# Further generalizations of the Cahn-Hilliard equation

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The Cahn–Hilliard equation : recent advances and applications

**Cahn-Hilliard-Gurtin models :**

Phenomenological derivation :

Mass balance : $\frac{\partial u}{\partial t} = -\mathrm{div} h$

$h$ : mass flux

Constitutive equation : $h = -\kappa \nabla w$

Ginzburg-Landau free energy : $\Psi_\Omega(u, \nabla u) = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))dx$

$\Omega$ : domain occupied by the material

Chemical potential : variational derivative of $\Psi_\Omega$ with respect to $u$

$\rightarrow w = -\alpha \Delta u + f(u)$

M. Gurtin :

• Phenomenological derivation : physically sound and important

• Should be regarded as a precursor of more complete theories

• Limits the manner in which rate terms enter the equation

• Requires a priori specifications of the constitutive equations (mass flux : $h = -\kappa \nabla w$)

• It is not clear how it can be generalized in the presence of processes such as deformations (e.g., for elastic solids) or heat transfers

• It is not clear whether there is an underlying balance law which can form a basis for more complete theories

Main idea : introduce a separate balance law for microforces, i.e., forces at an atomistic level

Microforce balance :

$$\mathrm{div}\xi + \pi = 0$$

$\pi$ (scalar) : microforces
$\xi$ (vector) : microstress

Mass balance :

$$\frac{d}{dt}\int_{\mathcal{R}} u\,dx \leq -\int_{\partial\mathcal{R}} h\cdot\nu\,d\Sigma$$

for every control volume $\mathcal{R} \subset \Omega$

Note that

$$\int_{\partial\mathcal{R}} h \cdot \nu \, d\Sigma = \int_{\mathcal{R}} \mathrm{div} h \, dx$$

$\rightarrow \frac{\partial u}{\partial t} = -\mathrm{div} h$

Second law of thermodynamics : the rate at which the free energy increases cannot exceed the sum of the work and the rate at which free energy is carried into the control volume $\mathcal{R}$ by mass transport

Integral formulation :

$$\frac{d}{dt} \int_{\mathcal{R}} \psi \, dx \leq \mathcal{W}(\mathcal{R}) + \mathcal{M}(\mathcal{R})$$

$\psi$ : free energy density

Work :

$$\mathcal{W}(\mathcal{R}) = \int_{\partial\mathcal{R}} (\xi \cdot \nu) \frac{\partial u}{\partial t} \, d\Sigma$$

Mass transport :

$$\mathcal{M}(\mathcal{R}) = -\int_{\partial\mathcal{R}} wh \cdot \nu \, d\Sigma$$

Note that

$$\int_{\partial\mathcal{R}} (\xi \cdot \nu) \frac{\partial u}{\partial t} \, d\Sigma = \int_{\mathcal{R}} \mathrm{div}(\frac{\partial u}{\partial t} \xi) \, dx$$

$$\int_{\partial\mathcal{R}} wh \cdot \nu \, d\Sigma = \int_{\mathcal{R}} \mathrm{div}(wh) \, dx$$

Thus :

$$\frac{\partial \psi}{\partial t} \leq -\mathrm{div}(wh) + \mathrm{div}(\frac{\partial u}{\partial t}\xi)$$

$$= -w\mathrm{divh} - h \cdot \nabla w + \frac{\partial u}{\partial t}\mathrm{div}\xi + \xi \cdot \nabla \frac{\partial u}{\partial t}$$

Dissipation inequality :

$$\frac{\partial \psi}{\partial t} + (\pi - w)\frac{\partial u}{\partial t} - \xi \cdot \nabla \frac{\partial u}{\partial t} + h \cdot \nabla w \leq 0$$

First models :

Set of independent constitutive variables :

$$Z = (u, \nabla u, w, \nabla w)$$

Assumptions : $\psi = \psi(Z), h = h(Z), \xi = \xi(Z), \pi = \pi(Z)$

Thus :

$$\frac{\partial \psi}{\partial t} = \partial_u \psi \frac{\partial u}{\partial t} + \partial_{\nabla u} \psi \cdot \nabla \frac{\partial u}{\partial t} + \partial_w \psi \frac{\partial w}{\partial t} + \partial_{\nabla w} \psi \cdot \nabla \frac{\partial w}{\partial t}$$

$\partial_X$ : differential with respect to $X$

Dissipation inequality :

$$(\partial_u \psi + \pi - w)\frac{\partial u}{\partial t} + (\partial_{\nabla u} \psi - \xi) \cdot \nabla \frac{\partial u}{\partial t} + \partial_w \psi \frac{\partial w}{\partial t} + \partial_{\nabla w} \psi \cdot \nabla \frac{\partial w}{\partial t} + h \cdot \nabla w \leq 0$$

We can choose $Z$ such that $Z$, $\frac{\partial u}{\partial t}$, $\nabla \frac{\partial u}{\partial t}$, $\frac{\partial w}{\partial t}$ and $\nabla \frac{\partial w}{\partial t}$ take arbitrary prescribed values at some chosen point $x$ and time $t$

$\frac{\partial u}{\partial t}$, $\nabla \frac{\partial u}{\partial t}$, $\frac{\partial w}{\partial t}$, $\nabla \frac{\partial w}{\partial t}$ appear linearly

Thus :

$$\partial_u \psi + \pi - w = 0$$

$$\partial_{\nabla u} \psi - \xi = 0$$

$$\partial_w \psi = 0,\ \partial_{\nabla w} \psi = 0$$

$\rightarrow \psi = \psi(u, \nabla u)$

Dissipation inequality :

$$h \cdot \nabla w \leq 0$$

$\rightarrow$ There exists a positive semidefinite matrix $B = B(Z)$ such that

$$h = -B \nabla w$$

Thus :

$$\frac{\partial u}{\partial t} = -\mathrm{div} h = \mathrm{div}(B(Z)\nabla w)$$

$$w = \partial_u \psi + \pi = \partial_u \psi - \mathrm{div}\xi = \partial_u \psi - \mathrm{div}\partial_{\nabla u}\psi$$

Take $\psi = \frac{\alpha}{2}|\nabla u|^2 + F(u)$ :

$$\partial_u \psi = F'(u) = f(u),\ \partial_{\nabla u}\psi = \alpha \nabla u$$

$\rightarrow$ Equations :

$$\frac{\partial u}{\partial t} = \mathrm{div}(B(Z)\nabla w)$$

$$w = -\alpha\Delta u + f(u)$$

Isotropic case ($B = \kappa I$, $\kappa > 0$) : Cahn-Hilliard system

Further generalizations :

Independent constitutive variables :

$$Z = (u, \nabla u, w, \nabla w, \frac{\partial u}{\partial t})$$

Then :

$$\frac{\partial \psi}{\partial t} = \partial_u \psi \frac{\partial u}{\partial t} + \partial_{\nabla u} \psi \cdot \nabla \frac{\partial u}{\partial t} + \partial_w \psi \frac{\partial w}{\partial t} + \partial_{\nabla w} \psi \cdot \nabla \frac{\partial w}{\partial t} + \partial_{\frac{\partial u}{\partial t}} \psi \frac{\partial^2 u}{\partial t^2}$$

Dissipation inequality :

$$(\partial_u \psi + \pi - w) \frac{\partial u}{\partial t} + (\partial_{\nabla u} \psi - \xi) \cdot \nabla \frac{\partial u}{\partial t} + \partial_w \psi \frac{\partial w}{\partial t}$$

$$+ \partial_{\nabla w} \psi \cdot \nabla \frac{\partial w}{\partial t} + \partial_{\frac{\partial u}{\partial t}} \psi \frac{\partial^2 u}{\partial t^2} + h \cdot \nabla w \leq 0$$

$\frac{\partial u}{\partial t}$ no longer appears linearly :

$$\psi = \psi(u, \nabla u)$$

There remains :

$$(\partial_u \psi + \pi - w)\frac{\partial u}{\partial t} + h \cdot \nabla w \leq 0$$

$\rightarrow$ There exist four constitutive moduli, $\beta = \beta(Z)$ (a scalar), $a = a(Z)$, $b = b(Z)$ (two vectors), $B = B(Z)$ (a matrix), such that

$$\partial_u \psi + \pi - w = -\beta\frac{\partial u}{\partial t} - b \cdot \nabla w$$

$$h = -a\frac{\partial u}{\partial t} - B\nabla u$$

$$\beta(Z)(\frac{\partial u}{\partial t})^2 + (a(Z) + b(Z)) \cdot \nabla w\frac{\partial u}{\partial t} + B(Z)\nabla w \cdot \nabla w \geq 0$$

$\rightarrow \beta \geq 0$, $B$ is positive semidefinite

Thus :

$$\frac{\partial u}{\partial t} = -\mathrm{div} h = \mathrm{div}(a\frac{\partial u}{\partial t}) + \mathrm{div}(B\nabla w)$$

$$w - b \cdot \nabla w = \beta\frac{\partial u}{\partial t} + \partial_u\psi - \mathrm{div}\partial_{\nabla u}\psi$$

Take $\psi = \frac{\alpha}{2}|\nabla u|^2 + F(u)$ :

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(Z)\frac{\partial u}{\partial t}) = \mathrm{div}(B(Z)\nabla w)$$

$$w - b(Z) \cdot \nabla w = \beta(Z)\frac{\partial u}{\partial t} - \alpha\Delta u + f(u)$$

Isotropic case ($\beta > 0$, $a = b = 0$, $B = \kappa I$, $\kappa > 0$) :

$$\frac{\partial u}{\partial t} = \kappa \Delta w$$

$$w = \beta \frac{\partial u}{\partial t} - \alpha \Delta u + f(u)$$

$\rightarrow$ Viscous Cahn–Hilliard equation

Constant constitutive moduli :

$$\frac{\partial u}{\partial t} - a \cdot \nabla \frac{\partial u}{\partial t} = \operatorname{div}(B \nabla w)$$

$$w - b \cdot \nabla w = \beta \frac{\partial u}{\partial t} - \alpha \Delta u + f(u)$$

$$\beta x^2 + (a+b) \cdot yx + By \cdot y \geq 0, \ \forall x \in \mathbb{R}, \ y \in \mathbb{R}^n$$

Multiply the first equation by $w$ and the second one by $\frac{\partial u}{\partial t}$ (periodic boundary conditions) :

$$((\frac{\partial u}{\partial t}, w)) + ((\frac{\partial u}{\partial t}, a \cdot \nabla w)) = -((B\nabla w, \nabla w))$$

$$((\frac{\partial u}{\partial t}, w)) - ((\frac{\partial u}{\partial t}, b.\nabla w)) = \beta \|\frac{\partial u}{\partial t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{d}{dt}\int_\Omega F(u)\, dx$$

Thus :

$$\frac{d}{dt}(\frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\, dx)$$

$$+\beta\|\frac{\partial u}{\partial t}\|^2 + ((\frac{\partial u}{\partial t}, (a+b)\cdot \nabla w)) + ((B\nabla w, \nabla w)) = 0$$

Dissipation inequality :

$$\frac{d}{dt}(\frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\, dx) \le 0$$

(total free energy decay)

Stronger coercivity relation :

$$\beta x^2 + (a+b)\cdot yx + By\cdot y \geq c_0(x^2+|y|^2),\ \forall x\in\mathbb{R},\ y\in\mathbb{R}^n,\ c_0>0$$

Then :

$$\frac{d}{dt}(\frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,dx) + c_0(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla w\|^2) \leq 0$$

Useful in view of the mathematical analysis

Mathematical analysis : periodic boundary conditions and regular nonlinear terms mostly

Extensions :

• Elastic effects

• Multicomponent alloys

• Thermal effects :

$$\frac{\partial u}{\partial t} = \mathrm{div}(A\nabla \frac{w}{\theta} + B\nabla \frac{1}{\theta})$$

$$\frac{\partial e}{\partial t} = -\mathrm{div}(C\nabla \frac{w}{\theta} + D\nabla \frac{1}{\theta} - \alpha(u,\theta)\frac{\partial u}{\partial t}\nabla u),\ \alpha > 0$$

$$w = 2c(\theta - \theta_c)u + \frac{1}{2}\partial_u \alpha(u,\theta)|\nabla u|^2 - \mathrm{div}(\alpha(u,\theta)\nabla u) + f(u),\ c,\ \theta_c > 0$$

$$e = c_V\theta - c\theta_c u^2 + \frac{1}{2}(\alpha(u,\theta) - \partial_\theta \alpha(u,\theta))|\nabla u|^2 + F(u),\ c_V > 0$$

$\theta$ : absolute temperature
$e$ : internal energy
$A$, $B$, $C$, $D$ : matrices such that $A$ and $D$ are positive semidefinite

Lead to mathematically challenging problems

Simplest problem ($A = D = I$, $B = C = 0$, $\alpha = 1$ constant, $\theta_c = 0$, $c_V = 1$, $c = 0$) :

$$\frac{\partial u}{\partial t} = \Delta \frac{w}{\theta}$$

$$w = -\Delta u + f(u)$$

$$\frac{\partial \theta}{\partial t} + \Delta \frac{1}{\theta} = -f'(u)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t}\Delta u$$

Set $\chi = \frac{w}{\theta}$ :

$$\frac{\partial u}{\partial t} = \Delta \chi$$

$$\theta \chi = -\Delta u + f(u)$$

$$\frac{\partial \theta}{\partial t} + \Delta \frac{1}{\theta} = -\theta \chi \Delta \chi$$

Energy conservation :

$$\frac{d}{dt} \int_{\Omega} (\frac{1}{2} |\nabla u|^2 + F(u) + \theta)\, dx = 0$$

No further result/estimate

Different approach : entropy principle (H.W. Alt-I. Pawlow)

Similar equations

Move convenient for the mathematical analysis :

$$\frac{w}{\theta} = -\Delta \frac{u}{\theta} + \frac{1}{\theta} f(u)$$

**Remark :** Conserved Caginalp phase-field system :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\theta$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t}$$

• Dynamic boundary conditions

A simple example (G.R. Goldstein-A. Miranville) :

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(Z)\frac{\partial u}{\partial t}) = \operatorname{div}(B(Z)\nabla w)$$

$$w - b(Z) \cdot \nabla w = \beta(Z)\frac{\partial u}{\partial t} - \alpha\Delta u + f(u)$$

$$\beta(Z)x^2 + (a(Z) + b(Z)) \cdot yx + (B(Z)y) \cdot y \geq 0,\ \forall x \in \mathbb{R},\ y \in \mathbb{R}^n,\ \forall Z$$

Integrate the first equation over $\Omega$ :

$$\int_\Omega \frac{\partial u}{\partial t}\, dx - \int_\Gamma [(a(Z) \cdot \nu)\frac{\partial u}{\partial t} + (B(Z)\nabla w) \cdot \nu]\, d\sigma = 0$$

Dynamic boundary condition :

$$\frac{\partial u}{\partial t} + (a(Z) \cdot \nu)\frac{\partial u}{\partial t} = \delta\Delta_\Gamma w - (B(Z)\nabla w) \cdot \nu \text{ on } \Gamma,\ \delta \geq 0$$

Ensures the conservation of mass :

$$\frac{d}{dt}(\int_\Omega u\,dx + \int_\Gamma u\,d\sigma) = 0$$

Total free energy :

$$\Psi = \Psi_\Omega + \Psi_\Gamma$$

$$\Psi_\Omega = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))\,dx$$

$$\Psi_\Gamma = \int_\Gamma (\frac{\eta}{2}|\nabla_\Gamma u|^2 + G(u))\,d\sigma,\ \eta \geq 0$$

Set $w = \partial_u \Psi$ (usual Cahn-Hilliard equation) :

$$w = -\eta\Delta_\Gamma u + \alpha\frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma,\ \eta \geq 0$$

Natural generalization for isotropic materials ($b = 0$, $B = \kappa(Z)I$, $\kappa \geq 0$) :

$$w = \beta(Z)\frac{\partial u}{\partial t} - \eta\Delta_\Gamma u + \alpha\frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma,\ \eta \geq 0$$

(accounts for the work of the microforces on the boundary)

Anisotropic materials :

$$w + (b(Z)\cdot\nu)w = \beta(Z)\frac{\partial u}{\partial t} - \eta\Delta_\Gamma u + \alpha\frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma,\ \eta \geq 0$$

Constant moduli ($B = \kappa I,\ \kappa > 0, \delta > 0, \eta > 0$) :

$$\frac{\partial u}{\partial t} - a \cdot \nabla \frac{\partial u}{\partial t} = \Delta w$$

$$w - b \cdot \nabla w = \beta \frac{\partial u}{\partial t} - \Delta u + f(u)$$

$$\frac{\partial u}{\partial t} + (a \cdot \nu) \frac{\partial u}{\partial t} = \Delta_\Gamma w - \frac{\partial w}{\partial \nu} \text{ on } \Gamma$$

$$w + (b \cdot \nu) w = \beta \frac{\partial u}{\partial t} - \Delta_\Gamma u + \frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

$$\beta x^2 + (a+b) \cdot yx + |y|^2 \geq c_0(x^2 + |y|^2),\ \forall x \in \mathbb{R},\ y \in \mathbb{R}^n,\ c_0 > 0$$

Well-posedness : $f(s) = s^3 - s,\ g(s) = s + \lambda,\ \lambda \in \mathbb{R}$

Mathematical formulation :

$$\frac{\partial U}{\partial t} - \mathcal{B}_a \frac{\partial U}{\partial t} = -AW \text{ in } \mathcal{V}'$$

$$W - \mathcal{B}_b W = \beta \frac{\partial U}{\partial t} + AU + \mathcal{F}(U) \text{ in } \mathcal{V}'$$

$$U = \begin{pmatrix} u \\ u|_\Gamma \end{pmatrix}, \; AU = \begin{pmatrix} -\Delta u \\ -\Delta_\Gamma u|_\Gamma + \frac{\partial u}{\partial \nu}|_\Gamma \end{pmatrix}, \; W = \begin{pmatrix} w \\ w|_\Gamma \end{pmatrix}$$

$$\mathcal{F}(U) = \begin{pmatrix} f(u) \\ g(u|_\Gamma) \end{pmatrix}$$

$$\mathcal{B}_c U = \begin{pmatrix} c \cdot \nabla u \\ -(c \cdot \nu) v \end{pmatrix}, \; \forall U = \begin{pmatrix} u \\ v \end{pmatrix}$$

Conservation of mass :

$$\langle U(t) \rangle = \langle U(0) \rangle$$

$$\langle U \rangle = \frac{1}{\int_\Omega dx + \int_\Gamma d\sigma} \left( \int_\Omega u\, dx + \int_\Gamma v\, d\sigma \right),\ U = \begin{pmatrix} u \\ v \end{pmatrix}$$

Furthermore :

$$\langle W \rangle = \langle \mathcal{F}(P) \rangle$$

Matrix formulation :

$$\mathcal{M} \begin{pmatrix} \frac{\partial U}{\partial t} \\ W \end{pmatrix} + \mathcal{N} \begin{pmatrix} U \\ 0 \end{pmatrix} + \mathcal{G}(U) = 0$$

$$\mathcal{M} = \begin{pmatrix} \beta I & -(I - \mathcal{B}_b) \\ I - \mathcal{B}_a & A \end{pmatrix}$$

$$\mathcal{N} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{G}(U) = \begin{pmatrix} \mathcal{F}(U) - \langle \mathcal{F}(P) \rangle + \mathcal{B}_b \langle \mathcal{F}(U) \rangle \\ 0 \end{pmatrix}$$

We have

$$\langle \mathcal{M} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \rangle = \beta \|U\|^2_{\mathcal{H}} - ((b \cdot \nabla u, \tilde{u})) + ((u, a \cdot \nabla \tilde{u})) + \|\tilde{U}\|^2_{\dot{\mathcal{V}}}$$

Thus :

$$\langle \mathcal{M} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \rangle = \beta \|U\|_{\mathcal{H}}^2 + ((u, (a+b) \cdot \nabla \tilde{u})) + \|\tilde{U}\|_{\dot{\mathcal{V}}}^2 - (((b \cdot \nu) u, \tilde{u}))_\Gamma$$

Coercivity assumption :

$$\langle \mathcal{M} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \rangle \geq c_1 (\|U\|_{\mathcal{H}}^2 + \|\tilde{U}\|_{\dot{\mathcal{V}}}^2), \ \forall U, \ \tilde{U} \in \mathcal{V}_0, \ c_1 > 0$$

$b = 0$ : follows from the dissipation inequality

Also holds when $b$ is small :

$$\langle \mathcal{M} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \rangle \geq c_0 (\|u\|^2 + \|\nabla \tilde{u}\|^2) + \beta \|u\|_\Gamma^2 + \|\nabla_\Gamma \tilde{u}\|_\Gamma^2 - (((b \cdot \nu) u, \tilde{u}))_\Gamma.$$

**Remark :** No anisotropy on the boundary :

$$w = \beta \frac{\partial u}{\partial t} - \Delta_\Gamma u + \frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma$$

No need for a stronger coercivity relation

**Theorem :** We assume that $U(0) \in \dot{\mathcal{V}}$. Then, the problem possesses a unique (weak) solution $(U, W)$ such that
$U \in L^\infty(0, T; \dot{\mathcal{V}}) \cap L^\infty(0, T; L^4(\Omega) \times L^2(\Gamma))$, $\frac{\partial U}{\partial t} \in L^2(0, T; \dot{\mathcal{H}})$ and
$W \in L^2(0, T; \mathcal{V})$, $\forall T > 0$.

**Hyperbolic relaxation of the Cahn-Hilliard equation :**

Equations :

$$\beta\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0,\ \beta>0$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \Delta u}{\partial \nu}=0 \text{ on } \Gamma$$

$$u|_{t=0}=u_0,\ \frac{\partial u}{\partial t}|_{t=0}=u_1$$

No mass conservation :

$$\beta\frac{d^2\langle u\rangle}{dt^2}+\frac{d\langle u\rangle}{dt}=0$$

$$\beta\frac{d\langle u\rangle}{dt}+\langle u\rangle=\langle \beta u_1+u_0\rangle$$

Thus :

$$\frac{d}{dt}(e^{\frac{t}{\beta}}\langle u\rangle) = \frac{1}{\beta}e^{\frac{t}{\beta}}\langle \beta u_1 + u_0\rangle$$

$$\langle u(t)\rangle = \langle u_0\rangle e^{-\frac{t}{\beta}} + \langle \beta u_1 + u_0\rangle(1 - e^{-\frac{t}{\beta}}),\ t \geq 0$$

Note that

$$\lim_{\beta\to 0^+}\langle u(t)\rangle = \langle u_0\rangle$$

Cubic nonlinear term :

1D : the problem is well understood

2-3D : find a good notion of a solution

One possibility : finite energy solution (total free energy belongs to $L^\infty(0,T)$, $T>0$)

No uniqueness in 3D

Logarithmic (singular nonlinear terms) : existence of weak solutions (no uniqueness) for the hyperbolic relaxation of the viscous Cahn-Hilliard equation

Hyperbolic relaxation of the Allen-Cahn equation (weakly damped wave equation) :

$$
\begin{aligned}
&\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = g, \ \epsilon > 0 \\
&u = 0 \text{ on } \Gamma \\
&u|_{t=0} = u_0, \ \frac{\partial u}{\partial t}|_{t=0} = u_1
\end{aligned}
$$

For simplicity : $g = g(x) \in L^\infty(\Omega)$

Here : $u_0 \in L^\infty(\Omega)$, $\|u_0\|_{L^\infty(\Omega)} < 1$

Existence of strong solutions only (when $\epsilon > 0$ is small and the initial data are not too large)

We are not able to prove the existence of weak solutions

Main ingredients :

• Perturbation argument : the solutions remain close to those of the limit parabolic problem

• Dissipativity provided by the equation

**Theorem :** There exist $\epsilon_0 > 0$ and a monotone decreasing function $R : (0, \epsilon_0] \to R^+$ satisfying

$$\lim_{\epsilon \to 0^+} R(\epsilon) = +\infty$$

s.t., for every initial data satisfying

$$D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq R(\epsilon),$$

there exists a unique global solution s.t.

$$\begin{aligned}
&D(u(t)) + \|u(t)\|^2_{H^2(\Omega)} + \epsilon\|\tfrac{\partial u}{\partial t}(t)\|^2_{H^1(\Omega)} + \|\tfrac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} \\
&+ \int_0^t e^{-\alpha(t-s)}\|\tfrac{\partial u}{\partial t}(s)\|^2_{H^1(\Omega)}\, ds \\
&\leq Q(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}})e^{-\alpha t} \\
&+Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0,
\end{aligned}$$

where $\alpha$ and $Q$ are independent of $\epsilon$ and $D(v) = \frac{1}{1-\|v\|_{L^\infty(\Omega)}}$.

Uniqueness : standard ($f' \geq -c_0$, $c_0 \geq 0$)

Existence : follows the following steps :

**Step 1 :** Dissipative estimate in $H^1(\Omega) \times L^2(\Omega)$ :

$$\begin{aligned}
&\|u(t)\|^2_{H^1(\Omega)} + \epsilon\|\frac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} + \int_0^t e^{-\alpha(t-s)}\|\frac{\partial u}{\partial t}(s)\|^2_{H^1(\Omega)}\, ds \\
&\leq Q(D(u_0) + (\|u_0\|^2_{H^1(\Omega)} + \epsilon\|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}})e^{-\alpha t} \\
&+Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0
\end{aligned}$$

$\alpha$ and $Q$ independent of $\epsilon$

**Step 2 :** Consider the limit parabolic problem ($\epsilon = 0$) :

$$\begin{array}{l} \frac{\partial u^0}{\partial t} - \Delta u^0 + f(u^0) = g \\ u^0 = 0 \text{ on } \Gamma \\ u^0|_{t=0} = u_0 \end{array}$$

We have :

$$D(u^0(t)) + \|u^0(t)\|^2_{H^2(\Omega)} \leq Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)})e^{-\alpha t} + Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0$$

**Step 3 :** Compare the solution to the hyperbolic problem to that to the limit parabolic problem :

$$\|u(t) - u^0(t)\|^2_{L^2(\Omega)} \leq \epsilon(Q(D(u_0) + \|u_0\|^2_{H^1(\Omega)} + \epsilon\|u_1\|^2_{L^2(\Omega)})e^{-\alpha t} + Q(\|g\|_{L^\infty(\Omega)}))$$

$\alpha > 0$ and $Q$ independent of $\epsilon$

**Step 4 :** Multiply the equation by $-\Delta(\beta u + \frac{\partial u}{\partial t})$, $\beta > 0$ small enough :

$$\frac{dE_u(t)}{dt} + \beta E_u(t) + \frac{\beta}{2}(\|\Delta u(t)\|^2_{L^2(\Omega)} + \|\nabla \frac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)}) \leq c\|f(u(t))\|^2_{H^1(\Omega)}$$

where

$$\begin{aligned} E_u(t) = &\epsilon\|\nabla \tfrac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} + \beta\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u(t)\|^2_{L^2(\Omega)} \\ &-2((g, \Delta u(t)))_{L^2(\Omega)} + \beta\epsilon((\nabla u(t), \nabla \tfrac{\partial u}{\partial t}(t)))_{L^2(\Omega)} \end{aligned}$$

**Step 5 :** Estimate $\|f(u(t))\|_{H^1(\Omega)}$

We have :

$$\|f(u(t)) - f(u^0(t))\|^2_{H^1(\Omega)} \leq M_f\left(\frac{1}{1-\|u^0(t)\|_{L^\infty(\Omega)}-\|u(t)-u^0(t)\|_{L^\infty(\Omega)}}\right)\times$$
$$\times(1 + \|u^0(t)\|^2_{H^2(\Omega)} + \|u(t)\|^2_{H^2(\Omega)})\|u(t) - u^0(t)\|^2_{H^1(\Omega)}$$

$M_f$ : smooth monotone increasing function only depending on $f$ and satisfying

$$\lim_{z\to+\infty} M_f(z) = +\infty$$

$u^0$ : solution to the limit parabolic problem

Consider the interpolation inequalities

$$\|u(t) - u^0(t)\|_{H^1(\Omega)} \leq c\|u(t) - u^0(t)\|^{\frac{1}{2}}_{L^2(\Omega)}\|u(t) - u^0(t)\|^{\frac{1}{2}}_{H^2(\Omega)}$$
$$\|u(t) - u^0(t)\|_{L^\infty(\Omega)} \leq c\|u(t) - u^0(t)\|^{\frac{1}{4}}_{L^2(\Omega)}\|u(t) - u^0(t)\|^{\frac{3}{4}}_{H^2(\Omega)}$$

This yields

$$\|f(u(t))\|^2_{H^1(\Omega)} \leq Q_0 \epsilon^{\frac{1}{2}}(1+E_u(t))^2 M_f\Big(\frac{1}{(\overline{Q}+Q_0)^{-1}-\epsilon^{\frac{1}{8}}(\overline{Q}+Q_0)(1+E_u(t))}\Big)$$
$$+Q_0 e^{-\alpha t}+\overline{Q}$$

$Q_0 = Q_0(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}})$

$\overline{Q} = \overline{Q}(\|g\|_{L^\infty(\Omega)})$

$\alpha > 0$, $Q_0$, $\overline{Q}$ : independent of $\epsilon$

Finally :

$$\frac{dE_u(t)}{dt} + \beta E_u(t) \leq Q_0 \epsilon^{\frac{1}{2}}(1+E_u(t))^2 M_f\Big(\frac{1}{(\overline{Q}+Q_0)^{-1}-\epsilon^{\frac{1}{8}}(\overline{Q}+Q_0)(1+E_u(t))}\Big)$$
$$+2Q_0 e^{-\alpha t}+2\overline{Q} \qquad (\beta < \alpha)$$

**Step 6 :** Assume that

$$D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq R(\epsilon)$$

$R = R(\epsilon)$ solves

$$\overline{Q} = Q_0(R)\epsilon^{\frac{1}{2}}(1 + 2(\beta-\alpha)^{-1}Q_0(R) + 3\beta^{-1}\overline{Q})^2 \times \\ \times M_f(\frac{1}{(\overline{Q}+Q_0(R))^{-1} - \epsilon^{\frac{1}{8}}(\overline{Q}+Q_0(R))(1+2(\beta-\alpha)^{-1}Q_0(R)+3\beta^{-1}\overline{Q})})$$

Then :

$$E_u(t) \leq E_0(t)$$

where

$$E_0(t) = 2(\beta-\alpha)^{-1}Q_0(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}}) \times \\ \times e^{-\alpha t} + 3\beta^{-1}\overline{Q}$$

Consequence of the comparison principle :

$E_0$ satisfies

$$\frac{dE_0(t)}{dt} + \beta E_0(t) \geq Q_0 \epsilon^{\frac{1}{2}} (1 + E_0(t))^2 M_f \left( \frac{1}{(\overline{Q}+Q_0)^{-1} - \epsilon^{\frac{1}{8}} (\overline{Q}+Q_0)(1+E_0(t))} \right)$$
$$+2Q_0 e^{-\alpha t} + 2\overline{Q}$$

We can take

$$E_u(0) \leq E_0(0)$$

We conclude by noting that

$$\lim_{\epsilon \to 0^+} R(\epsilon) = +\infty$$

Further results :

Additional regularity

Existence of finite-dimensional attractors

Extension : hyperbolic relaxation of the Caginalp phase-field system