The Cahn-Hilliard equation with logarithmic nonlinear terms (I)

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The Cahn–Hilliard equation : recent advances and applications

Equations :

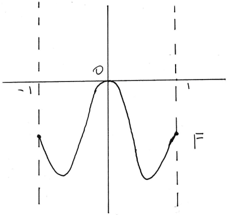
$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \frac{1+s}{1-s},\ s \in (-1,1)$$

$$0 < \theta < \theta_c$$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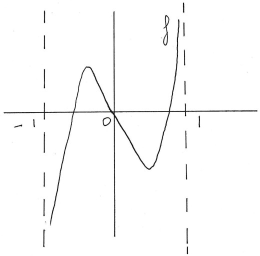
f
-1
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Properties of f :

• f is odd, $f(0) = 0$

• $f' \geq -c_0$, $c_0 = \theta_c > 0$

• $f(s) = f_1(s) - c_0 s$, $f_1' \geq 0$

Set $F(s) = \int_0^s f(\xi)\, d\xi$

• $f(s)s \geq F(s) - c_1 \geq -c_2,\ c_1,\ c_2 \geq 0$

Consider the function $\varphi(s) = f(s)s - F(s) + \frac{c_0}{2}s^2$, $\varphi'(s) = (f'(s) + c_0)s$

• $f(s)s \geq c_3|f(s)| - c_4,\ c_3 > 0,\ c_4 \geq 0$

Consider the function $\varphi(s) = f_1(s)s - \frac{1}{2}f_1(s)(= f_1(s)(s - \frac{1}{2}))\ (s \geq 0)$

$\lim_{s\to 1^-} \varphi(s) = +\infty$

$\to \varphi \geq -c,\ c \geq 0$

$\to f_1(s)s \geq \frac{1}{2}f_1(s) - c = \frac{1}{2}|f_1(s)| - c$

Thus :

$$f(s)s = f_1(s)s - c_0s^2 \geq \frac{1}{2}|f_1(s)| - c - c_0$$
$$\geq \frac{1}{2}|f(s)| - c - \frac{3}{2}c_0$$

• $f(s)(s-m) \geq F(s) - c_m,\ c_m \geq 0$, c_m depends continuously on m, $m \in (-1,1)$

Consider the function $\varphi(s) = f(s)(s-m) - F(s)$,
$\varphi'(s) = f'(s)(s-m) = f_1'(s)(s-m) - c_0(s-m)$

Thus :

$$\varphi(s) - \varphi(m) = \int_m^s f_1'(\xi)(\xi - m)\, d\xi - \frac{c_0}{2}(s-m)^2 \geq -2c_0$$

$\to f(s)(s-m) \geq F(s) - F(m) - 2c_0$

• $f(s)(s-m) \geq c_m|f(s)| - c_m',\ c_m > 0,\ c_m' \geq 0$, c_m, c_m' depend continuously on m, $m \in (-1,1)$

Note that $f(s)(s-m) = f_1(s)(s-m) - c_0 s(s-m) \geq f_1(s)(s-m) - 2c_0$

Consider the function $\varphi(s) = f_1(s)(s-m) - c_m f_1(s)s = f_1(s)((1-c_m)s - m)$, $c_m > 0$, $c_m < 1-m$ if $m \geq 0$, $c_m > 1+m$ if $m \leq 0$

Note that $\lim_{s\to\pm 1} \varphi(s) = +\infty$

$\to \varphi \geq -c'_m$

$\to f(s)(s-m) \geq c_m f_1(s)s - c'_m$

Approximated functions :

$$f_N(s) = f(-1+\frac{1}{N}) + f'(-1+\frac{1}{N})(s+1-\frac{1}{N}),\ s < -1+\frac{1}{N}$$

$$f_N(s) = f(s),\ |s| \leq 1-\frac{1}{N}$$

$$f_N(s) = f(1-\frac{1}{N}) + f'(1-\frac{1}{N})(s-1+\frac{1}{N}),\ s > 1-\frac{1}{N}$$

• f_N is odd, $f_N(0) = 0$

• $f_N(s) = f_{1,N}(s) - c_0 s, f'_{1,N} \geq 0$

Set $F_N(s) = \int_0^s f_N(\xi)\, d\xi$

$\bullet\, f_N(s)s \geq F_N(s) - c_5 \geq -c_6,\ c_5,\ c_6 \geq 0,\ s \in \mathbb{R}$ (N large enough)

It suffices to take $s \geq 0$

$s \in [0, 1 - \frac{1}{N}) : f_N(s) = f(s)$

$s \geq 1 - \frac{1}{N}$:

$$f_N(s)s - F_N(s) = f_N(s)s - F(1 - \frac{1}{N}) - \int_{1-\frac{1}{N}}^{s} f_N(\xi)\, d\xi$$

$$= f(1 - \frac{1}{N})(1 - \frac{1}{N}) + \frac{1}{2}f'(1 - \frac{1}{N})s^2 - \frac{1}{2}f'(1 - \frac{1}{N})(1 - \frac{1}{N})^2 - F(1 - \frac{1}{N})$$

Take $N \geq N_0 = N_0(f) \geq 2$ such that $f(1 - \frac{1}{N}) \geq 0, f'(1 - \frac{1}{N}) \geq 0$ (F bounded)

• $f_N(s)s \geq c_7|f_N(s)| - c_8,\ c_7 > 0,\ c_8 \geq 0,\ s \in \mathbb{R}$ (N large enough)

$s \in [0, 1 - \frac{1}{N}) : f_N(s) = f(s)$

$s \geq 1 - \frac{1}{N}$: set $\varphi_N(s) = f_N(s)(s - \frac{1}{2})$

$f_N(s) = f_1(1 - \frac{1}{N}) + f_1'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}) - c_0 s$

Take $N \geq N_1 = N_1(f) \geq N_0$ such that $f_1(1 - \frac{1}{N}) \geq c_0, f_1'(1 - \frac{1}{N}) \geq c_0$

$$f_N(s) = f_1(1 - \frac{1}{N}) + f_1'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}) - c_0(s - 1 + \frac{1}{N}) - c_0(1 - \frac{1}{N})$$

$$\geq f_1(1 - \frac{1}{N}) + f_1'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}) - c_0(s - 1 + \frac{1}{N}) - c_0 \geq 0$$

$\rightarrow \varphi_N \geq 0, f_N(s)s \geq \frac{1}{2} f_N(s) = \frac{1}{2}|f_N(s)|$

$\bullet\, f_N(s)(s-m) \geq c_m(|f_N(s)| + F_N(s)) - c'_m$
$c_m > 0,\ c'_m \geq 0,\ s \in \mathbb{R},\ m \in (-1,1)$ c_m, c'_m depend continuously on m (N large enough)

$s \in [0, 1-\frac{1}{N}) : f_N = f$

$s \geq 1 - \frac{1}{N}$: set $\varphi_N(s) = f_N(s)(s-m-c_m)$, $c_m > 0$ such that $|m + c_m| < 1$ ($m \in (-1,1)$)

Take $N \geq N_0 = N_0(f,m)$ such that $1 - \frac{1}{N} \geq m + c_m, f_N \geq 0 : \varphi_N \geq 0$

$\rightarrow f_N(s)(s-m) \geq c_m f_N(s) = c_m |f_N(s)|$

Recall that

$$f_N(s)s \geq F_N(s) - c_5$$

Take N large enough such that $f_N(s)(s-m) \geq \frac{1}{2} f_N(s)s$

Remark : F_N is bounded, independently of N, on $[-1, 1]$

Approximated problems :

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) = 0$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u_N|_{t=0} = u_0$$

Well-posedness : regular case

Mass conservation

Passage to the limit : estimates independent of N

Crucial step : estimate independent of N on $f_N(u_N)$ in $L^2(\Omega \times (0,T))$, $T > 0$

A priori estimates :

Assume that $\langle u_0 \rangle = \kappa = 0$, $-1 < u_0(x) < 1$, a.e. x

$\rightarrow F_N(u_0)$ is bounded, independently of N

Equivalent problem :

$$(-\Delta)^{-1}\frac{\partial u_N}{\partial t} - \Delta u_N + \overline{f_N(u_N)} = 0$$

$$\frac{\partial u_N}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u_N|_{t=0} = u_0$$

Multiply by $\frac{\partial u_N}{\partial t}$

$$\frac{d}{dt}(\|u_N\|_V^2 + 2\int_\Omega F_N(u_N)\,dx) + 2\|\frac{\partial u_N}{\partial t}\|_{-1}^2 = 0$$

Multiply by u $(((\overline{f_N(u_N)}, u_N)) = ((f_N(u_N), u_N)))$:

$$\frac{d}{dt}\|u_N\|_{-1}^2 + c(\|u_N\|_V^2 + 2\int_\Omega F_N(u_N)\,dx + \|f_N(u_N)\|_{L^1(\Omega)}) \leq c',\ c > 0$$

Sum :

$$\frac{dE_{1,N}}{dt} + c(E_{1,N} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\frac{\partial u_N}{\partial t}\|_{-1}^2) \leq c',\ c > 0$$

$$E_{1,N} = \|u_N\|_{-1}^2 + \|u_N\|_V^2 + 2\int_\Omega F_N(u_N)\,dx$$

Multiply by $\overline{f_N(u_N)}$:

$$(((-\Delta)^{-1}\frac{\partial u_N}{\partial t},\overline{f_N(u_N)})) + ((f_N'(u_N)\nabla u_N, \nabla u_N)) + \|\overline{f_N(u_N)}\|^2 = 0$$

Thus :

$$\|\overline{f_N(u_N)}\|^2 \leq c(\|u_N\|_V^2 + \|\frac{\partial u_N}{\partial t}\|_{-1}^2)$$

Combine with the the previous estimates :

$$\frac{dE_{1,N}}{dt} + c(E_{1,N} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\overline{f_N(u_N)}\|^2 + \|\frac{\partial u_N}{\partial t}\|_{-1}^2)$$
$$\leq c',\ c > 0$$

Multiply by $-\Delta u_N$:

$$\frac{d}{dt}\|u_N\|^2 + 2\|\Delta u_N\|^2 \leq 2c_0\|u_N\|^2$$

Combine with the previous estimates :

$$\frac{dE_{2,N}}{dt} + c(E_{2,N} + \|u_N\|^2_{H^2(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\overline{f_N(u_N)}\|^2 + \|\frac{\partial u_N}{\partial t}\|^2_{-1})$$

$$\leq c',\ c > 0$$

$$E_{2,N} = E_{1,N} + \delta\|u_N\|^2,\ \delta > 0$$

$$E_{2,N} \geq \|u_N\|^2_V - c$$

For N large enough :

$$|\langle f_N(u_N)\rangle| \leq c\|u_N\|\|\overline{f_N(u_N)}\| + c'$$

Note that ($T > 0$ given)

$$\|u_N(t)\| \leq c_T,\ t \in [0,T],$$
$$\|\overline{f_N(u_N)}\|_{L^2(\Omega\times(0,T))} \leq c_T$$

Thus :

$$\|f_N(u_N)\|_{L^2(\Omega\times(0,T))} \leq c_T$$

Existence and uniqueness of solutions :

Theorem : We assume that u_0 is given such that $u_0 \in H^1(\Omega)$ and $-1 < u_0(x) < 1$, a.e. x. Then, the problem possesses a unique (weak) solution such that, $\forall T > 0$,

$$u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)').$$

Furthermore, $-1 < u(x,t) < 1$, a.e. (x,t).

Uniqueness : $f' \geq -c_0$

Continuous dependence estimate :

$$\|u_1(t) - u_2(t)\|_{-1} \leq e^{ct}\|u_{1,0} - u_{2,0}\|_{-1},\ t \geq 0$$

for any two solutions u_1 and u_2 with initial data $u_{1,0}$ and $u_{2,0}$ such that $\langle u_{1,0} \rangle = \langle u_{2,0} \rangle = \kappa$

Existence :

$$u_N \to u \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak star and in } L^2(0,T;H^2(\Omega)) \text{ weakly}$$

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(0,T;H^1(\Omega)') \text{ weakly}$$

$$u_N \to u \text{ a.e. } (x,t) \text{ and in } L^2(\Omega \times (0,T))$$

Difficulty : pass to the limit in the nonlinear term $f_N(u_N)$

$f_N(u_N)$ is bounded, independently of N, in $L^1(\Omega \times (0,T))$

For N large enough :

$$\operatorname{meas}(E_{N,M}) \leq \varphi(\frac{1}{N}),\ N \leq M$$

$$E_{N,M} = \{(x,t) \in \Omega \times (0,T),\ |u_M(x,t)| > 1 - \frac{1}{N}\}$$

$$\varphi(s) = \frac{1}{|f(1-s)|},\ s \in (0,1)$$

Indeed (take N (and M) large enough) :

$$\int_0^T \int_\Omega |f_M(u_M)|\, dx\, dt \geq \int_{E_{N,M}} |f_M(u_M)|\, dx\, dt$$

$$\geq \text{meas}(E_{N,M}) f(1 - \frac{1}{N})$$

Pass to the limit $M \to +\infty$ (employ Fatou's lemma) and then $N \to +\infty$ (note that $\lim_{s\to 0} \varphi(s) = 0$) :

$$\text{meas}\{(x,t) \in \Omega \times (0,T),\ |u(x,t)| \geq 1\} = 0$$

$\to -1 < u(x,t) < 1,\ \text{a.e. } (x,t)$

Almost everywhere convergence of u_N to u and explicit expression of f_N :

$$f_N(u_N) \to f(u) \text{ a.e. } (x,t) \in \Omega \times (0,T)$$

$f_N(u_N)$ is bounded, independently of N, in $L^2(\Omega \times (0,T))$: $f_N(u_N) \to f(u)$ in $L^2(\Omega \times (0,T))$ weakly

Set

$$\Phi_\kappa = \{v \in H^1(\Omega) \cap L^\infty(\Omega),\ -1 < v(x) < 1,\ \text{a.e. } x \in \Omega,$$
$$\langle v \rangle = \kappa\},\ \kappa \in (-1,1)$$

We can define the continuous (for the $H^{-1}(\Omega)$-norm) semigroup

$$S(t) : \Phi_\kappa \to \Phi_\kappa,\ u_0 \mapsto u(t),\ t \geq 0$$

$S(t)$ is dissipative in Φ_κ

Continuous dependence estimate : we can extend (in a unique way and by continuity) $S(t)$ to a semigroup acting on the closure of Φ_κ in the $H^{-1}(\Omega)$-topology :

$$L_\kappa = \{v \in L^\infty(\Omega),\ \|v\|_{L^\infty(\Omega)} \leq 1,\ \langle v\rangle = \kappa\}$$

$\rightarrow$ We can now consider initial data which contain the pure states

$$S(t) : L_\kappa \rightarrow \Phi_\kappa,\ t > 0$$

The initial datum cannot be a pure state

If $u_0 \in L_\kappa$, consider a sequence of regular functions $u_{0,k} \in \Phi_\kappa$, $k \in \mathbb{N}$, such that

$$\lim_{k\rightarrow+\infty} \|u_0 - u_{0,k}\|_{-1} = 0$$

Then :

$$u(t) = S(t)u_0 = \lim_{k \to +\infty} S(t)u_{0,k}$$

$S(t)$ is dissipative in L_κ

Uniform Gronwall's lemma : we can choose the bounded absorbing set such that it is bounded in L_κ and compact in $H^1(\Omega)$

Theorem : The semigroup $S(t)$ possesses the global attractor $\mathcal{A}_\kappa$ on L_κ.

Finite fractal dimensionality of the global attractor : difficult problem

Main difficulty : We a priori only have a weak separation property from the pure states

We will prove a strict separation in 1D and 2D ($\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta$, $\delta \in (0, 1)$)

First proof of existence of the global attractor : A. Debussche-L. Dettori

Finite-dimensionality : based on the differentiability of the semigroup

$\rightarrow$ The strict separation from ± 1 was necessary

$\rightarrow$ Could be proved only for small domains

Here : no restriction on the size of Ω

Finite-dimensionality : construction of an exponential attractor

Exponential attractor : compact and positively invariant $(S(t)\mathcal{M}_m \subset \mathcal{M}_m,\ t \geq 0)$ set which contains the global attractor and has finite fractal dimension

Main tool : find a proper set $\mathcal{C}$ s.t.

$$\|S(t)u_1 - S(t)u_2\|_{L^2(\Omega)} \leq c(t)\|u_1 - u_2\|_{H^{-1}(\Omega)}$$

for some $t > 0$, $\forall u_1, u_2 \in \mathcal{C}$

$\mathcal{A}_\kappa$ is trivial if κ is large : $\exists M \in (0,1)$ s.t.

$$\mathcal{A}_\kappa = \{\kappa\} \text{ if } |\kappa| \geq M$$

Set $S(t)(\pm 1) = \pm 1$ (the pure states can be solutions)

Then

$$S(t)L = L,\ L = \cup_{|\kappa|\leq 1} L_\kappa = B_{L^\infty(\Omega)}(0,1)$$

Set $\mathcal{A}_{\pm 1} = \{\pm 1\}$

Theorem : The semigroup $S(t)$ possesses the finite-dimensional global attractor

$$\mathcal{A} = \cup_{|\kappa|\leq 1} \mathcal{A}_\kappa$$

on L.

Viscous Cahn-Hilliard equation :

$$-\beta\Delta\frac{\partial u}{\partial t}+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0,\ \beta>0$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \Delta u}{\partial \nu}=0 \text{ on } \Gamma$$

$$u|_{t=0}=u_0$$

Conservation of mass :

$$\langle u(t)\rangle=\langle u_0\rangle,\ t\geq 0$$

Equivalent formulation :

$$\beta\frac{\partial u}{\partial t}+(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Delta u+\overline{f(u)}=0$$

$$\frac{\partial u}{\partial \nu}=0 \text{ on } \Gamma$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + 2\beta\|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 = 0$$

Multiply by $-\Delta u$ ($f' \geq -c_0$) :

$$\frac{d}{dt}(\beta\|\nabla u\|^2 + \|\overline{u}\|^2) + 2\|\Delta u\|^2 \leq 2c_0\|\nabla u\|^2$$

Differentiate with respect to time :

$$\beta\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + (-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + \overline{f'(u)\frac{\partial u}{\partial t}} = 0$$

$$\frac{\partial}{\partial \nu}\frac{\partial u}{\partial t} = 0 \text{ on } \Gamma$$

Multiply by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}(\beta\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2) + 2\|\frac{\partial u}{\partial t}\|_V^2 \leq 2c_0\|\frac{\partial u}{\partial t}\|^2$$

$\rightarrow u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(\Omega \times (0,T))$, $T > 0$

$u_0 \in H^2(\Omega)$ (plus compatibility condition) : $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$

$\rightarrow \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$

Equivalent elliptic problem :

$$-\Delta u + \overline{f(u)} = h_u,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$h_u = -\beta\frac{\partial u}{\partial t} - (-\Delta)^{-1}\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))$$

Multiply by $-\Delta u$:

$$\|\Delta u\|^2 \leq 2\|h_u\|^2 + 2c_0\|\nabla u\|^2$$

$\to u \in L^\infty(0,T;H^2(\Omega))$

$\to \overline{f(u)} \in L^\infty(0,T;L^2(\Omega))$

As in the case of the Cahn–Hilliard equation : $\langle f(u)\rangle \in L^\infty(0,T)$, $f(u) \in L^\infty(0,T;L^2(\Omega))$

$\to$ Well-posedness, weak separation property

Equivalent second-order in space parabolic equation :

$$\beta \frac{\partial u}{\partial t} - \Delta u + f(u) = \tilde{h}_u$$

$$\tilde{h}_u = \langle f(u) \rangle - (-\Delta)^{-1} \frac{\partial u}{\partial t} \in L^\infty(0, T; H^2(\Omega))$$

Continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$: $\tilde{h}_u \in L^\infty(\Omega \times (0, T))$

Consider the ODE's

$$y'_\pm + f(y_\pm) = \pm \|\tilde{h}_u\|_{L^\infty(\Omega \times (0,T))}, \; y_\pm(0) = \pm \|u_0\|_{L^\infty(\Omega)}$$

Comparison principle :

$$y_-(t) \le u(x,t) \le y_+(t), \; (x,t) \in \Omega \times (0, T)$$

Set $z_+ = u - y_+$:

$$\beta \frac{\partial z_+}{\partial t} - \Delta z_+ + f(u) - f(y_+) \leq 0$$

$$\frac{\partial z_+}{\partial \nu} = 0 \text{ on } \Gamma$$

$$z_+(0) \leq 0$$

Multiply by $z_+^+ = \max(z_+, 0)$:

$$\frac{\beta}{2} \frac{d}{dt} \|z_+^+\|^2 + \|\nabla z_+^+\|^2 \leq c_0 \|z_+^+\|^2$$

Gronwall's lemma :

$$\|z_+^+(t)\|^2 \leq e^{\frac{2c_0}{\beta} t} \|z_+^+(0)\|^2 = 0, \ t \in [0, T]$$

$\rightarrow z_+^+(t) = 0$ and $u(x,t) \leq y_+(t)$

Second inequality : $z_- = y_- - u$

$$|y_{\pm}(t)| \leq 1 - Q(\|u_0\|_{L^\infty(\Omega)}, \|\tilde{h}_u\|_{L^\infty(\Omega\times(0,T))}),\ t \in [0,T]$$

$$Q(\|u_0\|_{L^\infty(\Omega)}, \|\tilde{h}_u\|_{L^\infty(\Omega\times(0,T))}) \in (0,1)$$

$\rightarrow$ strict separation property

Existence of finite-dimensional global attractors, existence of exponential attractors

Regularity and separation from the pure states :

We assume that $\kappa = 0$

Equations :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

Proposition : The weak solution u satisfies

$$\frac{\partial u}{\partial t} \in L^{\infty}(r, +\infty; H^{-1}(\Omega)) \cap L^{2}(r, T; H^{1}(\Omega)),$$

$\forall r < T, r > 0$ and $T > 0$ given.

We have :

$$\int_{t}^{t+r} \|\frac{\partial u}{\partial t}\|_{-1}^{2}\, d\tau \leq c(r),\ t \geq 0$$

Equivalent formulation :

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + \overline{f(u)} = 0$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

Differentiate with respect to time :

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+\overline{f'(u)\frac{\partial u}{\partial t}}=0$$
$$\frac{\partial}{\partial \nu}\frac{\partial u}{\partial t}=0 \text{ on } \Gamma$$

Multiplying by $\frac{\partial u}{\partial t}$:

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2+2\|\frac{\partial u}{\partial t}\|_V^2\leq 2c_0\|\frac{\partial u}{\partial t}\|^2$$

Interpolation inequality $\|\frac{\partial u}{\partial t}\|\leq\|\frac{\partial u}{\partial t}\|_{-1}^{\frac{1}{2}}\|\frac{\partial u}{\partial t}\|_V^{\frac{1}{2}}$:

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\frac{\partial u}{\partial t}\|_V^2\leq c\|\frac{\partial u}{\partial t}\|_{-1}^2$$

Uniform Gronwall's lemma

We have

$$\int_t^{t+r} \|\frac{\partial u}{\partial t}\|_V^2 \, d\tau \leq c(r), \; \forall t \geq r$$

Proposition : We assume that $2 \leq p < +\infty$, when $n = 2$, and $2 \leq p \leq 6$, when $n = 3$. Then, the solution u further satisfies

$$\|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \leq c,$$

$$\|u(t)\|_{W^{2,p}(\Omega)} \leq c,$$

$\forall t \geq r$, $r > 0$ given, where the constant c depends on the $H^1(\Omega)$-norm of u_0.

Equivalent elliptic equation :

$$-\Delta u + f_1(u) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + \langle f(u)\rangle + c_0 u \equiv h_u$$

$\overline{f(u)} \in L^\infty(r,t;L^2(\Omega))$

$\to \langle f(u)\rangle \in L^\infty(r,T)$

Thus :

$$\begin{aligned}\|h_u\|_{H^1(\Omega)} &\leq c(\|\frac{\partial u}{\partial t}\|_{-1} + |\langle f(u)\rangle| + \|u\|_{H^1(\Omega)})\\ &\leq c,\ t \geq r\end{aligned}$$

$\to \|h_u\|_{L^\infty(r,t;H^1(\Omega))} \leq c,\ t \geq r$

Multiply by $|f_1(u)|^{p-2}f_1(u)$:

$$-\int_\Omega \Delta u|f_1(u)|^{p-2}f_1(u)\,dx + \int_\Omega |f_1(u)|^p\,dx = \int_\Omega h_u|f_1(u)|^{p-2}f_1(u)\,dx$$

$$-\int_\Omega \Delta u|f_1(u)|^{p-2}f_1(u)\,dx = (p-1)\int_\Omega |f_1(u)|^{p-2}f_1'(u)|\nabla u|^2\,dx \geq 0$$

$$|\int_\Omega h_u|f_1(u)|^{p-2}f_1(u)\,dx| \leq \frac{1}{2}\|f_1(u)\|^p_{L^p(\Omega)} + c\|h_u\|^p_{H^1(\Omega)}$$

Thus :

$$\|f_1(u)\|^p_{L^p(\Omega)} \leq c\|h_u\|^p_{H^1(\Omega)}$$

$$\|f(u)\|_{L^p(\Omega)} \leq c(\|h_u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)})$$

$\rightarrow \|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \leq c,\ t \geq r$

Standard elliptic regularity results : $\|u\|_{L^\infty(r,t;W^{2,p}(\Omega))} \le c,\ t \ge r$

Remark : In 1D : $H^1(\Omega) \subset L^\infty(\Omega)$
Thus :

$$|\int_\Omega h_u|f_1(u)|^{p-2}f_1(u)\,dx| \le \|h_u\|_{L^p(\Omega)}\|f_1(u)\|^{p-1}_{L^p(\Omega)}$$

$$\le c\|h_u\|_{L^\infty(\Omega)}\|f_1(u)\|^{p-1}_{L^p(\Omega)}$$

$$\le c\|h_u\|_{H^1(\Omega)}\|f_1(u)\|^{p-1}_{L^p(\Omega)}$$

$\to \|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \le c,\ t \ge r$, c independent of $p \in [1, +\infty]$

Remark : Cannot be justified within a Galerkin scheme

Truncation functions, $k \in \mathbb{N}$:

$$\varphi_k(s) = 1 - \frac{1}{k},\ s > 1 - \frac{1}{k}$$

$$\varphi_k(s) = s,\ |s| \leq 1 - \frac{1}{k}$$

$$\varphi_k(s) = -1 + \frac{1}{k},\ s < -1 + \frac{1}{k}$$

Set $u_k = \varphi_k(u)$

$\rightarrow u_k \in \mathcal{C}([0,T]; H^1(\Omega))$ (φ_k is Lipschitz continuous)

$\rightarrow \nabla u_k = \chi_{(-1+\frac{1}{k},1-\frac{1}{k})}(u)\nabla u$ (χ : indicator function)

Set $\psi_k = |f_1(u_k)|^{p-2} f_1(u_k) \in \mathcal{C}([0,T]; H^1(\Omega))$

Multiply by ψ_k ($f_1(u_k)^2 \leq f_1(u_k) f_1(u),\ k \geq 2$)

$\rightarrow k \rightarrow +\infty$ (Fatou's lemma)

Proposition : We assume that $n = 1$. Then, there exists $\delta \in (0,1)$ depending on the $H^1(\Omega)$-norm of u_0 such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,\ t \geq r,$$

$r > 0$ given.

$p \rightarrow +\infty$ in Remark

Remark : Strict separation : physical and mathematical interpretations

Proposition : We assume that $n = 2$. Then, the following holds for every $t \geq r$, $r > 0$ given, and for every $p \in \mathbb{N}$:

$$\|f'(u)\|_{L^p(\Omega\times(r,t))} \leq c,$$

where the constant c depends on p.

Multiply $-\Delta u + f_1(u) = h_u$ by $f_1(u)e^{L|f_1(u)|}$:

$$\int_\Omega |\nabla u|^2 f_1'(u)(1 + |f_1(u)|e^{L|f_1(u)|})\, dx + \int_\Omega |f_1(u)|^2 e^{L|f_1(u)|}\, dx$$
$$= \int_\Omega h_u f_1(u) e^{L|f_1(u)|}\, dx$$

Thus :

$$\int_\Omega |f_1(u)|^2 e^{L|f_1(u)|}\, dx \leq \int_\Omega |h_u||f_1(u)|e^{L|f_1(u)|}\, dx$$

Generalized Young's inequality :

$$ab \leq e^a - a - 1 + (b+1)\ln(b+1)$$
$$\leq e^a + (b+1)\ln(b+1),\ a,\ b \geq 0$$

Thus :

$$\int_\Omega |h_u||f_1(u)|e^{L|f_1(u)|}\,dx \leq \frac{1}{2}\int_\Omega |f_1(u)|^2 e^{L|f_1(u)|}\,dx + \int_\Omega e^{c(L)|h_u|}\,dx + c'$$

$$a = k|h_u|,\ b = k^{-1}|f_1(u)|^2 e^{L|f_1(u)|}$$

(k properly chosen)

Moser-Trudinger's inequality :

$$\int_\Omega e^{|u(x)|}\,dx \le ce^{c\|u\|^2_{H^1(\Omega)}},\ \forall u \in H^1(\Omega)$$

Thus :

$$\int_\Omega |f_1(u)|^2 e^{L|f_1(u)|}\,dx \le c'$$

Note that

$$f_1'(s) \le e^{c|f_1(s)|+c'},\ s \in (-1,1).$$

Thus :

$$|f_1'(s)|^p \le e^{pc|f_1(s)|+pc'}$$

$\rightarrow$ Take $L = pc$

Proposition : We assume that $n=2$. Then, the weak solution u further satisfies

$$\frac{\partial u}{\partial t} \in L^\infty(r,+\infty;H) \cap L^2(r,T;H^2(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Multiply $(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + \overline{f'(u)\frac{\partial u}{\partial t}} = 0$ by $-\Delta\frac{\partial u}{\partial t}$:

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\frac{\partial u}{\partial t}\|^2 = ((f'(u)\frac{\partial u}{\partial t}, \Delta\frac{\partial u}{\partial t}))$$

Hölder, Young and Ladyzhenskaya's inequalities :

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\frac{\partial u}{\partial t}\|^2 \leq c\|f'(u)\|^4_{L^4(\Omega)}\|\frac{\partial u}{\partial t}\|^2$$

$\rightarrow$ Uniform Gronwall's lemma

Theorem : We assume that $n = 2$. Then, there exists $\delta \in (0, 1)$ depending on the $H^1(\Omega)$-norm of u_0 such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,\ t \geq r,$$

$r > 0$ given.

Elliptic equation :

$$-\Delta u + f_1(u) = h_u$$

Now : $\|h_u\|_{L^\infty(\Omega \times (r,t))} \leq c,\ t \geq r$

$\rightarrow$: We can conclude as in 1D

Remark : $\|u(t)\|_{H^4(\Omega)} \leq c,\ t \geq r, r > 0$

Remark : Finite dimensionality of global attractors, exponential attractors, convergence to equilibria (1 and 2D)

Remark : More general singular nonlinear terms :

$$\lim_{s \to \pm 1} f(s) = \pm\infty \text{ and } \lim_{s \to \pm 1} f'(s) = +\infty$$

Remark : Strict separation in 3D : open problem for logarithmic nonlinear terms

Can be proved if f behaves like $\frac{c}{(1-u^2)^\alpha}$, $\alpha > \frac{3}{7}$, close to ± 1

Other proof of existence :

Semigroup's theorey : H. Abels-M. Wilke