The Cahn-Hilliard-Oono equation

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The Cahn–Hilliard equation : recent advances and applications

$$\frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) + \beta u = 0,\ \beta > 0$$

Models long-ranged interactions

Based on the free energy :

$$\Psi_\Omega = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u) + \int_\Omega u(y)k(x,y)u(x)\ dy)\ dx$$

$$k(x,y) = \frac{\beta}{4\pi|x-y|}$$

Evolution equation :

$$\frac{\partial u}{\partial t} = \kappa\Delta\partial_u\Psi_\Omega$$

$$\Delta(-k(x,y)) = d_i(x-y)$$

$$\frac{\partial u}{\partial t} + \beta u + \Delta^2 u - \Delta f(u) = 0,\ \beta > 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

No mass conservation :

$$\frac{d\langle u\rangle}{dt} + \beta\langle u\rangle = 0$$

$\langle\cdot\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \cdot\, dx$ or $\frac{1}{\mathrm{Vol}(\Omega)}\langle\cdot, 1\rangle_{H^{-1}(\Omega),H^1(\Omega)}$

Thus :

$$\langle u(t)\rangle = e^{-\beta t}\langle u_0\rangle,\ t \geq 0$$

$\rightarrow$ Conservation of mass only when $\langle u_0\rangle = 0$

Note that

$$|\langle u(t)\rangle| \leq |\langle u_0\rangle|,\ t \geq 0$$

$$\lim_{t\to+\infty} \langle u(t)\rangle = 0\ (\beta > 0 \text{ fixed})$$

$$\lim_{\beta\to 0^+} \langle u(t)\rangle =< u_0 >\ \ (t \geq 0 \text{ fixed})$$

Simplest equation for which we do not have the conservation of mass

Already brings several difficulties

Cubic nonlinear term :

$f(s) = s^3 - s$ (can be generalized)

Assume that $|\langle u_0 \rangle| \leq M, \ M \geq 0$

A priori :

$$|\langle u(t) \rangle| \leq M, \ \forall t \geq 0$$

A priori estimates :

Equivalent equation :

$$\frac{\partial \overline{u}}{\partial t} + \beta \overline{u} + \Delta^2 u - \Delta f(u) = 0$$

$$\overline{u} = u - \langle u \rangle$$

Weak formulation :

$$(-\Delta)^{-1}\frac{\partial\overline{u}}{\partial t}+\beta(-\Delta)^{-1}\overline{u}-\Delta u+\overline{f(u)}=0$$

Multiply by $\overline{u}$:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^{2}+\beta\|\overline{u}\|_{-1}^{2}+\|\nabla u\|^{2}+((f(u),\overline{u}))=0$$

Hölder's inequality :

$$((f(u),u))\geq\frac{3}{4}\|u\|_{L^4(\Omega)}^{4}-c$$

$$|\langle u\rangle\int_{\Omega}f(u)\,dx|\leq\frac{1}{4}\|u\|_{L^4(\Omega)}^{4}+c_M$$

Thus :

$$\frac{d}{dt}\|\overline{u}\|_{-1}^{2}+\|\nabla u\|^{2}+\|u\|_{L^4(\Omega)}^{4}\leq c_M$$

Finally :

$$\|\overline{u}\|_{-1} \leq c\|\nabla u\|,$$

Thus :

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + c\|\overline{u}\|_{-1}^2 + \frac{1}{2}\|\nabla u\|^2 + \|u\|_{L^4(\Omega)}^4 \leq c'_M,\ c > 0$$

All constants are independent of β

Note that :

$$\frac{d\langle u\rangle^2}{dt} = 2\langle u\rangle\frac{d\langle u\rangle}{dt} = -2\beta\langle u\rangle^2 \leq 0$$

Thus :

$$\frac{d}{dt}(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2) + c(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2) + \frac{1}{2}\|\nabla u\|^2 + \|u\|_{L^4(\Omega)}^4 \leq c'_M,\ c > 0$$

$\rightarrow$ Dissipative estimate on $\|\overline{u}\|_{-1}^2 + \langle u \rangle^2$:

$$\|\overline{u}(t)\|_{-1}^2 + \langle u(t) \rangle^2 \leq e^{-ct}(\|\overline{u}_0\|_{-1}^2 + \langle u_0 \rangle^2) + c', \ c > 0$$

Bounded absorbing set in $H^{-1}(\Omega)$: $\forall R > 0$, $\|u_0\|_{H^{-1}(\Omega)} \leq R$,
$\exists t_0 = t_0(R) \geq 0$ such that $t \geq t_0$ implies

$$\|u(t)\|_{H^{-1}(\Omega)} \leq c$$

Furthermore :

$$\int_t^{t+r} \|\nabla u\|^2 \, ds \leq c_r, \ t \geq t_0, \ r > 0$$

$$\int_t^{t+r} \|u\|_{L^4(\Omega)}^4 \, ds \leq c_r, \ t \geq t_0, \ r > 0$$

Multiply by $-\Delta u$ ($f' \geq -1$) :

$$\frac{d}{dt}\|\overline{u}\|^2 + \|\Delta u\|^2 \leq 2\|\nabla u\|^2$$

Note that

$$\|u\|^2 \leq 2(\|\overline{u}\|^2 + \langle u\rangle^2) \leq c(\|\nabla u\|^2 + M^2),$$

Thus :

$$\|u(t)\| \leq c,\ t \geq t_1(\geq t_0)$$

$$\int_t^{t+r} \|\Delta u\|^2\, ds \leq c_r,\ t \geq t_1$$

$\rightarrow$ Existence of a bounded absorbing set in $L^2(\Omega)$

Multiply by $\frac{\partial \overline{u}}{\partial t}$:

$$\frac{d}{dt}(\beta\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + 2\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + 2\mathrm{Vol}(\Omega)\langle f(u)\rangle\langle\frac{\partial u}{\partial t}\rangle = 0$$

$$F(s) = \int_0^s f(\xi)\,d\xi$$

Young's inequality :

$$|2\mathrm{Vol}(\Omega)\langle f(u)\rangle\langle\frac{\partial u}{\partial t}\rangle| \le c\beta M(\int_\Omega u^4\,dx + 1)$$

Thus :

$$\frac{d}{dt}(\beta\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + \|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 \le c\beta M(\int_\Omega u^4\,dx + 1)$$

Uniform Gronwall's lemma :

$$\|u(t)\|_{H^1(\Omega)} \leq c,\ t \geq t_2(\geq t_1)$$

$\rightarrow$ Existence of a bounded absorbing set in $H^1(\Omega)$

Upper bounds : depends en β (depends continuously on β)

We can obtain more regularity

The dissipative semigroup :

Existence : follows from the a priori estimates

Uniqueness :

Let u_1, u_2 be two solutions with initial data $u_{1,0}$, $u_{2,0}$

Set $u = u_1 - u_2$, $u_0 = u_{1,0} - u_{2,0}$:

$$\frac{\partial u}{\partial t} + \beta u + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

Equivalent formulation :

$$\frac{\partial \overline{u}}{\partial t} + \beta \overline{u} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = 0$$

Multiply by $(-\Delta)^{-1}\overline{u}$:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^{2}+\beta\|\overline{u}\|_{-1}^{2}+\|\nabla u\|^{2}+((f(u_1)-f(u_2),\overline{u}))=0$$

Note that

$$((f(u_1)-f(u_2),\overline{u}))\geq -\|u\|^{2}-\langle u\rangle\int_{\Omega}(f(u_1)-f(u_2))\,dx$$

$$\|u\|^{2}\leq 2(\|\overline{u}\|^{2}+\langle u\rangle^{2})\leq c(\|\overline{u}\|_{-1}\|\nabla u\|+\langle u\rangle^{2})$$

$$\leq\gamma\|\nabla u\|^{2}+c(\|\overline{u}\|_{-1}^{2}+\langle u\rangle^{2}),\ \forall\gamma>0$$

Thus :

$$|\langle u\rangle\int_{\Omega}(f(u_1)-f(u_2))\,dx|\leq c|\langle u\rangle|\int_{\Omega}(|u_1|^{2}+|u_2|^{2}+1)|u|\,dx$$

$$\leq c(\|u\|^{2}+(\|u_1\|_{L^4(\Omega)}^{4}+\|u_2\|_{L^4(\Omega)}^{4}+1)\langle u\rangle^{2})$$

$$\leq\frac{1}{4}\|\nabla u\|^{2}+c(\|u_1\|_{L^4(\Omega)}^{4}+\|u_2\|_{L^4(\Omega)}^{4}+1)(\|\overline{u}\|_{-1}^{2}+\langle u\rangle^{2})$$

Thus :

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 \leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2)$$

Note that $\frac{d\langle u\rangle^2}{dt} \leq 0$:

$$\frac{d}{dt}(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2) + \|\nabla u\|^2 \leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2)$$

Gronwall's lemma :

$$\|u(t)\|_{H^{-1}(\Omega)} \leq ce^{c't}\|u_0\|_{H^{-1}(\Omega)}$$

The constants are independent of β

$\rightarrow$ Continuous dependence and uniqueness

We can define the continuous semigroups :

$$S_\beta(t) : L^2(\Omega) \to L^2(\Omega),\ u_0 \mapsto u(t),\ t \geq 0,\ \beta > 0$$
$$S_\beta(t) : L^2(\Omega) \to H^1(\Omega),\ t > 0$$

Set :

$$\Phi_M = \{v \in L^2(\Omega),\ |\langle v \rangle| \leq M\},\ M \geq 0$$

$$S_\beta(t) : \Phi_M \to \Phi_M,\ t \geq 0$$

Theorem : The semigroup $S_\beta(t)$ possesses the finite-dimensional (for the $H^{-1}(\Omega)$-topology) global attractor $\mathcal{A}_\beta^M$ on the phase space Φ_M which is compact in $L^2(\Omega)$ and bounded in $H^1(\Omega)$.

Finite fractal dimensionality :

Multiply

$$\frac{\partial u}{\partial t} + \beta u + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = 0$$

by tu :

$$\frac{d}{dt}(t\|u\|^2) \leq \|u\|^2 + ct\|\nabla u\|^2$$

Thus :

$$\|u(t)\|^2 \leq c\frac{1+t}{t}\int_0^t \|u\|^2_{H^1(\Omega)}\, ds,\ t > 0$$

Furthermore :

$$\int_0^t \|\nabla u\|^2\, ds \le c e^{c' t} \|u_0\|^2_{H^{-1}(\Omega)}$$

$$\|u\|^2 \le c(\|\nabla u\|^2 + \langle u \rangle^2)$$

$$\langle u \rangle^2 \le \langle u_0 \rangle^2$$

Thus :

$$\|u(t)\|^2 \le c\frac{1+t}{t} e^{c' t} \|u_0\|^2_{H^{-1}(\Omega)}$$

All constants are independent of β

Remarks :

a) The bounds on, as well as the upper bound on the fractal dimension of, $\mathcal{A}_\beta^M$ are independent of β, $\beta \leq \beta_0$

b) We can extend the semigroups to

$$\tilde{\Phi}_M = \{v \in H^{-1}(\Omega),\ |\langle v \rangle| \leq M\},\ M \geq 0$$

Note that

$$S_\beta(t) : \tilde{\Phi}_M \to \Phi_M,\ t > 0$$

Definition : A compact set $\mathcal{M} \subset E$ is an exponential attractor for the semigroup $S(t)$ acting on the Banach space E if

(i) It has finite fractal dimension, $\dim_F \mathcal{M} < +\infty$.

(ii) It is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geq 0$.

(iii) It attracts exponentially fast the bounded subsets of E in the following sense :

$$\forall B \subset E \text{ bounded},\ \mathrm{dist}_E(S(t)B, \mathcal{M}) \leq Q(\|B\|_E)e^{-\alpha t},\ t \geq 0,$$

where the positive constant α and the function Q are independent of B.

Construction of robust exponential attractors

Let E and E_1 be two Banach spaces such that the embedding $E_1 \subset E$ is compact, X be a bounded subset of E and $L_\epsilon : X \to X$, $\epsilon \in [0, \epsilon_0]$, $\epsilon_0 > 0$, be a family of operators such that

1. For every $x_1, x_2 \in X$ and every $\epsilon \in [0, \epsilon_0]$,

$$\|L_\epsilon x_1 - L_\epsilon x_2\|_{E_1} \leq c\|x_1 - x_2\|_E,$$

where the constant c is independent of ϵ.

2. For every $\epsilon \in [0, \epsilon_0]$, every $i \in \mathbb{N}$ and every $x \in X$,

$$\|L_\epsilon^i x - L_0^i x\|_E \leq c^i \epsilon,$$

where the constant c is independent of ϵ.

Then, there exists a family $\mathcal{M}_\epsilon \subset X$, $\epsilon \in [0, \epsilon_0]$, such that $\mathcal{M}_\epsilon$ is an exponential attractor for the discrete dynamical system generated by L_ϵ, i.e.,

(i) The set $\mathcal{M}_\epsilon$ is compact and has finite fractal dimension in E,

$$\dim_{\mathrm{F}} \mathcal{M}_\epsilon \leq c.$$

(ii) The set $\mathcal{M}_\epsilon$ is positively invariant,

$$L_\epsilon \mathcal{M}_\epsilon \subset \mathcal{M}_\epsilon.$$

(iii) The set $\mathcal{M}_\epsilon$ attracts X exponentially fast,

$$\mathrm{dist}_E(L^i X, \mathcal{M}_\epsilon) \leq c e^{-c' i},\ i \in \mathbb{N},\ c' > 0.$$

(iv) The family $\mathcal{M}_\epsilon$ is Hölder continuous at $\epsilon = 0$,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq c\epsilon^{c'},\ c' \in (0,1),$$

where $\mathrm{dist}_{\mathrm{sym}}$ denotes the Hausdorff symmetric distance between sets defined as

$$\mathrm{dist}_{\mathrm{sym}}(A,B) = \max(\mathrm{dist}_E(A,B), \mathrm{dist}_E(B,A)).$$

Finally, all constants are independent of ϵ and can be computed explicitly.

Theorem : For every $\beta \in [0, \beta_0]$, $\beta_0 > 0$ given, the semigroup $S_\beta(t)$ acting on $\tilde{\Phi}_M$ possesses an exponential attractor $\mathcal{M}_\beta^M$ on $\tilde{\Phi}_M$ such that

1. The set $\mathcal{M}_\beta^M$ has finite fractal dimension in $H^{-1}(\Omega)$,

$$\dim_{\mathrm{F}} \mathcal{M}_\beta^M \leq c.$$

2. The set $\mathcal{M}_\beta^M$ is positively invariant by $S_\beta(t)$,

$$S_\beta(t)\mathcal{M}_\beta^M \subset \mathcal{M}_\beta^M,\ t \geq 0.$$

3. The set $\mathcal{M}_\beta^M$ attracts all bounded subsets of $\tilde{\Phi}_M$ exponentially fast, i.e., for every bounded subset B of $\tilde{\Phi}_M$, there exists a constant $c = c(B)$ such that

$$\mathrm{dist}_{H^{-1}(\Omega)}(S_\beta(t)B, \mathcal{M}_\beta^M) \leq c e^{-c' t},\ t \geq 0,\ c' > 0.$$

4. The family of sets $\mathcal{M}_\beta^M$ is Hölder continuous at 0,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\beta^M, \mathcal{M}_0^M) \leq c\beta^{c'},\ c' \in (0,1).$$

Furthermore, all constants are independent of β and can be computed explicitly.

Existence of a uniform (with respect to β) bounded absorbing set $B_0 \subset \Phi_M \cap H^1(\Omega)$, i.e., $\forall B \subset \Phi_M$ bounded, $\exists t_0 = t_0(B) > 0$ independent of $\beta \in [0, \beta_0]$ such that

$$S_\beta(t)B \subset B_0,\ t \geq t_0,\ \beta \in [0, \beta_0]$$

$\rightarrow$ It suffices to construct the exponential attractor $\mathcal{M}_\beta^M$ on B_0

There exists $t_1 > 0$ independent of $\beta \in [0, \beta_0]$ such that

$$S_\beta(t)B_0 \subset B_0,\ t \geq t_1,\ \beta \in [0, \beta_0]$$

Let u^β, u^0 be two solutions to the problem for $\beta > 0$, $\beta = 0$ with the same initial datum u_0

Set $u = u^\beta - u^0$:

$$\frac{\partial u}{\partial t} + \beta u + \Delta^2 u - \Delta(f(u^\beta) - f(u^0)) = -\beta u^0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = 0$$

Recall that

$$\frac{d\langle u^\beta\rangle}{dt} + \beta\langle u^\beta\rangle = 0 \; (\beta > 0)$$

$$\frac{d\langle u^0\rangle}{dt} = 0$$

Thus :

$$\frac{\partial \overline{u}}{\partial t} + \beta\overline{u} + \Delta^2 u - \Delta(f(u^\beta) - f(u^0)) = -\beta\overline{u^0}$$

Multiply by $(-\Delta)^{-1}\overline{u}$:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \beta\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + ((f(u^\beta) - f(u^0), \overline{u}))$$

$$= -\beta(((-\Delta)^{-\frac{1}{2}}\overline{u^0}, (-\Delta)^{-\frac{1}{2}}\overline{u}))$$

Recall that

$$\|(-\Delta)^{-\frac{1}{2}}\overline{u}\| \leq c\|\nabla u\|$$

Thus :

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 \leq c\beta^2\|\overline{u^0}\|_{-1}^2 + 2\|u\|^2 + 2\langle u\rangle \int_\Omega (f(u^\beta) - f(u^0))\, dx$$

Recall that

$$\|u\|^2 \leq 2(\|\overline{u}\|^2 + \langle u\rangle^2) \leq c(\|\overline{u}\|_{-1}\|\nabla u\| + \langle u\rangle^2)$$
$$\leq \gamma\|\nabla u\|^2 + c(\|\overline{u}\|_{-1}^2 + \langle u\rangle^2),\ \forall\gamma > 0$$

Thus :

$$|2\langle u\rangle \int_\Omega (f(u^\beta) - f(u^0))\, dx| \leq c|\langle u\rangle| \int_\Omega (|u^\beta|^2 + |u^0|^2 + 1)|u|\, dx$$
$$\leq c(\|u\|^2 + (\|u^\beta\|_{L^4(\Omega)}^4 + \|u^0\|_{L^4(\Omega)}^4 + 1)\langle u\rangle^2)$$

$$\leq \frac{1}{2}\|\nabla u\|^2 + c(\|u^\beta\|^4_{L^4(\Omega)} + \|u^0\|^4_{L^4(\Omega)} + 1)(\|\overline{u}\|^2_{-1} + \langle u\rangle^2)$$

Thus :

$$\frac{d}{dt}\|\overline{u}\|^2_{-1} \leq c\beta^2\|\overline{u^0}\|^2_{-1} + c'(\|u^\beta\|^4_{L^4(\Omega)} + \|u^0\|^4_{L^4(\Omega)} + 1)(\|\overline{u}\|^2_{-1} + \langle u\rangle^2)$$

Finally :

$$\frac{d\langle u\rangle^2}{dt} = 2\langle u\rangle(-\beta\langle u\rangle - \beta\langle u^0\rangle) \leq -2\beta\langle u\rangle\langle u^0\rangle$$

$$\leq \beta^2\langle u^0\rangle^2 + \langle u\rangle^2$$

Thus :

$$\frac{d}{dt}(\|\overline{u}\|^2_{-1} + \langle u\rangle^2) \leq c\beta^2(\|\overline{u^0}\|^2_{-1} + \langle u^0\rangle^2)$$

$$+c'(\|u^\beta\|^4_{L^4(\Omega)} + \|u^0\|^4_{L^4(\Omega)} + 1)(\|\overline{u}\|^2_{-1} + \langle u\rangle^2)$$

We have :

$$\int_0^t \|u^\beta\|_{L^4(\Omega)}^4 ds \leq ce^{c't}$$

$$\int_0^t \|u^0\|_{L^4(\Omega)}^4 ds \leq ce^{c't}$$

$$\|u^0\|_{H^{-1}(\Omega)} \leq c$$

Gronwall's lemma :

$$\|u(t)\|_{H^{-1}(\Omega)} \leq c\beta e^{c't}$$

The constants are independent of β

Set $L_\beta = S_\beta(t_1)$, take $E = H^{-1}(\Omega)$, $E_1 = L^2(\Omega)$

$\rightarrow$ Existence of a robust family of exponential attractors $\mathcal{M}_\beta^{M,d}$ for the discrete dynamical systems generated by L_β

Set

$$\mathcal{M}_\beta^M = \cup_{t\in[0,t_1]} S_\beta(t)\mathcal{M}_\beta^{M,d}$$

We need to prove that the mapping $(t,x) \mapsto S_\beta(t)x$ is Hölder continuous on $[0,t_1]\times B_0$, uniformly with respect to $\beta \in [0,\beta_0]$

Hölder (Lipschitz) continuity with respect to x : continuous dependence estimate

Hölder continuity with respect to t :

$$\|S_\beta(t+s)u_0 - S_\beta(t)u_0\|_{H^{-1}(\Omega)} \leq |s|^{\frac{1}{2}}(\int_t^{t+s} \|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} d\tau)^{\frac{1}{2}}$$

$$\leq c|s|^{\frac{1}{2}}(\int_t^{t+s} (\|u\|^2_{H^1(\Omega)} + \|f(u)\|^2) d\tau)^{\frac{1}{2}}$$

Logarithmic nonlinear terms :

Equations :

$$\frac{\partial u}{\partial t} + \beta u = \Delta\mu$$

$$\mu = -\Delta u + f(u)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

$$f(s) = -\theta_c s + \frac{\theta}{2}\ln\frac{1+s}{1-s},\ s \in (-1,1)$$

Recall that ($c_0 = \theta_c$)

$$f' \geq -c_0,\ c_0 > 0$$

$$f(s) = f_1(s) - c_0 s,\ f_1' \geq 0$$

Set $F(s) = \int_0^s f(\xi)\,d\xi$

Approximated functions $f_N \in \mathcal{C}^1(\mathbb{R})$:

$$f_N(s) = f(-1+\frac{1}{N}) + f'(-1+\frac{1}{N})(s+1-\frac{1}{N}),\ s < -1+\frac{1}{N}$$

$$f_N(s) = f(s),\ |s| \le 1-\frac{1}{N}$$

$$f_N(s) = f(1-\frac{1}{N}) + f'(1-\frac{1}{N})(s-1+\frac{1}{N}),\ s > 1-\frac{1}{N}$$

Recall that

$$f_N' \ge -c_0$$

For N large enough, $F_N(s) = \int_0^s f_N(\xi)\, d\xi$:

$$-c_1 \le F_N(s) \le c_2 f_N(s)s + c_3,\ c_2 > 0,\ c_1,\ c_2 \ge 0,\ s \in \mathbb{R}$$

$$f_N(s)s \ge c_4|f_N(s)| - c_5,\ c_4 > 0,\ c_5 \ge 0$$

c_i, $i = 1, \cdots, 5$: independent of N

$$f_N(s)(s-m) \geq c_m(|f_N(s)| + F_N(s)) - c'_m,$$
$$c_m > 0,\ c'_m \geq 0,\ s \in \mathbb{R},\ m \in (-1,1)$$

c_m, c'_m : independent of N and depend continuously on m.

Approximated problems :

$$\frac{\partial u_N}{\partial t} + \beta u_N = \Delta \mu_N$$
$$\mu_N = -\Delta u_N + f_N(u_N)$$
$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \mu_N}{\partial \nu} = 0 \text{ on } \Gamma$$
$$u_N|_{t=0} = u_0$$

A priori estimates :

Crucial step : a priori estimate independent of N on $f_N(u_N)$ in $L^2((0,T) \times \Omega)$, $T > 0$

Assume that $-1 < u_0(x) < 1$, a.e. x, and $|\langle u_0 \rangle| \leq 1 - \delta,\ \delta \in (0,1)$
δ : fixed constant

Equation for the spatial average :

$$\frac{d\langle u_N \rangle}{dt} + \beta \langle u_N \rangle = 0$$

Thus :

$$\langle u_N(t) \rangle = e^{-\beta t} \langle u_0 \rangle,\ t \geq 0$$

$$|\langle u_N(t) \rangle| \leq |\langle u_0 \rangle|,\ t \geq 0$$

$$|\langle u_N(t) \rangle| \leq 1 - \delta,\ t \geq 0$$

Equivalent formulation :

$$\frac{\partial \overline{u}_N}{\partial t} + \beta \overline{u}_N = \Delta \mu_N$$
$$\mu_N = -\Delta \overline{u}_N + f_N(u_N)$$

Multiply the first equation by μ_N :

$$\|\nabla \mu_N\|^2 + ((\frac{\partial \overline{u}_N}{\partial t}, \mu_N)) + \beta((\overline{u}_N, \mu_N)) = 0$$

Second equation :

$$((\frac{\partial \overline{u}_N}{\partial t}, \mu_N)) + \beta((\overline{u}_N, \mu_N)) = \frac{1}{2}\frac{d}{dt}\|\overline{u}_N\|_V^2 + \frac{d}{dt}\int_\Omega F_N(u_N)\,dx$$
$$-((\frac{d\langle u_N\rangle}{dt}, f_N(u_N))) + \beta\|\overline{u}_N\|_V^2 + \beta((\overline{u}_N, f_N(u_N)))$$

Note that

$$-((\frac{d\langle u_N\rangle}{dt}, f_N(u_N))) = \beta((\langle u_N\rangle, f_N(u_N)))$$

Thus :

$$\frac{d}{dt}(\|\overline{u}_N\|_V^2 + 2\int_\Omega F_N(u_N)\,dx)$$

$$+c(\|\overline{u}_N\|_V^2 + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla\mu_N\|^2) \leq c',\ c > 0$$

Multiply the second equation by $\overline{u}_N$:

$$\|\overline{u}_N\|_V^2 + c(\|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\,dx) \leq c' + ((\mu_N, \overline{u}_N)) = c' + ((\overline{\mu}_N, \overline{u}_N))$$

$$\leq c' + c''\|\overline{u}_N\|_V\|\nabla\mu_N\|$$

Thus :

$$\|\overline{u}_N\|_V^2 + c(\|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\,dx)$$

$$\leq c' + c''(\|\nabla\mu_N\|^2 + \|\overline{u}_N\|_V^2),\ c > 0$$

Combine the two estimates :

$$\frac{dE_N}{dt} + c(E_N + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla \mu_N\|^2) \leq c',\ c > 0$$

$$E_N = \|\overline{u}_N\|_V^2 + 2\int_\Omega F_N(u_N)\,dx$$

Equivalent equation :

$$(-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t} + \beta(-\Delta)^{-1}\overline{u}_N = -\overline{\mu}_N$$

Multiply by $\frac{\partial \overline{u}_N}{\partial t}$:

$$\frac{\beta}{2}\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 = -((\overline{\mu}_N, \frac{\partial \overline{u}_N}{\partial t}))$$

Note that

$$|((\overline{\mu}_N, \frac{\partial \overline{u}_N}{\partial t}))| \leq \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}\|\nabla \mu_N\|$$
$$\leq \frac{1}{4}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + c\|\nabla \mu_N\|^2$$

Thus :

$$\beta \frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 \leq c\|\nabla \mu_N\|^2$$

Note that

$$\overline{\mu}_N = -\Delta \overline{u}_N + \overline{f_N(u_N)}$$

Multiply by $-\Delta \overline{u}_N$:

$$\|\Delta \overline{u}_N\|^2 \leq c_0\|\overline{u}_N\|_V^2 - ((\overline{\mu}_N, \Delta \overline{u}_N))$$
$$\leq c_0\|\overline{u}_N\|_V^2 + \frac{1}{2}\|\Delta \overline{u}_N\|^2 + c\|\nabla \mu_N\|^2$$

Thus :

$$\|\Delta \overline{u}_N\|^2 \leq 2c_0 \|\overline{u}_N\|_V^2 + c\|\nabla \mu_N\|^2$$

Combine with the previous estimate :

$$\frac{dE_N}{dt} + c(E_N + \|\overline{u}_N\|^2_{H^2(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\nabla \mu_N\|^2) \leq c',\ c > 0$$

Note that

$$\|\overline{f_N(u_N)}\| \leq c(\|\overline{u}_N\|_{H^2(\Omega)} + \|\nabla \mu_N\|)$$

Thus :

$$\frac{dE_N}{dt} + c(E_N + \|\overline{u}_N\|^2_{H^2(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\overline{f_N(u_N)}\|^2 + \|\nabla \mu_N\|^2)$$

$$\leq c',\ c > 0$$

Finally :

$$\int_\Omega |f_N(u_N)|\,dx \leq c|\int_\Omega f_N(u_N)\bar{u}_N\,dx| + c$$

Thus :

$$|\langle f_N(u_N)\rangle| \leq c\|\bar{u}_N\|\|\overline{f_N(u_N)}\| + c'$$

Note that

$$\|f_N(u)\|^2 \leq c(\|\overline{f_N(u_N)}\|^2 + |\langle f_N(u_N)\rangle|^2)$$

Thus :

$$\|f_N(u_N)\|_{L^2(\Omega\times(0,T))}$$

$$\leq c(\|\overline{f_N(u_N)}\|_{L^2(\Omega\times(0,T))} + \|\bar{u}_N\|_{L^\infty(0,T;L^2(\Omega))}\|\overline{f_N(u_N)}\|_{L^2(\Omega\times(0,T))})$$

$$+c',\ T>0$$

Poincaré's inequality :

$$\|\overline{f_N(u_N)}\|^2_{L^2(\Omega\times(0,T))} \leq cE_N(0) + c'T$$

$$\|\overline{u}_N\|^2_{L^\infty(0,T;L^2(\Omega))} \leq c\|\overline{u}_N\|^2_{L^\infty(0,T;V)} \leq c'E_N(0) + c''T$$

Thus :

$$\|f_N(u_N)\|_{L^2(\Omega\times(0,T))} \leq c_{T,\delta}(\|u_0\|^2_{H^1(\Omega)} + 1),\ T > 0$$

Furthermore :

$$E_N(t) \leq e^{-ct}E_N(0) + c',\ c > 0,\ t \geq 0$$

$$\|u_N(t)\|_{H^1(\Omega)} \leq c_\delta e^{-c't}(\|u_0\|_{H^1(\Omega)} + 1) + c'',\ c' > 0,\ t \geq 0$$

$$\|\mu_N\|_{L^2(0,T;H^1(\Omega))} \leq c_{T,\delta}(\|u_0\|^2_{H^1(\Omega)} + 1),\ T > 0$$

The dissipative semigroup :

Theorem : We assume that u_0 is given such that $u_0 \in H^1(\Omega)$ and $-1 < u_0(x) < 1$, a.e. x, with $|\langle u_0 \rangle| < 1$. Then, the problem possesses a unique (weak) solution such that, $\forall T > 0$,

$$u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)'),$$

$$\mu \in L^2(0,T; H^1(\Omega)).$$

Furthermore, $-1 < u(x,t) < 1$, a.e. (x,t).

Set

$$\Phi_M = \{v \in H^1(\Omega) \cap L^\infty(\Omega),\ -1 < v(x) < 1,\ \text{a.e. } x \in \Omega,\ \langle v \rangle = M\},$$

$$M \in (-1, 1)$$

We can define the continuous (for the $H^{-1}(\Omega)$-norm) semigroup

$$S_\beta(t) : \Phi_M \to \Phi_M,\ u_0 \mapsto u(t),\ t \geq 0$$

$S_\beta(t)$ is dissipative in Φ_M

Regularity and separation from the pure states :

Proposition : The solution u satisfies

$$\frac{\partial u}{\partial t} \in L^{\infty}(r,+\infty;H^{-1}(\Omega)) \cap L^{2}(r,T;H^{1}(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Equation :

$$(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} + \beta(-\Delta)^{-1}\overline{u} = -\overline{\mu}$$

Differentiate with respect to time :

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial \overline{u}}{\partial t} + \beta(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} = -\frac{\partial \overline{\mu}}{\partial t}$$

$$\frac{\partial \overline{\mu}}{\partial t} = -\Delta\frac{\partial \overline{u}}{\partial t} + \overline{f'(u)\frac{\partial u}{\partial t}}$$

Multiply by $\frac{\partial \overline{u}}{\partial t}$:

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \beta\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \|\frac{\partial \overline{u}}{\partial t}\|_V^2 + ((f'(u)\frac{\partial u}{\partial t}, \frac{\partial \overline{u}}{\partial t})) = 0$$

Furthermore :

$$((f'(u)\frac{\partial u}{\partial t}, \frac{\partial \overline{u}}{\partial t})) \geq -c_0\|\frac{\partial \overline{u}}{\partial t}\|^2 + ((f'(u)\frac{d\langle u\rangle}{dt}, \frac{\partial \overline{u}}{\partial t}))$$

Thus :

$$\frac{d}{dt}\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + 2\beta\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \|\frac{\partial \overline{u}}{\partial t}\|_V^2 \leq c\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 - 2((f'(u)\frac{d\langle u\rangle}{dt}, \frac{\partial \overline{u}}{\partial t}))$$

Difficulty : handle the term

$$((f'(u)\frac{d\langle u\rangle}{dt}, \frac{\partial \overline{u}}{\partial t})) = -\beta \mathrm{Vol}(\Omega)\langle u\rangle\langle f'(u)\frac{\partial \overline{u}}{\partial t}\rangle$$

To avoid it :

$$\begin{aligned}((f'(u)\frac{\partial u}{\partial t}, \frac{\partial \overline{u}}{\partial t})) &= ((f'(u)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t})) - \frac{d\langle u\rangle}{dt}((f'(u)\frac{\partial u}{\partial t}, 1)) \\ &\geq -c_0\|\frac{\partial u}{\partial t}\|^2 - \frac{d\langle u\rangle}{dt}\frac{d}{dt}\int_\Omega F(u)\,dx \\ &= -c_0\|\frac{\partial u}{\partial t}\|^2 - \frac{d}{dt}(\frac{d\langle u\rangle}{dt}\int_\Omega F(u)\,dx) + \frac{d^2\langle u\rangle}{dt^2}\int_\Omega F(u)\,dx\end{aligned}$$

Set

$$\Lambda = \frac{1}{2}\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 - \frac{d\langle u\rangle}{dt}\int_\Omega F(u)\,dx$$

Thus :

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \|\frac{\partial \overline{u}}{\partial t}\|_V^2 \leq c_0\|\frac{\partial u}{\partial t}\|^2 - \frac{d^2\langle u\rangle}{dt^2}\int_\Omega F(u)\,dx$$

and

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \frac{1}{2}\|\frac{\partial \overline{u}}{\partial t}\|_V^2 \leq c(\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + |\frac{d\langle u\rangle}{dt}|^2) - \frac{d^2\langle u\rangle}{dt^2}\int_\Omega F(u)\,dx$$

Λ is bounded from below :

$$\Lambda \geq \frac{1}{2}\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 - c$$

The last two terms in the right-hand side are bounded from above :

$$\frac{d\Lambda}{dt} + \beta\|\frac{\partial\overline{u}}{\partial t}\|_{-1}^2 + \frac{1}{2}\|\frac{\partial\overline{u}}{\partial t}\|_V^2 \leq c(\|\frac{\partial\overline{u}}{\partial t}\|_{-1}^2 + \langle u_0\rangle^2 + E(0)^2 + 1)$$

$\rightarrow$ Uniform Gronwall's lemma ($|\langle\frac{\partial u}{\partial t}\rangle| = \beta|\langle u\rangle| \leq \beta$)

Proposition : We assume that $2 \leq p < +\infty$, when $n = 2$, and $2 \leq p \leq 6$, when $n = 3$. Then, the solution u further satisfies

$$\|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \leq c,$$

$$\|u(t)\|_{W^{2,p}(\Omega)} \leq c,$$

$\forall t \geq r$, $r > 0$ given, where the constant c depends on the $H^1(\Omega)$-norm of u_0.

Consider the elliptic equation

$$-\Delta u + f_1(u) = \mu + c_0 u, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

It suffices to prove that $\mu \in L^\infty(r,t;H^1(\Omega))$

Note that $\overline{\mu} \in L^\infty(r,t;H^1(\Omega))$

Multiply

$$-\Delta \overline{u} + \overline{f(u)} = \overline{\mu}$$

by $\overline{f(u)}$:

$$\|\overline{f(u)}\|^2 \leq c(\|\nabla u\|^2 + \|\nabla \mu\|^2)$$

$\rightarrow \overline{f(u)} \in L^\infty(r,t;L^2(\Omega))$

Finally : $\langle \mu \rangle = \langle f(u) \rangle$

Proposition : We assume that $n = 1$. Then there exists $\delta \in (0, 1)$ depending on the $H^1(\Omega)$-norm of u_0 such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \ t \geq r,$$

$r > 0$ given.

Proposition : We assume that $n = 2$. Then, the following holds for every $t \geq r$, $r > 0$ given, and for every $p \in \mathbb{N}$:

$$\|f'(u)\|_{L^p(\Omega \times (r,t))} \leq c,$$

where the constant c depends on p.

Proposition : We assume that $n = 2$. Then, the weak solution u further satisfies

$$\frac{\partial u}{\partial t} \in L^{\infty}(r, +\infty; H) \cap L^2(r, T; H^2(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Theorem : We assume that $n = 2$. Then, there exists $\delta \in (0, 1)$ depending on the $H^1(\Omega)$-norm of u_0 such that

$$\|u(t)\|_{L^{\infty}(\Omega)} \leq 1 - \delta, \ t \geq r,$$

$r > 0$ given.

Remarks :

a) Strict separation property : existence of finite-dimensional attractors, convergence to steady states

b) To avoid differentiating with respect to time : quotient differences

Useful when one cannot differentiate with respect to time (e.g. : coupling with the incompressible Navier-Stokes equations)