The Cahn-Hilliard equation in image inpainting

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The Cahn–Hilliard equation : recent advances and applications

Applications of the Cahn-Hilliard equation in image processing :

- Image denoising
- Image inpainting

A Cahn-Hilliard model for binary image inpainting (A. Bertozzi-S. Esedoglu-A. Gillette, 2007)

Image inpainting : consists in filling in parts of an image/video from the surrounding area (interpolation)

Applications : restoration of old paintings, removing scratches, altering scenes, restoration of motion pictures, ...

PDE's for image inpainting : M. Bertalmio et al. (Navier-Stokes like model)

Other models : second-order models (S. Esedoglu-J. Shen)

$$g(x,s) = \lambda_0(s - h(x))\chi_{\Omega\setminus D}(x),\ \lambda_0 > 0,\ D \subset\subset \Omega$$

Equation :

$$\frac{\partial u}{\partial t} + \epsilon\Delta^2 u - \frac{1}{\epsilon}\Delta f(u) + g(x,u) = 0,\ \epsilon > 0$$

$h(x)$: given image ($h \in L^2(\Omega)$)
D : inpainting domain (damaged region)
$f(s) = 4s^3 - 6s^2 + 2s$
$g(x,u)$: added to keep u close to the image $h(x)$ outside the inpainting region (fidelity term)

Advantages (over, e.g., $u = h$ outside D) : no regularity assumption on D, no perfect h outside D

Algorithm : dynamic two-steps algorithm involving ϵ

First step : large value of ϵ to connect the edges

Second step : small value of ϵ (depending on the mesh size) ; solution obtained in the first step as initial datum

Idea : solve the equation to steady state to construct an inpainting version $u(x)$ of $h(x)$

Advantage : fast in numerical simulations

Remark : Limit problem when $\lambda_0 = +\infty$ (g of class $\mathcal{C}^2$) :

$$\Delta(\epsilon\Delta u - \frac{1}{\epsilon}f(u)) = 0 \text{ in } D$$

$$u = h \text{ on } \partial D$$

$$\nabla u = \nabla h \text{ on } \partial D$$

$\rightarrow$ Continuation of the image gradient into the missing domain

$\rightarrow$ λ_0 large in numerical simulations

Mathematical analysis (L. Cherfils-H. Fakih-A. Miranville, IPI, SIIMS) :

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega\setminus D}(x)u = 0\\
&\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma\\
&u\big|_{t=0} = u_0
\end{aligned}$$

We take $f(s) = s^3 - s$ (more generally : $f(s) = \sum_{i=1}^{2p+1} a_i s^i$, $a_{2p+1} > 0$), $h \equiv 0$

First well-posedness result : A. Bertozzi et al.

To go further : global in time/dissipative estimate

First step : obtain an estimate in $H^{-1}(\Omega)$

$\rightarrow$ We need to estimate $\langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega u\,dx$

Classical Cahn-Hilliard equation : conservation of mass

If $|\langle u_0\rangle| \leq M$, then $|\langle u(t)\rangle| \leq M$, $t \geq 0$

Equation for $\langle u\rangle$:

$$\frac{d\langle u\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} u\,dx = 0$$

We set

$$u = \langle u \rangle + v$$

We find

$$\frac{d\langle u \rangle}{dt} + c_0 \langle u \rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega \backslash D} v \, dx, \; c_0 = \frac{\mathrm{Vol}(\Omega \backslash D)}{\mathrm{Vol}(\Omega)}$$

v is solution to

$$\begin{array}{l} \frac{\partial}{\partial t}(-\Delta)^{-1} v - \Delta v + f(\langle u \rangle + v) - \langle f(\langle u \rangle + v) \rangle \\ +(-\Delta)^{-1}(\chi_{\Omega \backslash D}(x) u - \langle \chi_{\Omega \backslash D}(x) u \rangle) = 0 \end{array}$$

$(-\Delta)^{-1}$: inverse minus Laplacian acting on functions with null average ($\langle v \rangle = 0$)

Multiply the equation by v

Use the inequality

$$
\begin{aligned}
&((f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle, v))_{L^2} \\
&= ((f(\langle u\rangle + v) - f(\langle u\rangle), v))_{L^2} \\
&\geq \frac{c_0}{2}\int_\Omega (v^4 + v^2\langle u\rangle^2)\, dx - \|v\|_{L^2}^2
\end{aligned}
$$

We obtain

$$\frac{d}{dt}\|v\|_{H^{-1}}^2 + \|\nabla v\|_{L^2}^2 + c_0\int_\Omega (v^4 + v^2\langle u\rangle^2)\, dx \leq c$$

Consequence : $\|v\|_{H^{-1}}^2 \leq e^{-ct}\|v_0\|_{H^{-1}}^2 + c',\ c > 0,\ t \geq 0$

Multiply the equation by $-\Delta v$:

$$\|v(t)\|_{L^2} \leq Q(\|u_0\|_{L^2}),\ t \geq 0$$
$$\|v(t)\|_{L^2} \leq c,\ t \geq t_0,\ t_0 > 0$$

Q : monotone increasing function
c : independent of u_0 and t, $\|u_0\|_{L^2} \leq R$, $t_0 = t_0(R)$

Equation for $\langle u \rangle$:

$$\frac{d\langle u \rangle}{dt} + c_0 \langle u \rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega \backslash D} v\, dx$$

$\to |\langle u(t) \rangle| \leq Q(\|u_0\|_{L^2}) e^{-ct} + c',\ c > 0,\ t \geq 0$

Well-posedness, further regularity

Existence of finite-dimensional attractors

Open problem : convergence of solutions to steady states

Numerical simulations : one-step algorithm with threshold

Two-step algorithm : $\epsilon = 0.1$ and then $\epsilon = 0.01$

Here : $\epsilon = 0.05$ and then threshold

If $u \geq \frac{1}{2}$, then we take $u = 1$

If $u < \frac{1}{2}$, then we take $u = 0$

When D is not "too large" : results comparable with the two-steps algorithm, computation time divided by two

Logarithmic nonlinear terms :

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega \setminus D}(x) u = 0 \\
&\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma \\
&u\big|_{t=0} = u_0
\end{aligned}$$

$f(s) = -2\theta_0 s + \theta_1 \ln \frac{1+s}{1-s},\ s \in (-1,1),\ 0 < \theta_1 < \theta_0$

For $h \neq 0$, we need $\int_{\Omega \setminus D} h \, dx = 0$

We have a local (in time) existence result

Theorem : We assume that $u_0 \in H^1(\Omega)$, $|\langle u_0 \rangle| < 1$ and $-1 < u_0(x) < 1$ a.e. $x \in \Omega$. Then, there exists $T_0 = T_0(u_0)$ and a solution u such that $u \in \mathcal{C}([0, T_0]; H^{-1}(\Omega)) \cap L^\infty(0, T_0; H^1(\Omega)) \cap L^2(0, T_0; H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0, T_0; H^{-1}(\Omega))$. Furthermore,

$$-1 < u(t, x) < 1 \text{ a.e. } (t, x) \in (0, T_0) \times \Omega.$$

We approximate the singular nonlinear term by regular ones

The approximated functions need to satisfy a (uniform) coercivity relation

We set $F_N(s) = F_{1,N}(s) - \theta_0 s^2$

$$F_{1,N}(s) = \begin{cases} \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(1-\frac{1}{N})(s-1+\frac{1}{N})^k, \ s \geq 1-\frac{1}{N}, \\ F_1(s), \ |s| \leq 1-\frac{1}{N}, \\ \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(-1+\frac{1}{N})(s+1-\frac{1}{N})^k, \ s \leq -1+\frac{1}{N}. \end{cases}$$

$F(s) = F_1(s) - \theta_0 s^2$

$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(1-\frac{1}{N})(s-1+\frac{1}{N})^k, \ s > 1-\frac{1}{N}$$

$$F_{1,N}(s) = F_1(s), \ |s| \leq 1-\frac{1}{N}$$

$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(-1+\frac{1}{N})(s+1-\frac{1}{N})^k, \ s < -1+\frac{1}{N}$$

The approximated functions $f_N = F'_N$ satisfy

- $F_N \in \mathcal{C}^4(R)$
- $f_N(0) = 0$
- $f'_N \geq -\theta_0, F_N \geq -c_1, c_1 \geq 0$
- $f_N(s)s \geq c_2(F_N(s) + |f_N(s)|) - c_3,\ c_2 > 0,\ c_3 \geq 0,\ s \in R$
- $(f_N(s+a) - f_N(a))s \geq c_4(s^4 + a^2 s^2) - c_5,\ c_4 > 0,\ c_5 \geq 0,\ s,\ a \in R$

All the constants are independent of N

Approximated problems :

$$\begin{aligned}
&\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) + \chi_{\Omega\setminus D}(x)(u_N - h) = 0\\
&\frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0 \text{ on } \Gamma\\
&u_N\big|_{t=0} = u_0
\end{aligned}$$

We have the well-posedness and the regularity of the solutions to the approximated problems

A priori estimates :

Equation for the spatial average :

$$\frac{d\langle u_N\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\backslash D} u_N\,dx = 0$$

Set $u_N = \langle u_N\rangle + \overline{u}_N$:

$$\frac{d\langle u_N\rangle}{dt} + c\langle u_N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\backslash D} \overline{u}_N\,dx$$

$c = \frac{\mathrm{Vol}(\Omega\backslash D)}{\mathrm{Vol}(\Omega)}$

$\overline{u}_N$ is solution to

$$\frac{\partial \overline{u}_N}{\partial t} + \Delta^2 \overline{u}_N - \Delta(f_N(u_N) - \langle f_N(u_N)\rangle) + \chi_{\Omega\setminus D}(x)u_N - \langle \chi_{\Omega\setminus D}(x)u_N\rangle = 0$$

$$\frac{\partial \overline{u}_N}{\partial \nu} = \frac{\partial \Delta \overline{u}_N}{\partial \nu} = 0 \text{ on } \Gamma$$

$$\overline{u}_N|_{t=0} = v_0 = u_0 - \langle u_0\rangle$$

Equivalent formulation :

$$(-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t} - \Delta \overline{u}_N + f_N(u_N) - \langle f_N(u_N)\rangle$$

$$+(-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)u_N - \langle \chi_{\Omega\setminus D}(x)u_N\rangle) = 0$$

$$\frac{\partial \overline{u}_N}{\partial \nu} = 0 \text{ on } \Gamma$$

Multiply by $\overline{u}_N$:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\overline{u}_N\|_V^2$$

$$+((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) + ((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}\overline{u}_N)) = 0$$

Note that

$$((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) = ((f_N(u_N) - f_N(\langle u_N\rangle), \overline{u}_N))$$

Thus :

$$((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) \geq c_4(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) - c$$

Furthermore :

$$|((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}\overline{u}_N))| \leq c(\|\overline{u}_N\|^2 + |\langle u_N\rangle|\|\overline{u}_N\|)$$

$$\leq \frac{c_4}{2}(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) + c$$

Thus :

$$\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\overline{u}_N\|_V^2 + c_4(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) \leq c$$

Next :

$$\frac{d\langle u_N\rangle^2}{dt} + c_6\langle u_N\rangle^2 \leq c\|\overline{u}_N\|^2$$

Thus :

$$\frac{d\langle u_N\rangle^2}{dt} + c_6\langle u_N\rangle^2 \leq \frac{c_4}{2}(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) + c$$

Sum :

$$\frac{dE_{1,N}}{dt} + c(\|u_N\|_{H^1(\Omega)}^2 + \|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) \leq c',\ c > 0$$

$$E_{1,N} = \langle u_N\rangle^2 + \|\overline{u}_N\|_{-1}^2$$

$$E_{1,N} \geq c\|u_N\|_{H^{-1}(\Omega)}^2,\ c > 0$$

Multiply the original equation by u_N :

$$\frac{d}{dt}\|u_N\|^2 + \|\Delta u_N\|^2 \leq 2c_0\|\nabla u_N\|^2 + c\|u_N\|^2$$

Combine the two estimates :

$$\frac{dE_{2,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|\bar{u}_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|\bar{u}_N\|^2) \leq c',\ c > 0$$

$$E_{2,N} = \delta_1\|u_N\|^2 + E_{1,N}$$

$$E_{2,N} \geq c\|u_N\|^2,\ c > 0$$

Equivalent equations :

$$\frac{\partial u_N}{\partial t} + \chi_{\Omega\setminus D}(x)u_N = \Delta\mu_N$$

$$\mu_N = -\Delta u_N + f_N(u_N)$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \mu_N}{\partial \nu} = 0 \text{ on } \Gamma$$

Multiply the first equation by μ_N and the second by $\frac{\partial u_N}{\partial t}$:

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + \|\nabla \mu_N\|^2 = -((u_N, \chi_{\Omega\backslash D}(x)\mu_N))$$

Multiply the second equation by $\chi_{\Omega\backslash D}(x)u_N$:

$$((u_N, \chi_{\Omega\backslash D}(x)\mu_N)) = -((\Delta u_N, \chi_{\Omega\backslash D}(x)u_N)) + \int_{\Omega\backslash D} f_N(u_N)u_N\,dx$$

Thus :

$$\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx)$$

$$+c(\|\nabla \mu_N\|^2 + \int_{\Omega\backslash D} |f_N(u_N)|\,dx + \int_{\Omega\backslash D} F_N(u_N)\,dx) \leq c'\|u_N\|^2_{H^2(\Omega)} + c'',\ c > 0$$

Combining :

$$\frac{dE_{3,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|\overline{u}_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|\overline{u}_N\|^2$$

$$+ \int_{\Omega\backslash D} |f_N(u_N)|\, dx + \int_{\Omega\backslash D} F_N(u_N)\, dx + \|\nabla\mu_N\|^2) \leq c',\ c > 0$$

$$E_{3,N} = \delta_2(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\, dx) + E_{2,N}$$

$$E_{3,N} \geq c\|u_N\|^2_{H^1(\Omega)} - c',\ c > 0$$

Equivalent equations :

$$(-\Delta)^{-1}\frac{\partial\overline{u}_N}{\partial t} + (-\Delta)^{-1}(\chi_{\Omega\backslash D}(x)u_N - \langle\chi_{\Omega\backslash D}(x)u_N\rangle) = -(\mu_N - \langle\mu_N\rangle)$$

$$\mu_N - \langle\mu_N\rangle = -\Delta\overline{u}_N + f_N(u_N) - \langle f_N(u_N)\rangle$$

Thus :

$$\|\frac{\partial \bar{u}_N}{\partial t}\|_{-1} \leq c(\|u_N\| + \|\nabla \mu_N\|)$$

and

$$\|\frac{\partial u_N}{\partial t}\|_{H^{-1}(\Omega)} \leq c(\|u_N\| + \|\nabla \mu_N\|)$$

Furthermore :

$$\|f_N(u_N) - \langle f_N(u_N)\rangle\| \leq c(\|u_N\|_{H^2(\Omega)} + \|\nabla \mu_N\|)$$

Finally :

$$\frac{dE_{3,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|\overline{u}_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2 + \|\overline{u}_N\|^2$$
$$+ \|\frac{\partial u_N}{\partial t}\|^2_{-1} + \|f_N(u_N) - \langle f_N(u_N)\rangle\|^2$$
$$+ \int_{\Omega\setminus D} |f_N(u_N)|\,dx + \int_{\Omega\setminus D} F_N(u_N)\,dx) \leq c',\ c > 0$$

$$\overline{u}_N = u_N - \langle u_N\rangle$$

$$E_{3,N} = \|u_N\|^2_{-1} + \delta_1\|u_N\|^2 + \delta_2(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx)$$
$\delta_1,\ \delta_2 > 0$ small

$$E_{3,N} \geq c\|u_N\|^2_{H^1(\Omega)} - c',\ c > 0$$

Not sufficient to pass to the limit

We need to estimate $|\langle f_N(u_N)\rangle|$ (hence an estimate on the L^2-norm of $f_N(u_N)$)

To do so, we need an estimate of the form $|\langle u_N\rangle| \leq 1-\delta$, $\delta \in (0,1)$ independent of N

We can prove this only locally in time

We assume that $|\langle u_0\rangle| \leq 1-2\delta$, $\delta > 0$ given

We have

$$\frac{d\langle u_N\rangle}{dt} + c\langle u_N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} \overline{u}_N \, dx$$

This yields

$$\langle u_N(t)\rangle = e^{-ct}\langle u_0\rangle - e^{-ct}\int_0^t e^{cs}\, ds \int_{\Omega\setminus D} \overline{u}_N \, dx$$

and

$$\begin{aligned}&|\langle u_N(t)\rangle| \le |\langle u_0\rangle| + ce^{-ct}\textstyle\int_0^t e^{cs}\|u_N\|\, ds\\ &\le 1-2\delta + c'(1-e^{-ct})\end{aligned}$$

$\rightarrow$ There exists $T_0 = T_0(u_0,\delta) > 0$ independent of N such that

$$|\langle u_N(t)\rangle| \le 1-\delta,\ t\in[0,T_0]$$

We can then prove that

$$|\langle f_N(u_N)\rangle| \leq c_\delta \|\overline{u}_N\|_{L^2} \|f_N(u_N) - \langle f_N(u_N)\rangle\|_{L^2} + c'_\delta$$

$\rightarrow$ Uniform estimate on the L^2-norm of $f_N(u_N)$ on $[0, T_0]$

This allows to pass to the limit on $[0, T_0]$

Existence of a local (in time) solution

Remark : We rewrite the equation in the form

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + u - \chi_D(x)u = 0$$

We then have

$$\frac{d\langle u\rangle}{dt} + \langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_D u\, dx$$

and

$$|\langle u(t)\rangle| \leq e^{-t}|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)} e^{-t} \int_0^t e^s\, ds$$

This yields

$$|\langle u(t)\rangle| \leq \max(|\langle u_0\rangle|, \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)})$$

whence

$$|\langle u(t)\rangle| \leq 1-\delta$$

where $\delta = \delta(u_0) \in (0,1)$ is independent of time

$\rightarrow$ The solutions are global in time

Numerical simulations :

One-step algorithm with threshold

The convergence time is faster

The results are better with large inpainting domains

Extensions of the model :

Cahn-Hilliard inpainting for multicolor images (L. Cherfils-H. Fakih-A. Miranville, JMIV) :

Multiphasic Cahn-Hilliard system

Each phase corresponds to a color

We consider the hyperplane

$$S = \{c \in R^n \text{ such that } \sum_{i=1}^{n} c_i = 1\}$$

$h = (h_1, ..., h_n) \in S$: damaged image, known on $\Omega \backslash D$

We look for $u = (u_1, ..., u_n) \in S$ such that

$$
\begin{aligned}
&\frac{\partial u_i}{\partial t} = \Delta \mu_i + \lambda_0 \chi_{\Omega \setminus D}(x)(h_i - u_i), \quad i = 1, ..., n \\
&\mu_i = f_i(u) - \epsilon^2 \Delta u_i, \quad i = 1, ..., n \\
&\frac{\partial u_i}{\partial \nu} = \frac{\partial \mu_i}{\partial \nu} = 0 \text{ on } \Gamma, \quad i = 1, ..., n \\
&u_i|_{t=0} = u_{i,0}, \quad i = 1, ..., n
\end{aligned}
$$

$$
\begin{aligned}
&f_i(u) = \frac{\partial F(u)}{\partial u_i} - \frac{1}{n} \sum_{j=1}^{n} \frac{\partial F(u)}{\partial u_j}, \quad i = 1, ..., n \\
&F(u) = \frac{1}{n} \sum_{i=1}^{n} u_i^2 (1 - u_i^2)
\end{aligned}
$$

Lagrange multiplier to ensure $u \in S$

Well-posedness and regularity of the solutions

Existence of finite-dimensional attractors

The model is algebraically consistent with the diphasic model

Numerical simulations :

One-step algorithm with threshold

Drawback : not efficient when the number of colors n is large

Grayscale Cahn-Hilliard inpainting : (L. Cherfils-H. Fakih-A. Miranville, MMS)

Aim : propose a simple model

Known models : heavy to implement numerically

J. Bosch et al. : multiphasic Cahn-Hilliard system ; n : number of shades of gray (not efficient when n is large)

Other models : total variation in H^{-1}, Low Curvature Image Simplifier (similar drawbacks)

Idea : consider a complex version of the Bertozzi et al. model (H. Grossauer-O. Sherzer : complex Allen-Cahn equation)

$h_1 \in L^2(\Omega)$: damaged image ($h_1 : \Omega \to [-1, 1]$)

We introduce $h : \Omega \to C$ defined by

$$h = h_1 + ih_2,\ h_2(x) = \sqrt{1 - h_1(x)^2}$$

$\to h \in L^2(\Omega; C)$, $|h| = 1$

Complex version of the Bertozzi et al. model :

$$\begin{array}{l} \frac{\partial u}{\partial t} + \epsilon \Delta^2 u - \frac{1}{\epsilon} \Delta f(u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma \\ u\big|_{t=0} = u_0 \end{array}$$

$f(z) = |z|^2 z - z$, $z \in C$

Well-posedness and regularity of the solutions

Existence of finite-dimensional attractors

Numerical simulations :

Two-steps algorithm

We use the information on the image known outside the inpainting region

Inpainting result : real part of the solution

We only need to compute two functions whatever the number of shades of gray is