The Cahn-Hilliard equation with regular nonlinear terms (II)

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The Cahn–Hilliard equation : recent advances and applications

Improved regularity result :

Equations :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

Assumptions on f :

$$f \text{ is of class } \mathcal{C}^2,\ f(0) = 0$$

$$f'(s) \geq -c_0,\ c_0 \geq 0,\ s \in \mathbb{R}$$

$$f(s)s \geq c_1 F(s) - c_2,\ F(s) \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R}$$

$$|f(s)| \leq \epsilon F(s) + c_\varepsilon,\ \forall \varepsilon > 0,\ s \in \mathbb{R}$$

Last assumption : handle the case $\langle u_0 \rangle = \kappa \neq 0$

Theorem : We assume that $u_0 \in H^2(\Omega)$, with $\frac{\partial u_0}{\partial \nu} = 0$ on Γ. Then, the problem possesses a unique solution u such that $u(t) \in H^2(\Omega)$, $\forall t \geq 0$.

Weak formulation ($\langle u_0 \rangle = 0$) :

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u) - \langle f(u) \rangle = 0$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{dE_1}{dt} + 2 \| \frac{\partial u}{\partial t} \|_{-1}^2 = 0$$

$$E_1 = \|u\|_V^2 + 2 \int_\Omega F(u)\, dx$$

Multiply by u :

$$\frac{d}{dt}\|u\|_{-1}^2 + \|u\|_V^2 + c\int_\Omega F(u)\,dx \leq c',\ c>0$$

Sum the two differential inequalities :

$$\frac{dE_2}{dt} + c(E_2 + \|\frac{\partial u}{\partial t}\|_{-1}^2) \leq c',\ c>0$$

$$E_2 = E_1 + \|u\|_{-1}^2$$

This yields :

$$E_2(t) \leq ce^{-c't}(\|u_0\|_{H^1(\Omega)}^2 + \int_\Omega F(u_0)\,dx) + c'',\ c'>0,\ t\geq 0$$

$$\int_t^{t+1} \|\frac{\partial u}{\partial t}\|_{-1}^2\,ds \leq ce^{-c't}(\|u_0\|_{H^1(\Omega)}^2 + \int_\Omega F(u_0)\,dx)$$

$$+c'',\ c'>0,\ t\geq 0$$

Multiply by $-\Delta \frac{\partial u}{\partial t}$:

$$\|\Delta f(u)\| \leq Q(\|u\|_{L^\infty(\Omega)})$$

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leq Q(\|u\|_{H^2(\Omega)})$$

Set $y = \|\Delta u\|^2$:

$$y' \leq Q(y)$$

Consider the solution z to the ODE :

$$z' = Q(z),\ z(0) = y(0)$$

Comparison principle : $\exists T_0 = T_0(\|u_0\|_{H^2(\Omega)}) \in (0, \frac{1}{2})$ s.t.

$$y(t) \leq z(t),\ \forall t \in [0, T_0]$$

Thus :

$$\|u(t)\|_{H^2(\Omega)} \leq Q(\|u_0\|_{H^2(\Omega)}),\ t \leq T_0$$

Differentiate the equation w.r.t. time :

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} - \langle f'(u)\frac{\partial u}{\partial t}\rangle = 0$$

Multiply by $t\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}(t\|\frac{\partial u}{\partial t}\|_{-1}^2) + t\|\frac{\partial u}{\partial t}\|_V^2 \leq ct\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2$$

Gronwall's lemma :

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq \frac{1}{t}Q(\|u_0\|_{H^2(\Omega)}),\ t \in (0, T_0]$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial u}{\partial t}\|_V^2 \leq c\|\frac{\partial u}{\partial t}\|_{-1}^2$$

Gronwall's lemma :

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq ce^{ct}\|\frac{\partial u}{\partial t}(T_0)\|_{-1}^2,\ t \geq T_0$$

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}),\ t \geq T_0$$

Rewrite the problem in elliptic form :

$$-\Delta u + \overline{f(u)} = h_u(t),\ \frac{\partial}{\partial \nu}\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma,\ t \geq T_0$$

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t},\ \|h_u(t)\| \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}),\ t \geq T_0$$

Multiply by u :

$$\|u\|_V^2 \leq c\|h_u(t)\|^2 + c'$$

Multiply by $-\Delta u$:

$$\|\Delta u\|^2 \leq c\|h_u(t)\|^2 + c'$$

Thus :

$$\|u(t)\|_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}),\ t \geq T_0$$

Finally :

$$\|u(t)\|_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}),\ t \geq 0$$

Dissipative H^2-estimate :

Multiply the Cahn-Hilliard equation by u :

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \leq c\|u\|_V^2$$

Thus :

$$\int_0^1 \|u\|_{H^2(\Omega)}^2 \, dt \leq c(\|u_0\|_{H^1(\Omega)}^2 + \int_\Omega F(u_0)\, dx) + c'$$

$\to \exists T \in (0,1)$ s.t.

$$\|u(T)\|_{H^2(\Omega)}^2 \leq c(\|u_0\|_{H^1(\Omega)}^2 + \int_\Omega F(u_0)\, dx) + c'$$

Repeat the estimates, starting from $t = T$ instead of $t = 0$:

$$\|u(1)\|^2_{H^2(\Omega)} \leq Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c'$$

(smoothing property)

Next :

$$\|u(t)\|^2_{H^2(\Omega)} \leq Q(\|u(t-1)\|^2_{H^1(\Omega)} + \int_\Omega F(u(t-1))\, dx) + c',\ t \geq 1$$

Thus :

$$\|u(t)\|^2_{H^2(\Omega)} \leq e^{-c't} Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\, dx) + c'',\ c' > 0,\ t \geq 1$$

Finally :

$$\|u(t)\|_{H^2(\Omega)} \leq e^{-ct} Q(\|u_0\|_{H^2(\Omega)}) + c',\ c > 0,\ t \geq 0$$

Remark : When $\kappa \neq 0$:

$$f(s)(s-\kappa) \geq c_1 F(s) - \kappa f(s) - c_2 \geq \frac{c_1}{2} F(s) - c, \; c = c(\kappa)$$

Remark : Existence of the finite-dimensional attractor $\mathcal{A}$ on $D(A)$, for the topology of V

Defaults of the global attractor :

May attract the trajectories at a slow rate

Very difficult to express the convergence rate in terms of the physical parameters of the problem

May be sensitive to perturbations

Upper semicontinuity :

$$\operatorname{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) \to 0 \text{ as } \epsilon \to 0^+$$

Lower semicontinuity :

$$\operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\epsilon) \to 0 \text{ as } \epsilon \to 0^+$$

More difficult to prove

$\rightarrow$ We need to construct and study larger objects which contain the global attractor, are more robust under perturbations, attract the trajectories at a fast rate and are still finite-dimensional

Possible objects : inertial manifolds and exponential attractors

Definition : A Lipschitz finite-dimensional manifold $\mathcal{M} \subset E$ is an inertial manifold for the semigroup $S(t)$ acting on the Banach space E if

(i) It is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geq 0$
(ii) It satisfies the following asymptotic completeness property :

$$\forall u_0 \in E,\ \exists v_0 \in \mathcal{M} \text{ such that}$$

$$\|S(t)u_0 - S(t)v_0\|_E \leq Q(\|u_0\|_E)e^{-\alpha t},\ t \geq 0,$$

where the positive constant α and the function Q are independent of u_0

An inertial manifold, if it exists, contains the global attractor and attracts the trajectories exponentially fast (and uniformly with respect to bounded sets of initial data)

The existence of such a set would confirm, in a perfect way, the heuristic conjecture on a finite-dimensional reduction principle of infinite-dimensional dissipative dynamical systems

The dynamics, restricted to an inertial manifold, can be described by a system of ODEs which is Lipschitz continuous : inertial form of the system

The asymptotic completeness property gives, in a particularly strong form, the equivalence of the initial dynamical system $(E, S(t))$ with its inertial form $(\mathcal{M}, S(t))$

Robust under small perturbations : normal hyperbolicity

Constructions of inertial manifolds : by the Lyapunov–Perron method, by constructing converging sequences of approximate inertial manifolds, by the so-called graph-transform method, ...

Drawback : all the known constructions of inertial manifolds make use of a restrictive condition (spectral gap condition)

$\rightarrow$ Can only be verified in one space dimension in general

Cahn-Hilliard equation : periodic boundary conditions (also in 3D)

Definition : A compact set $\mathcal{M} \subset E$ is an exponential attractor for the semigroup $S(t)$ acting on the Banach space E if

(i) It has finite fractal dimension, $\dim_F \mathcal{M} < +\infty$
(ii) It is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}, \ \forall t \geq 0$
(iii) It attracts exponentially fast the bounded subsets of E in the following sense :

$$\forall B \subset E \text{ bounded}, \ \operatorname{dist}_E(S(t)B, \mathcal{M}) \leq Q(\|B\|_E)e^{-\alpha t}, \ t \geq 0,$$

where the positive constant α and the function Q are independent of B

An exponential attractor, if it exists, contains the global attractor

The existence of an exponential attractor $\mathcal{M}$ yields the existence of the global attractor $\mathcal{A} \subset \mathcal{M}$: it is a compact attracting set

Finite-dimensional reduction principle given by the modified Hölder–Mañé theorem

Proving the existence of an exponential attractor is also one way of proving that the global attractor has finite fractal dimension

Compared with an inertial manifold : not smooth in general ; one still has a uniform exponential control on the rate of attraction of the trajectories

Main drawback : relaxation to positive invariance makes these objects nonunique

Family of exponential attractors

Question of the best choice ; find a simple algorithm

Theorem : Let X be a bounded subset of E. We assume that the mapping $S : X \to X$ enjoys the following smoothing property :

$$\|Sx_1 - Sx_2\|_{E_1} \le c\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in E,$$

where E_1 is a second Banach space such that the embedding $E_1 \subset E$ is compact. Then the discrete dynamical system generated by the iterations of S possesses an exponential attractor $\mathcal{M} \subset X$ i.e.,

(i) It is compact in E and has finite fractal dimension.

(ii) It is positively invariant, i.e., $S\mathcal{M} \subset \mathcal{M}$.

(iii) There holds

$$\mathrm{dist}_E(S^N X, \mathcal{M}) \le ce^{-\alpha N}, \ N \in \mathbb{N},$$

where c and $\alpha > 0$ only depend on X. Furthermore, all constants can be computed explicitly.

Continuous semigroup $S(t)$ acting on X :

Prove that $S(t_\star)$ satisfies the smoothing property for some $t_\star > 0$ (typically, $t_\star = 1$)

$\rightarrow$ Exponential attractor $\mathcal{M}_\star$ for the discrete dynamical system generated by the mapping $S_\star := S(t_\star)$

Set

$$\mathcal{M} := \cup_{t\in[0,t_\star]} S(t)\mathcal{M}_\star$$

If $(t,x) \mapsto S(t)x$ is Lipschitz (or even Hölder) continuous on $[0,t_\star] \times X$, $\mathcal{M}$ is an exponential attractor for $S(t)$ on X

Exponential attractors are robust under small perturbations : construction of robust families of exponential attractors

Cahn-Hilliard equation : prescribed average

Convergence of single trajectories to steady states (coarsening) : S. Zheng (1D), P. Rybka-K.H. Hoffmann (2-3D)

Continuum of steady states

Lojasiewicz-Simon's inequality

Numerical analysis : Cahn-Hilliard system (two second-order in space parabolic equations)

$$\frac{\partial u}{\partial t} = \Delta \mu$$

$$\mu = -\Delta u + f(u)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma$$

Multiply the first equation by μ and the second one by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}\Big(\frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,dx\Big) + \|\nabla \mu\|^2 = 0$$

(Energy identity/decay)

Variational formulation :

Find $(u, \mu) : [0, T] \to H^1(\Omega)^2$, $T > 0$ given, such that

$$((\frac{\partial u}{\partial t}, v)) + ((\nabla u, \nabla v)) = 0,\ \forall v \in H^1(\Omega)$$

$$((\nabla u, \nabla v)) + ((f(u) - \mu, v)) = 0,\ \forall v \in H^1(\Omega)$$

Consider a quasi-uniform family $\mathcal{T}^h$ of polygonal decomposition of Ω, $h \in (0, 1)$ $(n = 2)$

Finite element space $\mathcal{V}^h \subset H^1(\Omega)$:

$$\mathcal{V}^h = \{v \in \mathcal{C}(\overline{\Omega}),\ v|_\tau \in \mathcal{P}_m,\ \tau \in \mathcal{T}^h\}$$

$\mathcal{P}_m$: set of polynomials with degree less than or equal to m

Splitting method :

Find $(u^h, \mu^h) : [0, T] \to \mathcal{V}^h \times \mathcal{V}^h$ such that

$$((\frac{\partial u^h}{\partial t}, v)) + ((\nabla \mu^h, \nabla v)) = 0,\ \forall v \in \mathcal{V}^h$$

$$((\nabla u^h, \nabla v)) + ((f((u^h) - \mu_h, v)) = 0,\ \forall v \in \mathcal{V}^h$$

$u_0^h \in \mathcal{V}^h$: proper approximation of u_0

w_i, $i = 1, ..., N^h$: basis of $\mathcal{V}^h$

M : matrix with entries $M_{ij} = ((w_i, w_j))$ (mass matrix)

K : matrix with entries are $K_{ij} = ((\nabla w_i, \nabla w_j))$ (stiffness matrix)

Set

$$u^h(t) = \sum_{i=1}^{N^h} c_i(t) w_i \text{ and } \mu^h(t) = \sum_{i=1}^{N^h} d_i(t) w_i$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{N^h} \end{pmatrix}$$

$$D = \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_{N^h} \end{pmatrix}$$

Then :

$$M\frac{dC}{dt} = -KD$$

$$MD = KC + \mathcal{F}(C)$$

$$\mathcal{F}(C) = \begin{pmatrix} ((f(u^h), w_1)) \\ ((f(u^h), w_2)) \\ . \\ . \\ . \\ ((f(u^h), w_{N^h})) \end{pmatrix}$$

Thus (M is invertible) :

$$M\frac{dC}{dt} + KM^{-1}KC + KM^{-1}\mathcal{F}(C) = 0$$

Set $A = M^{-1}K$ ($-A$: finite element approximation of the Laplacian) :

$$\frac{dC}{dt} + A^2C + A(M^{-1}\mathcal{F}(C)) = 0$$

(resembles the Cahn–Hilliard equation)

Furthermore :

$$\frac{dC}{dt} = -AD$$

$$D = AC + M^{-1}\mathcal{F}(C)$$

(resembles the Cahn–Hilliard system)

Taking $v = 1$:

$$\frac{d}{dt}\int_{\Omega} u^h \, dx = 0$$

(Discrete mass conservation)

Discrete energy identity :

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u^h\|^2 + \int_{\Omega} F(u^h)\, dx\right) + \|\nabla \mu^h\|^2 = 0$$

Stability, consistency and convergence : C.M. Elliott-D.A. French

Higher-order Cahn–Hilliard models :

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{M} (-1)^i \sum_{|k|=i} a_k \mathcal{D}^{2k} u - \Delta f(u) = 0,\ M \in \mathbb{N},\ a_k > 0,\ |k| = M$$

For $k = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$ (n=3)

$$|k| = k_1 + k_2 + k_3$$

$$\mathcal{D}^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

Boundary conditions : Dirichlet boundary conditions

$$\mathcal{D}^k u = 0 \text{ on } \Gamma,\ |k| \leq M$$

Functional setting and linear operators :

Introduce, for $N \in \mathbb{N}$, the elliptic operator A_N defined as

$$\langle A_N v, w \rangle_{H^{-N}(\Omega), H_0^N(\Omega)} = \sum_{|k|=N} a_k((\mathcal{D}^k v, \mathcal{D}^k w))$$

$H^{-N}(\Omega)$: topological dual of $H_0^N(\Omega)$, $(-\Delta)^{-1}$: inverse minus Laplace operator associated with Dirichlet boundary conditions
Note that

$$(v, w) \in H_0^N(\Omega)^2 \mapsto \sum_{|k|=N} a_k((\mathcal{D}^k v, \mathcal{D}^k w))$$

is bilinear, symmetric, continuous and coercive. Thus :

$$A_N : H_0^N(\Omega) \to H^{-N}(\Omega)$$

is well defined

Elliptic regularity results for linear elliptic operators of order $2N$: A_N is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A_N) = H^{2N}(\Omega) \cap H_0^N(\Omega),$$

For $v \in D(A_N)$:

$$A_N v = (-1)^N \sum_{|k|=N} a_k \mathcal{D}^{2k} v$$

Furthermore : $D(A_N^{\frac{1}{2}}) = H_0^N(\Omega)$ and, for $(v, w) \in D(A_N^{\frac{1}{2}})^2$:

$$((A_N^{\frac{1}{2}} v, A_N^{\frac{1}{2}} w)) = \sum_{|k|=N} a_k ((\mathcal{D}^k v, \mathcal{D}^k w))$$

Finally : $\|A_N \cdot \|$ (resp., $\|A_N^{\frac{1}{2}} \cdot \|$) is equivalent to the usual H^{2N}-norm (resp., H^N-norm) on $D(A_N)$ (resp., $D(A_N^{\frac{1}{2}})$)

Define the linear operator $\overline{A}_N = -\Delta A_N$:

$$\overline{A}_N : H_0^{N+1}(\Omega) \to H^{-N-1}(\Omega)$$

Strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(\overline{A}_N) = H^{2N+2}(\Omega) \cap H_0^{N+1}(\Omega)$$

For $v \in D(\overline{A}_N)$:

$$\overline{A}_N v = (-1)^{N+1} \Delta \sum_{|k|=N} a_k \mathcal{D}^{2k} v$$

Furthermore : $D(\overline{A}_N^{\frac{1}{2}}) = H_0^{N+1}(\Omega)$ and, for $(v, w) \in D(\overline{A}_N^{\frac{1}{2}})^2$:

$$((\overline{A}_N^{\frac{1}{2}} v, \overline{A}_N^{\frac{1}{2}} w)) = \sum_{|k|=N} a_k ((\nabla \mathcal{D}^k v, \nabla \mathcal{D}^k w))$$

$\|\overline{A}_N \cdot \|$ (resp., $\|\overline{A}_N^{\frac{1}{2}} \cdot \|$) is equivalent to the usual H^{2N+2}-norm (resp., H^{N+1}-norm) on $D(\overline{A}_N)$ (resp., $D(\overline{A}_N^{\frac{1}{2}})$)

Define the operator $\tilde{A}_N = (-\Delta)^{-1} A_N$:

$$\tilde{A}_N : H_0^{N-1}(\Omega) \to H^{-N+1}(\Omega)$$

$-\Delta$ and A_N commute : the same holds for $(-\Delta)^{-1}$ and A_N, so that $\tilde{A}_N = A_N(-\Delta)^{-1}$

Lemma : The operator $\tilde{A}_N$ is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(\tilde{A}_N) = H^{2N-2}(\Omega) \cap H_0^{N-1}(\Omega),$$

where, for $v \in D(\tilde{A}_N)$,

$$\tilde{A}_N v = (-1)^N \sum_{|k|=N} a_k \mathcal{D}^{2k}(-\Delta)^{-1} v.$$

Furthermore, $D(\tilde{A}_N^{\frac{1}{2}}) = H_0^{N-1}(\Omega)$ and, for $(v, w) \in D(\tilde{A}_N^{\frac{1}{2}})^2$,

$$((\tilde{A}_N^{\frac{1}{2}} v, \tilde{A}_N^{\frac{1}{2}} w)) = \sum_{|k|=N} a_k ((\mathcal{D}^k(-\Delta)^{-\frac{1}{2}} v, \mathcal{D}^k(-\Delta)^{-\frac{1}{2}} w)).$$

Besides, $\|\tilde{A}_N \cdot \|$ (resp., $\|\tilde{A}_N^{\frac{1}{2}} \cdot \|$) is equivalent to the usual H^{2N-2}-norm (resp., H^{N-1}-norm) on $D(\tilde{A}_N)$ (resp., $D(\tilde{A}_N^{\frac{1}{2}})$).

Functional form of the equation :

$$\frac{du}{dt} - \overline{A}_M u - \Delta B_M u - \Delta f(u) = 0$$

$$B_M v = \sum_{i=1}^{M-1} (-1)^i \sum_{|k|=i} a_k \mathcal{D}^{2k} v$$

Weak form :

$$(-\Delta)^{-1} \frac{du}{dt} + A_M u + B_M u + f(u) = 0$$

Assumptions on f :

$$f \in \mathcal{C}^1(\mathbb{R}),\ f(0) = 0$$

$$f' \geq -c_0,\ c_0 \geq 0$$

$$f(s)s \geq c_1 F(s) - c_2 \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R}$$

$$F(s) \geq c_4 s^4 - c_5,\ c_4 > 0,\ c_5 \geq 0,\ s \in \mathbb{R}$$

Scalar product by $\frac{du}{dt}$ and u :

$$\frac{dE_1}{dt} + c(E_1 + \|\frac{\partial u}{\partial t}\|_{-1}^2) \leq c', \; c > 0,$$

$$E_1 = \|A_M^{\frac{1}{2}} u\|^2 + B_M^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\,dx + \|u\|_{-1}^2$$

$$E_1 \geq c(\|u\|_{H^M(\Omega)}^2 + \int_\Omega F(u)\,dx) - c', \; c > 0$$

$$B_M^{\frac{1}{2}}[u] = \sum_{i=1}^{M-1} \sum_{|k|=i} a_k \|\mathcal{D}^k u\|^2$$

Note that :

$$|B_M^{\frac{1}{2}}[u]| \leq \frac{1}{2}\|A_M^{\frac{1}{2}} u\|^2 + c\|u\|^2$$

Thus :

$$\|u(t)\|^2_{H^M(\Omega)} \leq ce^{-c't}(\|u_0\|^2_{H^M(\Omega)} + \int_\Omega F(u_0)\,dx) + c'',\ c' > 0,\ t \geq 0$$

Scalar product by $A_M u$:

$$\frac{d}{dt}\|\tilde{A}_M^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^{2M}(\Omega)} \leq c(\|u\|^2 + \|f(u)\|^2)$$

Note that

$$\|u\|^2 + \|f(u)\|^2 \leq Q(\|u\|_{H^M(\Omega)})$$

Thus :

$$\frac{d}{dt}\|\tilde{A}_M^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^{2M}(\Omega)} \leq e^{-c't}Q(\|u_0\|_{H^M(\Omega)}) + c'',\ c,\ c' > 0,\ t \geq 0$$

Scalar product by $-\Delta \frac{du}{dt}$:

$$\frac{d}{dt}(\|\overline{A}_M^{\frac{1}{2}}u\|^2 + \overline{B}_M^{\frac{1}{2}}[u]) + \|\frac{\partial u}{\partial t}\|^2 \leq e^{-c't}Q(\|u_0\|_{H^M(\Omega)}) + c'',\ c,\ c' > 0$$

$$\overline{B}_M^{\frac{1}{2}}[u] = \sum_{i=1}^{M-1}\sum_{|k|=i} a_k\|\nabla\mathcal{D}^k u\|^2$$

Combine the estimates :

$$\frac{dE_2}{dt}+c(E_2+\|u\|^2_{H^{2M}(\Omega)}+\|\frac{\partial u}{\partial t}\|^2) \leq e^{-c't}Q(\|u_0\|_{H^M(\Omega)})+c'',\ c,\ c' > 0,\ t \geq 0$$

$$E_2 = E_1 + \|\tilde{A}_M^{\frac{1}{2}}u\|^2 + \|\overline{A}_M^{\frac{1}{2}}u\|^2 + \overline{B}_M^{\frac{1}{2}}[u]$$

$$E_2 \geq c(\|u\|^2_{H^{M+1}(\Omega)} + \int_\Omega F(u)\,dx) - c',\ c > 0$$

Thus :

$$\|u(t)\|_{H^{M+1}(\Omega)} \leq e^{-ct}Q(\|u_0\|_{H^{M+1}(\Omega)}) + c',\ c > 0,\ t \geq 0$$

We can also obtain a dissipative H^{2M}-estimate

We can also consider periodic boundary conditions (conservation of mass)

Viscous Cahn-Hilliard equation :

$$-\beta\Delta\frac{\partial u}{\partial t}+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0,\ \beta>0$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \Delta u}{\partial \nu}=0 \text{ on } \Gamma$$

$$u|_{t=0}=u_0$$

Conservation of mass :

$$\langle u(t)\rangle=\langle u_0\rangle,\ t\geq 0$$

Equivalent formulation :

$$\beta\frac{\partial u}{\partial t}+(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Delta u+\overline{f(u)}=0$$

$$\frac{\partial u}{\partial \nu}=0 \text{ on } \Gamma$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + 2\beta\|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 = 0$$

Multiply by $-\Delta u$ $(f' \geq -1)$:

$$\frac{d}{dt}(\beta\|\nabla u\|^2 + \|\overline{u}\|^2) + 2\|\Delta u\|^2 \leq 2\|\nabla u\|^2$$

Differentiate with respect to time :

$$\beta\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + (-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + \overline{f'(u)\frac{\partial u}{\partial t}} = 0$$

$$\frac{\partial}{\partial \nu}\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}(\beta\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|^2_{-1}) + 2\|\frac{\partial u}{\partial t}\|^2_V \leq 2\|\frac{\partial u}{\partial t}\|^2$$

$\to u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(\Omega \times (0,T))$, $T > 0$

$u_0 \in H^2(\Omega)$ (plus compatibility condition) : $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$

$\to \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$

Equivalent elliptic problem :

$$-\Delta u + \overline{f(u)} = h_u,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$h_u = -\beta\frac{\partial u}{\partial t} - (-\Delta)^{-1}\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))$$

Multiply by $-\Delta u$:

$$\|\Delta u\|^2 \leq 2\|h_u\|^2 + 2\|\nabla u\|^2$$

$\rightarrow u \in L^\infty(0, T; H^2(\Omega))$

Well-posedness, existence of finite-dimensional global attractors

Existence of exponential attractors (prescribed spatial average)

Uniform estimates as $\beta \rightarrow 0^+$

Construction of robust families of exponential attractors

Equivalent second-order in space parabolic equation :

$$\beta \frac{\partial u}{\partial t} - \Delta u + f(u) = \tilde{h}_u$$

$$\tilde{h}_u = \langle f(u) \rangle - (-\Delta)^{-1} \frac{\partial u}{\partial t} \in L^\infty(0, T; H^2(\Omega))$$

Continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega}) : \tilde{h}_u \in L^\infty(\Omega \times (0, T))$

Consider the ODE's

$$y'_\pm + f(y_\pm) = \pm \|\tilde{h}_u\|_{L^\infty(\Omega \times (0,T))},\ y_\pm(0) = \pm \|u_0\|_{L^\infty(\Omega)}$$

Comparison principle :

$$y_-(t) \leq u(x,t) \leq y_+(t),\ (x,t) \in \Omega \times (0, T)$$

Set $z_+ = u - y_+$:

$$\beta \frac{\partial z_+}{\partial t} - \Delta z_+ + f(u) - f(y_+) \leq 0$$
$$\frac{\partial z_+}{\partial \nu} = 0 \text{ on } \Gamma$$
$$z_+(0) \leq 0$$

Multiply by $z_+^+ = \max(z_+, 0)$:

$$\frac{\beta}{2} \frac{d}{dt} \|z_+^+\|^2 + \|\nabla z_+^+\|^2 \leq \|z_+^+\|^2$$

Gronwall's lemma :

$$\|z_+^+(t)\|^2 \leq e^{\frac{2}{\beta} t} \|z_+^+(0)\|^2 = 0,\ t \in [0, T]$$

$\rightarrow z_+^+(t) = 0$ and $u(x, t) \leq y_+(t)$

Second inequality : $z_- = y_- - u$

Viscous Cahn-Hilliard equation : behaves as a second-order in space parabolic equation

Does not allow to prove that $u \in [0, 1]$

Allen-Cahn equation (ordering of atoms) :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = 0$$

$$u = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

Comparison principle

Separable ODE :

$$y' - y^3 - y = 0,\ y(0) \in [-1, 1],$$

$\rightarrow u \in [-1, 1]$