $\|A^2u_m\| \geq c_0\|Au_m\|$ and depends on λ_1 (see the previous subsection), this yields

$$\frac{d}{dt}\|Au_m\|^2 + \|A^2u_m\|^2 \leq \frac{1}{c_0^2}\|f(t)\|^2.$$

Therefore, u_m is bounded, independently of m, in $L^2(0,T;D(A))$ and $L^2(0,T;D(A^2))$ and we can pass to the limit as above.

□

Note that, if $u \in L^\infty(0,T;H) \cap L^2(0,T;D(A))$ is solution to (2.2), then

$$A^{-1}\frac{du}{dt} + Au = A^{-1}f(t) \text{ in } L^2(0,T;H)$$

and conversely.

We will actually consider a weaker formulation of (2.2), namely,

$$A^{-1}\frac{du}{dt} + Au = A^{-1}f(t) \text{ in } L^2(0,T;V'), \tag{2.11}$$

$$u(0) = u_0 \text{ in } V', \tag{2.12}$$

which is associated with the variational formulation

Find $u : [0,T] \to V$ such that

$$\frac{d}{dt}((A^{-1}u, v)) + \langle Au, v\rangle = \langle A^{-1}f(t), v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in V, \tag{2.13}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product in V'. Note that $A^{-1}\frac{du}{dt} = \frac{dA^{-1}u}{dt}$ and recall that $V' = D(A^{-\frac{1}{2}})$.

We have the following result (existence and uniqueness of very weak solutions).

Theorem 2.12. *We assume that $u_0 \in V'$ and $f \in L^2(0,T;D(A^{-\frac{3}{2}}))$. Then, (2.11)-(2.12) possesses a unique solution u such that $u \in \mathcal{C}([0,T];V') \cap L^2(0,T;V)$ and $\frac{du}{dt} \in L^2(0,T;D(A^{-\frac{3}{2}}))$.*

Remark 2.13. Note that $V \subset V' \subset D(A^{-\frac{3}{2}})$ with continuous injections. Therefore, $u \in L^2(0,T;V)$ and $\frac{du}{dt} \in L^2(0,T;D(A^{-\frac{3}{2}}))$ imply that $u \in \mathcal{C}([0,T];V')$. Therefore, $u(0)$ makes sense in V'. Also note that, if $f \in L^2(0,T;D(A^{-\frac{3}{2}}))$, then $A^{-1}f \in L^2(0,T;V')$; the same holds for $\frac{du}{dt}$.

Proof. **Uniqueness:**

Let u_1 and u_2 be solutions to

$$A^{-1}\frac{du_1}{dt} + Au_1 = A^{-1}f_1(t) \text{ in } \mathcal{D}'(0,T;V'),$$
$$u_1(0) = u_{0,1} \text{ in } V',$$

and

$$A^{-1}\frac{du_2}{dt} + Au_2 = A^{-1}f_2(t) \text{ in } \mathcal{D}'(0,T;V'),$$
$$u_2(0) = u_{0,2} \text{ in } V'.$$

Then, $u = u_1 - u_2$, $f = f_1 - f_2$ and $u_0 = u_{1,0} - u_{2,0}$ satisfy

$$A^{-1}\frac{du}{dt} + Au = A^{-1}f(t) \text{ in } \mathcal{D}'(0,T;V'), \tag{2.14}$$

$$u(0) = u_0 \text{ in } V'. \tag{2.15}$$

The above equation is associated with the variational formulation

$$\frac{d}{dt}((A^{-1}u,v)) + \langle Au, v\rangle = \langle A^{-1}f(t), v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in V. \tag{2.16}$$

Taking $v = u$ in (2.16), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1}^2 + \|u\|_V^2 = \langle A^{-1}f(t), u\rangle,$$

where $\|\cdot\|_{-1} = \|A^{-\frac{1}{2}}\cdot\|$. Noting that $|\langle A^{-1}f(t), u\rangle| \le \|f(t)\|_{D(A^{-\frac{3}{2}})}\|u\|_V$, this yields the differential inequality

$$\frac{d}{dt}\|u\|_{-1}^2 + \|u\|_V^2 \le \|f(t)\|_{D(A^{-\frac{3}{2}})}^2,$$

so that, integrating over $[0,t]$, $0 \le t \le T$,

$$\|u(t)\|_{-1}^2 \le \|u_0\|_{-1}^2 + \|f\|_{L^2(0,T;D(A^{-\frac{3}{2}})}^2.$$

This gives the continuous dependence with respect to the data (i.e., $(u_0, f) \mapsto u(t)$ is continuous from $V' \times L^2(0,T; D(A^{-\frac{3}{2}}))$ onto V'), as well as the uniqueness.

Existence:

We again employ a Galerkin scheme.

We consider, for $m \in \mathbb{N}$ given, the approximated problem

Find $u_m = \sum_{i=1}^m u_i w_i$, $u_i = u_i(t)$, $i = 1, \cdots, m$, such that

$$\frac{d}{dt}((A^{-1}u_m, v)) + ((Au_m, v)) = \langle f(t), v\rangle, \ \forall v \in W_m,$$

$$u_m(0) = u_{0,m},$$

where $u_{0,m}$ is the projection (in V') of u_0 onto W_m. Note that here we have enough regularity to write a scalar product in H, instead of a duality product.

Taking $v = w_i$, multiplying the resulting equation by u_i and summing over $i = 1, \cdots, m$, we find

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{-1}^2 + \|u_m\|_V^2 = \langle f(t), u_m\rangle,$$

which yields

$$\frac{d}{dt}\|u_m\|_{-1}^2 + \|u_m\|_V^2 \leq \|f(t)\|^2_{D(A^{-\frac{3}{2}})}.$$

Therefore, u_m is bounded independently of m in $L^\infty(0,T;V')$ and $L^2(0,T;V)$. We can then pass to the limit in the approximated problems and finish the proof of the theorem.

□

Remark 2.14. Assuming more regularity on u_0 and f, we can obtain more regularity on the solutions and recover the strong formulation, in particular.

3. Aubin–Lions compactness results [383]

Let X_0, X and X_1 be three Banach spaces such that $X \subset X_0 \subset X_1$ with dense and continuous injections, where $X \subset X_0$ is compact. Let $p \geq 1$, $1 \leq q \leq +\infty$ and $T > 0$ be given. We set

$$W = \{u \in L^p(0,T;X), \ \frac{du}{dt} \in L^q(0,T;X_1)\}.$$

- If $p < +\infty$, then the embedding $W \in L^p(0,T;X_0)$ is compact.
- If $p = +\infty$, then the embedding $W \in \mathcal{C}([0,T];X_0)$ is compact.

Remark 3.1. When $p = q = 2$ and the injections are only dense and continuous, we recover Lions' theorem, namely, $W \subset \mathcal{C}([0,T];X_0)$ with continuous injection (see [290]; see also [384] for insightful discussions on Lions' theorem).

4. Some useful inequalities

4.1. **Interpolation inequalities** [388]**.** Let $m \in \mathbb{N} \cup \{0\}$ and $s \in (0,1)$. Then,

$$\|u\|_{H^{m+s}(\Omega)} \leq c\|u\|^{\theta}_{H^m(\Omega)}\|u\|^{1-\theta}_{H^{m+1}(\Omega)}, \ \forall u \in H^{m+1}(\Omega), \tag{4.1}$$

where $\theta = 1 - s$ and $1 - \theta = s$. It is understood here that $H^0(\Omega) = L^2(\Omega)$.

Example 4.1. There holds, $\forall u \in H^1(\Omega)$,

$$\|u\|_{H^{\frac{1}{2}}(\Omega)} \leq c\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}. \tag{4.2}$$

Let $m_2 > m_1 \geq 0$, m_1, $m_2 \in \mathbb{N} \cup \{0\}$, and $s \in (m_1, m_2)$. Then,

$$\|u\|_{H^s(\Omega)} \leq c\|u\|^{\theta}_{H^{m_1}(\Omega)}\|u\|^{1-\theta}_{H^{m_2}(\Omega)}, \ \forall u \in H^{m_2}(\Omega), \tag{4.3}$$

where $\theta = \frac{m_2 - s}{m_2 - m_1}$ and $1 - \theta = \frac{s - m_1}{m_2 - m_1}$.

Example 4.2. (i) There holds, $\forall u \in H^4(\Omega)$,

$$\|u\|_{H^2(\Omega)} \leq c\|u\|^{\frac{2}{3}}_{H^1(\Omega)}\|u\|^{\frac{1}{3}}_{H^4(\Omega)}. \tag{4.4}$$

(ii) There holds, $\forall u \in H^4(\Omega)$,

$$\|u\|_{H^{\frac{7}{4}}(\Omega)} \leq c\|u\|^{\frac{3}{4}}_{H^1(\Omega)}\|u\|^{\frac{1}{4}}_{H^4(\Omega)}. \tag{4.5}$$

Remark 4.3. a) One important question is to find in an easy way the coefficients in the above interpolation inequalities. This can be done in several ways. One possibility is via a dimensional analysis, based on lengths. Let us consider the interpolation inequality (4.3). In $\mathbb{R}^n$, a volume corresponds to the dimension L^n. Then, a derivative corresponds to L^{-1}. Thus, for instance, the norm of u in $H^{m_1}(\Omega)$ corresponds to $(L^n L^{-2m_1})^{\frac{1}{2}}$ and, by analogy, the norm of u in $H^s(\Omega)$, even for s noninteger, to $(L^n L^{-2s})^{\frac{1}{2}}$. Therefore, we have, in (4.3), $(L^n L^{-2s})^{\frac{1}{2}} = (L^n L^{-2m_1})^{\frac{\theta}{2}}(L^n L^{-2m_2})^{\frac{1-\theta}{2}}$, so that $s = \theta m_1 + (1-\theta)m_2$ and the result follows.

b) In particular, we can see that the coefficients are independent of the space dimension n.

4.2. **Agmon's inequalities** [388]**.** There holds, in two space dimensions,

$$\|u\|_{\mathbb{L}^\infty(\Omega)} \leq c\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^2(\Omega)},\ \forall u \in H^2(\Omega). \tag{4.6}$$

Furthermore, there holds, both in two and three space dimensions,

$$\|u\|_{\mathbb{L}^\infty(\Omega)} \leq c\|u\|^{\frac{1}{2}}_{H^1(\Omega)}\|u\|^{\frac{1}{2}}_{H^2(\Omega)},\ \forall u \in H^2(\Omega). \tag{4.7}$$

Remark 4.4. Note that, interpolating as above, i.e., with exponents $\frac{1}{2}$, one obtains the spaces $H^1(\Omega)$ and $H^{\frac{3}{2}}(\Omega)$, respectively, and that, in two space dimensions (resp., three space dimensions), $L^\infty(\Omega)$ is not continuously embedded into $H^1(\Omega)$ (resp., $H^{\frac{3}{2}}(\Omega)$).

4.3. **Moser–Trudinger's inequality** [340]**.** Let Ω be a bounded smooth domain of $\mathbb{R}^2$. Then, there exists a positive constant C such that

$$\int_\Omega e^{|u(x)|}\,dx \leq Ce^{C\|u\|^2_{H^1(\Omega)}},\quad \forall u \in H^1(\Omega). \tag{4.8}$$

Remark 4.5. This inequality is also related to the limit case in the continuous embedding of $H^1(\Omega)$ into $L^p(\Omega)$ in two space dimensions and to an Orlicz space. Note that such an inequality is not available in three space dimensions.

5. Asymptotic behavior of dissipative systems: global attractors

Our aim in this section is to review some basic results in the theory of attractors for dissipative (parabolic) partial differential equations. More details and developments on this subject can be found, e.g., in [19], [335] and [388].

Let E be a Banach space endowed with the norm $\|.\|_E$. Let then $\{S(t),\ t \geq 0\}$ be a family of (nonlinear) operators acting on E,

$$S(t) : E \to E,\ t \geq 0.$$

We assume that this family of operators satisfies the following properties:

$$S(0) = \mathrm{I}\ \text{(identity operator)},$$

$$S(t+s) = S(t) \circ S(s),\ \forall s,\ t \geq 0,$$

and we say that it forms a semigroup acting on E. We will also need some continuity property and we assume that the mapping

$$S(t) : E \to E,\ x \mapsto S(t)x,$$

is continuous, $\forall t \geq 0$.

We first start with some terminology.

• For $u_0 \in E$, the trajectory starting at u_0 (or positive trajectory) is the set $\cup_{t\geq 0} S(t)u_0$.
• For $u_0 \in E$, a trajectory ending at u_0 (or negative trajectory), if it exists, is a set of the form

$$\{u(t),\ u(t) \in S(-t)^{-1}u_0,\ t \leq 0\}$$

(note that such a set, if it exists, is not necessarily unique).
• For $u_0 \in E$, a complete trajectory containing u_0, if it exists, is the union of the trajectory starting at u_0 and of a trajectory ending at u_0. In particular, fixed points are complete trajectories ($x \in E$ is a fixed point if $S(t)x = x,\ \forall t \geq 0$).
• A set $X \subset E$ is positively invariant by $S(t)$ if $S(t)X \subset X,\ \forall t \geq 0$ (thus, any trajectory starting from X remains in X).
• A set $X \subset E$ is negatively invariant by $S(t)$ if $X \subset S(t)X,\ \forall t \geq 0$ (thus, for every $x \in X$, there exists a trajectory ending at x which is contained in X).
• A set $X \subset E$ is invariant by $S(t)$ if $S(t)X = X,\ \forall t \geq 0$.

For instance, complete trajectories or unions of complete trajectories are invariant by $S(t)$. A more complicated (and essential in view of the construction of the global attractor) invariant set is an ω-limit set.

Definition 5.1. Let $x \in E$. The ω-limit set of x is the set

$$\omega(x) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)x}.$$

Similarly, if $B \subset E$ is a nonempty set, the ω-limit set of B is the set

$$\omega(B) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)B}.$$

An important (and easy to see) characterization of ω-limit sets is the following:

$$x \in \omega(B) \text{ iff } \exists (x_k)_{k\in\mathbb{N}},\ x_k \in B,\ \forall k \in \mathbb{N},\ \exists (t_k)_{k\in\mathbb{N}},\ t_k \to +\infty \text{ as } k \to +\infty,$$
$$\text{such that } S(t_k)x_k \to x \text{ as } k \to +\infty.$$

We have the following result (which is again essential in view of the construction of the global attractor).

Proposition 5.2. *Let $X \subset E$ be such that $X \neq \emptyset$ and $\cup_{t\geq t_0} S(t)X$ is relatively compact in E for some $t_0 > 0$. Then, $\omega(X)$ is nonempty, compact and invariant.*

Proof. First, note that, since $X \neq \emptyset$, $\cup_{t\geq s} S(t)X \neq \emptyset$, $\forall s \geq 0$. Therefore, the sets $\overline{\cup_{t\geq s} S(t)X}$, $s \geq t_0$, are nonempty and compact sets which decrease as s increases: their intersection, namely, $\omega(X)$, is a nonempty and compact set.

Let us now prove that $\omega(X)$ is invariant by $S(t)$.

Let $y \in S(t)\omega(X)$, for a given $t \geq 0$. Then, there exists $x \in \omega(X)$ such that $y = S(t)x$. Furthermore, owing to the characterization of ω-limit sets, there exist $x_k \in X$ and $t_k \to +\infty$ such that $S(t_k)x_k \to x$ as $k \to +\infty$. Then, by continuity

of $S(t)$, $S(t) \circ S(t_k)x_k \to S(t)x = y$ as $k \to +\infty$. Since, owing to the semigroup properties, $S(t) \circ S(t_k)x_k = S(t + t_k)x_k$, where $t + t_k \to +\infty$ as $k \to +\infty$, it follows that $y \in \omega(X)$ and $S(t)\omega(X) \subset \omega(X)$, $\forall t \geq 0$.

Let now $x \in \omega(X)$. Then, there exists $t_k \to +\infty$ and $x_k \in X$ such that $S(t_k)x_k \to x$ as $k \to +\infty$. Since $\cup_{t \geq t_0} S(t)X$ is relatively compact in E, it is easy to see that the sequence $S(t_k - t)x_k$ is relatively compact in E for $t_k - t \geq 0$ (i.e., for k large enough). Therefore, there exists a subsequence k_i and $y \in E$ such that

$$S(t_{k_i} - t)x_{k_i} \to y \text{ as } k_i \to +\infty.$$

This yields that $y \in \omega(X)$ (indeed, $t_{k_i} - t \to +\infty$ as $k_i \to +\infty$). Finally, since

$$S(t) \circ S(t_{k_i} - t)x_{k_i} = S(t_{k_i})x_{k_i}$$

and, by continuity,

$$S(t) \circ S(t_{k_i} - t)x_{k_i} \to S(t)y \text{ as } k_i \to +\infty,$$

we have (of course, $S(t_{k_i})x_{k_i} \to x$ as $k_i \to +\infty$)

$$x = S(t)y,$$

i.e., $x \in S(t)\omega(X)$. Therefore, $\omega(X) \subset S(t)\omega(X)$, $\forall t \geq 0$, which finishes the proof of the proposition. □

• A set $X \subset E$ attracts the bounded sets of E (we also say that it is an attracting set) if, $\forall B \subset E$ bounded,

$$\text{dist}_E(S(t)B, X) \to 0 \text{ as } t \to +\infty,$$

where dist_E denotes the Hausdorff semidistance between sets, defined by

$$\text{dist}_E(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_E.$$

Note in particular that $\text{dist}_E(A, B) \neq \text{dist}_E(B, A)$. Furthermore, $\text{dist}_E(A, B) = 0$ only implies $A \subset \overline{B}$. This attraction property is equivalent to the following (natural) property:

$$\forall \epsilon > 0,\ \forall B \subset E \text{ bounded},\ \exists t_0 = t_0(B) \text{ such that } t \geq t_0 \Longrightarrow S(t)B \subset \mathcal{U}_\epsilon,$$

where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of X, i.e., the union of all open balls with radius ϵ and with centers in X.

• A bounded set $\mathcal{B}_0 \subset E$ is a bounded absorbing set for $S(t)$ if, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$.

Note that the existence of a bounded absorbing set is sometimes used as a mathematical definition of dissipativity (from a physical point of view, roughly speaking, a dissipative system exhibits some kind of energy dissipation phenomenon).

We are now ready to give the definition of a global attractor.

Definition 5.3. A set $\mathcal{A} \subset E$ is a global attractor for the semigroup $S(t)$ if the following properties hold:
(i) $\mathcal{A}$ is compact in E.
This essentially says that a global attractor is much thinner than the phase space E; indeed, in applications, E usually is an infinite-dimensional function space (typically, $L^2(\Omega)$, $L^\infty(\Omega)$ or $H_0^1(\Omega)$) and, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense.
(ii) $\mathcal{A}$ is invariant by $S(t)$, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$.
(iii) $\mathcal{A}$ is an attracting set.

We first note that a global attractor, if it exists, is unique. Indeed, if $\mathcal{A}'$ is a second global attractor, then, since $\mathcal{A}'$ is bounded in E, $\mathrm{dist}_E(S(t)\mathcal{A}', \mathcal{A}) \to 0$ as $t \to +\infty$ and, since, owing to the invariance, $S(t)\mathcal{A}' = \mathcal{A}'$, there remains $\mathrm{dist}_E(\mathcal{A}', \mathcal{A}) = 0$, so that $\mathcal{A}' \subset \overline{\mathcal{A}} = \mathcal{A}$. Similarly, $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A} = \mathcal{A}'$.

Furthermore, it is easy to see that $\mathcal{A}$ is the smallest (for the inclusion) closed set which enjoys the attraction property and thus appears as a suitable set in view of the study of the asymptotic behavior of the system. Indeed, let $X \subset E$ be a closed attracting set. Then, $\mathrm{dist}_E(S(t)\mathcal{A}, X) \to 0$ as $t \to +\infty$, i.e., $\mathrm{dist}_E(\mathcal{A}, X) = 0$, so that $\mathcal{A} \subset \overline{X} = X$.

Finally, $\mathcal{A}$ is the largest (again, for the inclusion) bounded invariant set. Indeed, let X be a bounded invariant set. Then, $\mathrm{dist}_E(S(t)X, \mathcal{A}) \to 0$ as $t \to +\infty$, which implies, since X is invariant and $\mathcal{A}$ is closed, that $X \subset \mathcal{A}$.

Remark 5.4. a) The global attractor is also called the universal attractor (in the sense that it attracts all the bounded subsets of the phase space) or the maximal attractor.

b) The existence of the global attractor implies the existence of a bounded absorbing set. Indeed, any ϵ-neighborhood of the global attractor is a bounded absorbing set.

We have, concerning the existence of the global attractor, the following result.

Theorem 5.5. *We assume that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0$ and that, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B) > 0$ such that $\cup_{t \geq t_0} S(t)B$ is relatively compact in E. Then, $S(t)$ possesses the global attractor $\mathcal{A}$ and $\mathcal{A} = \omega(\mathcal{B}_0)$.*

Proof. It actually suffices, in view of Proposition 5.2, to prove that $\omega(\mathcal{B}_0)$ is an attracting set (indeed, we already know that $\omega(\mathcal{B}_0)$ is a nonempty, compact and invariant set).

We will argue by contradiction.

Let us assume that $\omega(\mathcal{B}_0)$ is not an attracting set. Then, there exists $B_1 \subset E$ bounded such that $\mathrm{dist}_E(S(t)B_1, \omega(\mathcal{B}_0))$ does not tend to 0 as t tends to $+\infty$.

Therefore, there exist $\delta > 0$ and $t_k \to +\infty$ such that

$$\forall k \in \mathbb{N},\ \mathrm{dist}_E(S(t_k)B_1, \omega(\mathcal{B}_0)) \geq \delta.$$

Thus, by definition of the Hausdorff semidistance, necessarily, $\forall k \in \mathbb{N}$, $\exists b_k \in B_1$ such that

$$\mathrm{dist}_E(S(t_k)b_k, \omega(\mathcal{B}_0)) \geq \frac{\delta}{2}. \tag{5.1}$$

We recall that $\mathcal{B}_0$ is an absorbing set. Therefore, if $t_k \geq t_1 = t_1(B_1)$, $t_1 \geq 0$, $S(t_k)B_1 \subset \mathcal{B}_0$ (indeed, $t_k \to +\infty$ as $k \to +\infty$) and $S(t_k)b_k \in \mathcal{B}_0$ if k is large enough.

Using now the fact that $\cup_{t\geq t_0} S(t)\mathcal{B}_0$ is relatively compact for some $t_0 > 0$, we deduce that the sequence $\{S(t_k)b_k,\ k \in \mathbb{N}\}$ is relatively compact. Therefore, there exists a subsequence (t_{k_i}, b_{k_i}) and $b \in E$ such that

$$S(t_{k_i})b_{k_i} \to b \text{ as } k_i \to +\infty.$$

Writing, for k_i large enough so that $t_{k_i} - t_1 \geq 0$,

$$S(t_{k_i})b_{k_i} = S(t_{k_i} - t_1) \circ S(t_1)b_{k_i}$$

and noting that $S(t_1)b_{k_i} \in \mathcal{B}_0$, we deduce that $b \in \omega(\mathcal{B}_0)$ (indeed, $t_{k_i} - t_1 \to +\infty$ as $k_i \to +\infty$). This leads to a contradiction, since, by continuity, (5.1) yields, at the limit,

$$\mathrm{dist}_E(b, \omega(\mathcal{B}_0)) \geq \frac{\delta}{2}.$$

This finishes the proof of the theorem. □

In particular, if $S(t)$ possesses a bounded absorbing set $\mathcal{B}_1$ such that $\mathcal{B}_1$ is relatively compact in E, then, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B) > 0$ such that $S(t)B \subset \mathcal{B}_1$, $\forall t \geq t_0$, which implies that $\cup_{t\geq t_0} S(t)B \subset \mathcal{B}_1$ is relatively compact. Therefore, owing to Theorem 5.5, $S(t)$ possesses the global attractor $\mathcal{A}$. In particular, this is the case for many parabolic systems. However, this cannot work for, e.g., weakly damped wave equations (indeed, in that case, there is no regularizing effect). In such a case, we have, for instance, the following result.

Theorem 5.6. *We assume that $S(t)$ possesses a compact attracting set K. Then, $S(t)$ possesses the global attractor $\mathcal{A}$ and $\mathcal{A} = \omega(K)$.*

Remark 5.7. a) We can also prove, in Theorems 5.5 and 5.6, that the global attractor $\mathcal{A}$ is connected.

b) It follows from the invariance property that the global attractor $\mathcal{A}$ has the following structure:

$$\mathcal{A} = \{x \in E,\ \exists \text{ a bounded complete trajectory containing } x\}.$$

Note that any bounded complete trajectory lies in $\mathcal{A}$ (indeed, it is a bounded invariant set).

It has been early conjectured that the global attractor should be, in some proper sense, finite-dimensional, meaning that the dynamics, restricted to this set, should be described by a finite number of parameters, even though the initial phase space is infinite-dimensional. However, the global attractor is expected to have a very complicated geometric structure (this should be compared with strange attractors in finite dimensions (i.e., for systems of ODE's in $\mathbb{R}^n$, $n \geq 3$) which are, typically, homeomorphic to the product of some $\mathbb{R}^m$ and some Cantor set). Therefore, we need to use other dimensions than the usual notion of dimension to measure the dimension of the global attractor; in particular, one usually considers covering dimensions such as the Hausdorff dimension or the entropy (or box-counting) dimension. Actually, we will only consider here the entropy dimension which we will call the fractal dimension. This dimension is defined as follows.

Definition 5.8. Let $X \subset E$ be a (relatively) compact set. For $\epsilon > 0$, let $N_\epsilon(X)$ (if it is necessary to precise the topology, we will also use the notation $N_\epsilon(X, E)$) be the minimal number of balls of radius ϵ which are necessary to cover X. Then, the fractal dimension of X is the quantity (which belongs to $[0, +\infty]$)

$$\dim_{\mathrm{F}} X = \limsup_{\epsilon \to 0^+} \frac{\log_2 N_\epsilon(X)}{\log_2 \frac{1}{\epsilon}} (= \limsup_{\epsilon \to 0^+} \frac{\ln N_\epsilon(X)}{\ln \frac{1}{\epsilon}}).$$

Furthermore, the quantity $\mathcal{H}_\epsilon(X) (= \mathcal{H}_\epsilon(X, E)) = \log_2 N_\epsilon(X)$ is called the Kolmogorov ϵ-entropy of X.

The fractal dimension satisfies the following properties:

- $\dim_{\mathrm{F}}(X_1 \times X_2) \leq \dim_{\mathrm{F}}(X_1) + \dim_{\mathrm{F}}(X_2)$.
- If $f : X_1 \to X_2$ is Lipschitz continuous, then $\dim_{\mathrm{F}} X_2 \leq \dim_{\mathrm{F}} X_1$.
- If X is a smooth m-dimensional manifold, then $\dim_{\mathrm{F}} X = m$.

It is also important to note that, for sets which are not manifolds, the fractal dimension can be noninteger; for instance, we have, in the case of the ternary Cantor set,

$$0 < \dim_{\mathrm{F}} X = \frac{\ln 2}{\ln 3} < 1.$$

Furthermore, it follows from the definition that, if the minimal number of balls of radius ϵ which are necessary to cover X satisfies

$$N_\epsilon(X) \leq c(\frac{1}{\epsilon})^d \text{ (i.e., } \mathcal{H}_\epsilon(X) \leq d \ln \frac{1}{\epsilon} + c', \ c' = \ln c),$$

where c and d are independent of ϵ, then

$$\dim_{\mathrm{F}} X \leq d.$$

Now, an interest for considering the fractal dimension over other dimensions is given by the (modified) Hölder-Mañé theorem (see [244]). First, we need the following definition (see [245]).

Definition 5.9. A Borel set X of a Banach space E is prevalent if there exists a compactly supported probability measure μ such that $\mu(X + x) = 1$, $\forall x \in E$. A non-Borel set which contains a prevalent set is also prevalent.

We do not wish to go into details here and will just mention that prevalence extends the notion of "Lebesgue almost every" from Euclidean to infinite-dimensional spaces.

Theorem 5.10. *(Modified Hölder-Mañé theorem) Assume that E is a Hilbert space and let $X \subset E$ be a compact set such that $\dim_{\mathrm{F}} X = d$ and $m > 2d$ be an integer. Then, almost every (in the sense of prevalence) bounded linear projector $P : E \to \mathbb{R}^m$ is one-to-one on X and has a Hölder continuous inverse.*

It follows from this theorem that, if the global attractor $\mathcal{A}$ has finite fractal dimension, then, fixing a projector P satisfying the assumptions of the theorem, the reduced dynamical system $(\overline{S}(t), \overline{\mathcal{A}})$, $\overline{S}(t) = P \circ S(t) \circ P^{-1}$, $\overline{\mathcal{A}} = P(\mathcal{A})$, is a finite-dimensional dynamical system (in $\mathbb{R}^m$) which is Hölder continuous with respect to the initial data (note that such a result does not hold for other dimensions and, in particular, for the Hausdorff dimension which is also often considered in the theory of global attractors; see however [365] for partial results for the Hausdorff dimension). This shows that the fractal dimension plays a fundamental role in the finite-dimensional reduction of infinite-dimensional dynamical systems, although Theorem 5.10 is not completely satisfactory, since the reduced dynamical system is not Lipschitz continuous and cannot thus be realized as a well-posed system of ODE's (indeed, one would need the Mañé projectors to be Lipschitz continuous and sufficient conditions on $\mathcal{A}$ which would ensure that this holds are not known).

We now give a general method to prove the finite fractal dimensionality of a compact set (see [420]).

Theorem 5.11. *Let X be a compact subset of E. We assume that there exist a Banach space E_1, with norm $\|.\|_{E_1}$, such that E_1 is compactly embedded into E and a mapping $L : X \to X$ such that $L(X) = X$ and*

$$\|Lx_1 - Lx_2\|_{E_1} \leq c\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in X, \ c > 0. \tag{5.2}$$

Then, the fractal dimension of X is finite and satisfies

$$\dim_{\mathrm{F}} X \leq \mathcal{H}_{\frac{1}{4c}}(B_{E_1}(0,1), E), \tag{5.3}$$

where $B_{E_1}(0,1)$ is the unit ball in E_1 (note that it is relatively compact in E, so that its $\frac{1}{4c}$-entropy is finite).

Proof. Let $\{B_E(k_i,\epsilon),\ i=1,\ \cdots,\ N_\epsilon\}$ be an ϵ-covering of X (note that X is compact) with balls in E with centers $k_i \in X,\ i=1,\ \cdots,\ N_\epsilon$. Then, owing to (5.2), the system

$$\{B_{E_1}(L(k_i),c\epsilon),\ i=1,\ \cdots,\ N_\epsilon\}$$

of balls of radius $c\epsilon$ in E_1 covers the set $L(X)$, i.e., it also covers X, since $L(X)=X$.

In a second step, we cover each of these balls with radius $c\epsilon$ in E_1 by a finite number of balls with radius $\frac{\epsilon}{4}$ in E. By definition, the minimal number of such balls is given by

$$N_{\frac{\epsilon}{4}}(B_{E_1}(L(k_i),c\epsilon),E) = N_{\frac{\epsilon}{4}}(B_{E_1}(0,c\epsilon),E) = N_{\frac{1}{4c}}(B_{E_1}(0,1),E) = \mathcal{N}.$$

Here, we have used the fact that E is a Banach, and thus a vector, space. If such a ball of radius $\frac{\epsilon}{4}$ does not intersect X, there is nothing to say and we forget it. Let us now consider the balls of radius $\frac{\epsilon}{4}$ which intersect X (and which still form a covering of X). Note that the centers of these balls of radius $\frac{\epsilon}{4}$ do not necessarily belong to X. Now, by considering balls of radius $\frac{\epsilon}{2}$, we can construct balls which still cover X, but now with centers in X (we need this to apply (5.2)), in E and with the same number of balls.

We thus deduce that

$$N_{\frac{\epsilon}{2}}(X,E) \leq \mathcal{N} N_\epsilon(X,E),$$

so that

$$\mathcal{H}_{\frac{\epsilon}{2}}(X,E) \leq \mathcal{H}_\epsilon(X,E) + \log_2 \mathcal{N}. \tag{5.4}$$

We will now prove (5.3) by proceeding recursively (i.e., by using (5.4) recursively).

First, note that, since X is compact in E, then there exist $x_0 \in X$ and $\epsilon_0 > 0$ such that $X \subset B_E(x_0,\epsilon_0)$, so that

$$\mathcal{H}_{\epsilon_0}(X,E) = 0. \tag{5.5}$$

Iterating (5.4) k times (starting from the ϵ_0-covering by one ball) and using (5.5), we have

$$\mathcal{H}_{\frac{\epsilon_0}{2^k}}(X,E) \leq k \log_2 \mathcal{N}. \tag{5.6}$$

Let now $\epsilon > 0$ be arbitrary and fix $k = k(\epsilon)$ such that

$$\frac{\epsilon_0}{2^k} \leq \epsilon \leq \frac{\epsilon_0}{2^{k-1}}. \tag{5.7}$$

Then, obviously, owing to the first of (5.7),

$$N_\epsilon(X,E) \leq N_{\frac{\epsilon_0}{2^k}}(X,E),$$

so that, owing to (5.6) and the second of (5.7),

$$\mathcal{H}_\epsilon(X,E) \le k \log_2 \mathcal{N} \le (\log_2 \frac{\epsilon_0}{\epsilon} + 1) \log_2 \mathcal{N},$$

which yields (5.3). This finishes the proof of the theorem. □

Remark 5.12. a) In applications to parabolic PDE's, one usually proves (5.2) for, say, $L = S(1)$. Then, owing to the invariance property (i.e., $S(1)\mathcal{A} = \mathcal{A}$), one deduces that the global attractor $\mathcal{A}$ has finite fractal dimension. For weakly damped hyperbolic systems, however, (5.2) cannot hold (we again recall that there is no regularizing effect), but one can prove weaker versions of (5.2).

b) It is also important to obtain sharp estimates on the fractal dimension of the global attractor in terms of the physical parameters of the problem. Actually, the above approach does not give the best estimates in general. Sharp estimates are usually obtained by the so-called volume contraction method, based on the Lyapunov exponents. This second approach, however, requires additional assumptions, namely, the differentiability of the semigroup, which may be difficult to prove (and may even not hold) in certain situations.

c) The existence of the finite-dimensional global attractor $\mathcal{A}$ has been proved for a large number of important (dissipative) equations in mathematical physics, such as the two-dimensional incompressible Navier-Stokes equations, reaction-diffusion systems, weakly damped wave equations, pattern-formation equations, etc.

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