The Cahn-Hilliard equation with regular nonlinear terms (I)

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The Cahn–Hilliard equation : recent advances and applications

Equations :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

$\Gamma = \partial\Omega$, Ω : bounded and regular domain of R^n
$f(s) = s^3 - s$, $F(s) = \int_0^s f(\xi)\, d\xi = \frac{1}{4}s^4 - \frac{1}{2}s^2$
$f' \geq -1$

Linear operators :

Spaces :

$H = \dot{L}^2(\Omega) = \{u \in L^2(\Omega),\ \langle u\rangle = 0\}, ((\cdot,\cdot)), \|\cdot\|$
$V = \dot{H}^1(\Omega) = H^1(\Omega) \cap H, ((\cdot,\cdot))_V = ((\nabla\cdot,\nabla\cdot))$
$V' = \{u \in H^{-1}(\Omega),\ \langle u\rangle = 0\}$

$\langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u\,dx,\ u \in L^1(\Omega)$
$\langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle u, 1\rangle,\ u \in H^{-1}(\Omega)$

$\|\cdot\|_X$: norm on the Banach space X

$V \subset H \equiv H' \subset V'$, with dense, continuous and compact embeddings
$H^1(\Omega) \subset L^2(\Omega),\ L^2(\Omega)' \subset H^{-1}(\Omega)$, with dense, continuous and compact embeddings

We define the linear operator $A : V \to V'$ by

$$\langle Au, v\rangle = ((u, v))_V, \ \forall u, \ v \in V$$

Isomorphism from V onto V'

$$D(A) = A^{-1}(H) = \{u \in V, \ Au \in H\} = \{u \in \dot{H}^1(\Omega), \ -\Delta u \in \dot{L}^2(\Omega)\}$$

$$D(A) = \{u \in H^2(\Omega) \cap V, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\}$$

$Au = f$, $u \in D(A)$ and $f \in H$, is equivalent to

$$-\Delta u = f \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

$A^{-1} : H \to H$: compact, selfadjoint and positive

There exists an orthonormal basis (w_j), $j \in \mathbb{N}$, of H formed of eigenvectors of A^{-1} :

$$A^{-1} w_j = \mu_j w_j,\ \mu_j \to 0 \text{ as } j \to +\infty,\ \mu_j > 0$$

$$A w_j = \lambda_j w_j,\ \lambda_j = \frac{1}{\mu_j}$$

w_j, λ_j : eigenvectors/eigenvalues of A, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to +\infty$ as $j \to +\infty$

$$((w_j, w_k))_V = \langle A w_j, w_k \rangle = \lambda_j ((w_j, w_k)) = 0$$

$$\langle A w_j, w_j \rangle = ((w_j, w_j))_V = \lambda_j \| w_j \|^2 = \lambda_j$$

$D(A^\alpha) = \{u \in H,\ u = \sum_{j=1}^{\infty} u_j w_j,\ \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2 < +\infty\},\ \alpha > 0$

$((u,v))_{D(A^\alpha)} = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} u_j v_j,\ u = \sum_{j=1}^{\infty} u_j w_j,\ v = \sum_{j=1}^{\infty} v_j w_j$

$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^{\alpha} u_j w_j,\ u = \sum_{j=1}^{\infty} u_j w_j$

Graph norm : $\| \cdot \|_{D(A^\alpha)} = \|A^\alpha \cdot \|$

$D(A^{\frac{1}{2}}) = V$

$D(A^2) = \{u \in H^4(\Omega) \cap V,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma\}$
$A^2 u = f$, $u \in D(A^2)$ and $f \in H$, is equivalent to

$$\Delta^2 u = f \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$\rightarrow$ We recover the Neumann boundary conditions

$\alpha < 0 : D(A^{\alpha}) = D(A^{-\alpha})'$

If $\alpha > \alpha'$, $\alpha, \alpha' \in \mathbb{R}$, then $D(A^{\alpha}) \subset D(A^{\alpha'})$ with continuous, dense and compact injection

$\|A^{-\frac{1}{2}} \cdot \|$, is equivalent to the usual $H^{-1}(\Omega)$-norm on $D(A^{-\frac{1}{2}}) = V'$
$\| \cdot \|_{-1} = \|A^{-\frac{1}{2}} \cdot \|$

$u \mapsto (\|A^{-\frac{1}{2}}(u - \langle u \rangle)\|^2 + \langle u \rangle^2)^{\frac{1}{2}}$ is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual $H^{-1}(\Omega)$-one

The linear Cahn-Hilliard equation :

$$\frac{\partial u}{\partial t} + \Delta^2 u = f(x,t)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

Functional form :

$$\frac{du}{dt} + A^2 u = f(t) \text{ in } \mathcal{D}'(0,T;H),\ T > 0$$

If $f \in L^2(0,T;H)$, $u \in L^2(0,T;D(A^2))$ and $\frac{du}{dt} \in L^2(0,T;H)$, the equation makes sense in $L^2(0,T;H)$

Variational formulation :

Find $u : [0,T] \to D(A)$ such that

$$\frac{d}{dt}((u,v)) + ((Au,Av)) = ((f(t),v)) \text{ in } \mathcal{D}'(0,T),\ \forall v \in D(A)$$

Theorem : (Existence and uniqueness of weak solutions) We assume that $f \in L^2(0,T;D(A^{-1}))$ and $u_0 \in H$, $T > 0$ given. Then, the linear initial value problem

$$\frac{du}{dt} + A^2 u = f(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1}))$$
$$u(0) = u_0 \text{ in } H$$

possesses a unique solution u such that $u \in \mathcal{C}([0,T];H) \cap L^2(0,T;D(A))$ and $\frac{du}{dt} \in L^2(0,T;D(A^{-1}))$.

Let u_1 and u_2 be solutions to

$$\frac{du_1}{dt} + A^2 u_1 = f_1(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1}))$$

$$u_1(0) = u_{1,0} \text{ in } H$$

$$\frac{du_2}{dt} + A^2 u_2 = f_2(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1})),$$

$$u_2(0) = u_{2,0} \text{ in } H$$

$u = u_1 - u_2, f = f_1 - f_2$ and $u_0 = u_{1,0} - u_{2,0}$ satisfy

$$\frac{du}{dt} + A^2 u = f(t) \text{ in } \mathcal{D}'(0,T;D(A^{-1}))$$

$$u(0) = u_0 \text{ in } H$$

Variational formulation :

$$\frac{d}{dt}((u,v)) + ((Au,Av)) = \langle f(t), v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in D(A)$$

Take $v = u(t)$:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|Au\|^2 = \langle f(t), u\rangle$$

This yields

$$\frac{d}{dt}\|u\|^2 + \|Au\|^2 \leq \|f(t)\|^2_{D(A^{-1})}$$

$$\|u(t)\|^2 \leq \|u_0\|^2 + \|f\|^2_{L^2(0,T;D(A^{-1}))}$$

Existence : Galerkin scheme

Approximated problems :

Find $u_m = \sum_{i=1}^{m} \overline{u}_i w_i$, $\overline{u}_i = \overline{u}_i(t)$, $i = 1, \cdots, m$, such that

$$\frac{d}{dt}((u_m, v)) + ((Au_m, Av)) = \langle f(t), v \rangle, \ \forall v \in W_m \equiv \mathrm{Span}(w_1, \cdots, w_m)$$

$$u_m(0) = u_{0,m}$$

$u_{0,m}$: projection (in H) of u_0 onto W_m

$$u_{0,m} = \sum_{i=1}^{m} ((u_0, w_i)) w_i$$

Equivalent formulation :

$$\frac{d\overline{u}_i}{dt} + \lambda_i^2 \overline{u}_i = \langle f(t), w_i \rangle, \ i = 1, \ \cdots, \ m$$

$$\overline{u}_i(t) = e^{-\lambda_i^2 t} u_{i,0} + e^{-\lambda_i^2 t} \int_0^t e^{\lambda_i^2 s} \langle f(s), w_i \rangle \, ds, \ i = 1, \ \cdots, \ m$$

$u_{i,0} = ((u_0, w_i))$

Take $v = w_i$, multiply by $\overline{u}_i$ and sum over i

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \|Au_m\|^2 = \langle f(t), u_m\rangle$$

This yields

$$\frac{d}{dt}\|u_m\|^2 + \|Au_m\|^2 \leq \|f(t)\|^2_{D(A^{-1})}$$

u_m is bounded, independently of m, in $L^2(0,T;D(A))$ and $L^\infty(0,T;H)$

There exists $u \in L^\infty(0,T;H) \cap L^2(0,T;D(A))$ such that $u_m \to u$ in $L^\infty(0,T;H)$ weak star and in $L^2(0,T;D(A))$ weak

Theorem : (Existence of strong solutions) We assume that $u_0 \in D(A)$ and $f \in L^2(0,T;H)$. Then, the solution u satisfies $u \in \mathcal{C}([0,T];D(A)) \cap L^2(0,T;D(A^2))$ and $\frac{du}{dt} \in L^2(0,T;H)$ and is a strong solution.

$$\frac{d}{dt}((u_m, v)) + ((Au_m, Av)) = ((f(t), v)),\ \forall v \in W_m$$

Take $v = w_i$, multiply by $\lambda_i^2 \overline{u}_i$ and sum over i :

$$\frac{d}{dt}\|Au_m\|^2 + \|A^2 u_m\|^2 \leq \frac{1}{c_0^2}\|f(t)\|^2$$

$(\|A^2 u_m\| \geq c_0 \|Au_m\|)$

Weaker formulation :

$$A^{-1}\frac{du}{dt} + Au = A^{-1}f(t) \text{ in } L^2(0,T;V')$$
$$u(0) = u_0 \text{ in } V'$$

Variational formulation :

Find $u : [0,T] \to V$ such that

$$\frac{d}{dt}((A^{-1}u, v)) + ((u,v))_V = \langle A^{-1}f(t), v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in V$$

Theorem : (Existence and uniqueness of very weak solutions) We assume that $u_0 \in V'$ and $f \in L^2(0,T;D(A^{-\frac{3}{2}}))$. Then, the problem possesses a unique solution u such that $u \in \mathcal{C}([0,T];V') \cap L^2(0,T;V)$ and $\frac{du}{dt} \in L^2(0,T;D(A^{-\frac{3}{2}}))$.

The Cahn-Hilliard equation with a cubic nonlinear term

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

$f(s) = s^3 - s$, $F(s) = \int_0^s f(\xi)\, d\xi = \frac{1}{4}s^4 - \frac{1}{2}s^2$

$f' \geq -1$

Mass conservation :

$$\frac{d\langle u\rangle}{dt} = 0$$

$$\langle u(t)\rangle = \langle u_0\rangle,\ t \geq 0$$

We assume that

$$\langle u_0\rangle = \kappa,\ \kappa \in \mathbb{R} \text{ given}$$

Therefore

$$\langle u(t)\rangle = \kappa,\ t \geq 0$$

Set $\overline{u} = u - u_0 (= u - \kappa), \overline{f(u)} = f(u) - \langle f(u) \rangle$:

$$\frac{\partial \overline{u}}{\partial t} + \Delta^2 \overline{u} - \Delta \overline{f(u)} = 0$$

$$\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial \Delta \overline{u}}{\partial \nu} = 0 \text{ on } \Gamma$$

$$\overline{u}|_{t=0} = \overline{u}_0 (= u_0 - \kappa)$$

Functional formulation :

$$\frac{d\overline{u}}{dt} + A^2 \overline{u} + A\overline{f(u)} = 0 \text{ in } \mathcal{D}'(0, T; D(A^{-1}))$$

$$\overline{u}(0) = \overline{u}_0 \text{ in } H$$

Variational formulation :

Find $\overline{u} : [0, T] \to D(A)$ such that

$$\frac{d}{dt}((\overline{u}, v)) + ((A\overline{u}, Av)) + ((\overline{f(u)}, Av)) = 0 \text{ in } \mathcal{D}'(0, T), \ \forall v \in D(A)$$

$$\overline{u}(0) = \overline{u}_0 \text{ in } H$$

$u = \overline{u} + \kappa$

Equivalent formulation :

$$\frac{\partial \overline{u}}{\partial t} = -\Delta \overline{\mu}$$

$$\overline{\mu} = -\Delta \overline{u} + \overline{f(u)}$$

$$\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial \overline{\mu}}{\partial \nu} = 0 \text{ on } \Gamma$$

$\overline{\mu} = \mu - \langle \mu \rangle$, $\langle \mu \rangle = \langle f(u) \rangle$

Functional formulation :

$$\frac{d\overline{u}}{dt} = -A\overline{\mu} \text{ in } \mathcal{D}'(0,T;D(A^{-1}))$$

$$\overline{\mu} = A\overline{u} + \overline{f(u)} \text{ in } \mathcal{D}'(0,T;H)$$

Variational formulation :

Find $(\overline{u}, \overline{\mu}) : [0,T] \to D(A) \times H$ such that

$$\frac{d}{dt}((\overline{u}, v)) = -((\overline{\mu}, Av)) \text{ in } \mathcal{D}'(0,T), \ \forall v \in D(A)$$

$$((\overline{\mu}, v)) = ((A\overline{u}, v)) + ((\overline{f(u)}, v)) \text{ in } \mathcal{D}'(0,T), \ \forall v \in H$$

with $u = \overline{u} + \kappa$, $\mu = \overline{\mu} + \langle f(u) \rangle$

Weaker formulation :

$$A^{-1}\frac{d\overline{u}}{dt} + A\overline{u} + \overline{f(u)} = 0 \text{ in } \mathcal{D}'(0,T;V')$$

$$\overline{u}(0) = \overline{u}_0 \text{ in } V'$$

Variational formulation :

Find $\overline{u} : [0,T] \to V$ such that

$$\frac{d}{dt}((A^{-1}\overline{u}, v)) + ((\overline{u}, v))_V + ((\overline{f(u)}, v)) = 0 \text{ in } \mathcal{D}'(0,T),\ \forall v \in V$$

$$\overline{u}(0) = \overline{u}_0 \text{ in } V'$$

$u = \overline{u} + \kappa$

Theorem : We assume that $u_0 \in H^1(\Omega)$, i.e., $\overline{u}_0 \in V$. Then, the problem possesses a unique weak solution $u = \overline{u} + \kappa$ such that $\overline{u} \in L^\infty(\mathbb{R}^+; V) \cap \mathcal{C}([0,T]; V) \cap L^2(0,T; D(A^{\frac{3}{2}}))$ and $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}^+; V')$, $\forall T > 0$.

Uniqueness :

Similar to the linear case, with $f' \geq -1$

$$\frac{d}{dt}\|u\|_{-1}^2 \leq 2\|u\|^2$$

Interpolation inequality :

$$\|u\|^2 = ((A^{-\frac{1}{2}}u, A^{\frac{1}{2}}u)) \leq \|u\|_{-1}\|u\|_V$$

$$\frac{d}{dt}\|u\|_{-1}^2 \leq \|u\|_{-1}^2$$

Existence :

Galerkin scheme

Formal estimates

Equation :

$$(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} - \Delta\overline{u} + f(u) - \overline{f(u)} = 0$$

Multiply the equation by $\overline{u}$:

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\overline{u}\|_V^2 + \|u\|_{L^4(\Omega)}^4 \leq \mathrm{Vol}(\Omega)$$

Multiply the equation by $-\Delta\overline{u}$ $(f' \geq -1)$:

$$\frac{d}{dt}\|\overline{u}\|^2 + c\|\overline{u}\|_{H^2(\Omega)}^2 \leq \|u\|^2,\ c > 0$$

Multiply the equation by $\frac{\partial \overline{u}}{\partial t}$:

$$\frac{d}{dt}(\|\overline{u}\|_V^2 + 2\int_\Omega F(u)\,dx) + 2\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 = 0$$

Multiply the equation by $\Delta^2 u$:

$$\frac{d}{dt}\|\overline{u}\|_V^2 + c\|\overline{u}\|_{H^3(\Omega)}^2 \leq c'(\|\overline{u}\|_V^4 + 1)\|\overline{u}\|_{H^2(\Omega)}^2 + c'',\ c > 0.$$

Note that

$$v \mapsto (\|A^{-\frac{1}{2}}\overline{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

$$v \mapsto (\|\overline{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

$$v \mapsto (\|\overline{v}\|_V^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

$$v \mapsto (\|A\overline{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

are norms on $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual ones

Passage to the limit in the nonlinear term :

$$\int_0^T \int_\Omega ((u_m^3, v))\varphi(t)\, dx\, dt$$

$\varphi \in \mathcal{D}(0, T)$, with $u_m^3 \to u^3$ in $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega))$ and $u_m \to u$ a.e. (Aubin-Lions compactness results)

There exists $g \in L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega))$ such that

$$|u_m^3| \leq g \text{ a.e.}$$

Lebesgue's theorem

Theorem : We assume that $u_0 \in H^2(\Omega)$, with $\frac{\partial u_0}{\partial \nu} = 0$ on Γ, i.e., $\overline{u}_0 \in D(A)$. Then, the solution $u = \overline{u} + \kappa$ satisfies $\overline{u} \in \mathcal{C}([0,T]; D(A)) \cap L^2(0,T; D(A^2))$ and $\frac{\partial u}{\partial t} \in L^2(0,T;H)$, $\forall T > 0$. Furthermore, it is a strong solution :

$$\frac{d\overline{u}}{dt} + A^2\overline{u} + A\overline{f(u)} = 0 \text{ in } L^2(0,T;H).$$

Multiply the equation

$$(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} - \Delta \overline{u} + f(u) - \overline{f(u)} = 0$$

by $(-\Delta)^3 u$ ($\kappa = 0$, $\overline{u} = u$, for simplicity) :

$$\frac{d}{dt}\|\Delta u\|^2 + 2\|\Delta^2 u\|^2 \leq 2\|\Delta f(u)\|\|\Delta^2 u\|$$

$$\Delta f(u) = f'(u)\Delta u + f''(u)\nabla u \cdot \nabla u$$

$$\|\Delta f(u)\| \leq c(\|u^2\Delta u\| + \|\Delta u\| + \|u\nabla u \cdot \nabla u\|)$$

Interpolation inequality :

$$\|u\|_{H^2(\Omega)} \leq c\|u\|_{H^1(\Omega)}^{\frac{2}{3}}\|u\|_{H^4(\Omega)}^{\frac{1}{3}}$$

Agmon's inequality :

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}}$$

Then :

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{H^1(\Omega)}^{\frac{5}{6}}\|u\|_{H^4(\Omega)}^{\frac{1}{6}}.$$

and

$$\|u^2\Delta u\| \leq c\|u\|_{H^1(\Omega)}^{\frac{7}{3}}\|u\|_{H^4(\Omega)}^{\frac{2}{3}}$$

We have :

$$\|u\nabla u \cdot \nabla u\| \leq \|u\|_{L^\infty(\Omega)} \||\nabla u|^2\| \leq c\|u\|^{\frac{5}{6}}_{H^1(\Omega)} \|u\|^{\frac{1}{6}}_{H^4(\Omega)} \||\nabla u|\|^2_{L^4(\Omega)}$$

$H^{\frac{3}{4}}(\Omega) \subset L^4(\Omega)$ with continuous embedding :

$$\|\nabla u\|_{L^4(\Omega)} \leq c\|u\|_{H^{\frac{7}{4}}(\Omega)}$$

Interpolation inequality :

$$\|u\|_{H^{\frac{7}{4}}(\Omega)} \leq c\|u\|^{\frac{3}{4}}_{H^1(\Omega)} \|u\|^{\frac{1}{4}}_{H^4(\Omega)}$$

Thus :

$$\|u\nabla u \cdot \nabla u\| \leq c\|u\|^{\frac{5}{6}}_{H^1(\Omega)} \|u\|^{\frac{1}{6}}_{H^4(\Omega)} \|u\|^{\frac{3}{2}}_{H^1(\Omega)} \|u\|^{\frac{1}{2}}_{H^4(\Omega)}$$

$$\|u\nabla u \cdot \nabla u\| \leq c\|u\|^{\frac{7}{3}}_{H^1(\Omega)} \|u\|^{\frac{2}{3}}_{H^4(\Omega)}$$

Young's inequality,

$$\|\Delta f(u)\| \leq c(\|u\|_{H^1(\Omega)}^{\frac{7}{3}}\|u\|_{H^4(\Omega)}^{\frac{2}{3}} + \|u\|_{H^1(\Omega)}^{\frac{1}{3}}\|u\|_{H^4(\Omega)}^{\frac{2}{3}})$$

$$\leq c(1 + \|u\|_{H^1(\Omega)}^{\frac{7}{3}})\|u\|_{H^4(\Omega)}^{\frac{2}{3}}$$

Finally :

$$|((\Delta\overline{f(u)}, \Delta^2 u))| \leq c(1 + \|u\|_{H^1(\Omega)}^{\frac{7}{3}})\|u\|_{H^4(\Omega)}^{\frac{2}{3}}\|\Delta^2 u\|$$

$$\leq c(1 + \|u\|_{H^1(\Omega)}^{\frac{7}{3}})\|\Delta^2 u\|^{\frac{5}{3}}$$

and

$$\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 \leq c(1 + \|u\|_V^{14})$$

Remark : In 2D : $f(s) = \sum_{i=1}^{2p+1} a_i s^i$, $a_{2p+1} > 0$, $p \in N$ (in 2D : $p = 1$).

Existence of finite-dimensional global attractors :

E : Banach space endowed with the norm $\|.\|_E$

$\{S(t),\ t \geq 0\}$: family of (nonlinear) operators acting on E

$$S(t) : E \to E,\ t \geq 0$$

We assume that this family of operators satisfies the following properties :

$$S(0) = \mathrm{I} \text{ (identity operator)}$$

$$S(t+\tau) = S(t) \circ S(\tau),\ \forall t,\ \tau \geq 0$$

We say that it forms a semigroup acting on E

Continuity property :

$$S(t) : E \to E,\ x \mapsto S(t)x,$$

is continuous, $\forall t \geq 0$

A bounded set $\mathcal{B}_0 \subset E$ is a bounded absorbing set for $S(t)$ if, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$

Mathematical definition of dissipation

Definition : A set $\mathcal{A} \subset E$ is a global attractor for the semigroup $S(t)$ if the following properties hold :

(i) $\mathcal{A}$ is compact in E.

(ii) $\mathcal{A}$ is invariant by $S(t)$, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$.

(iii) $\mathcal{A}$ is an attracting set : $\forall B \subset E$ bounded,

$$\mathrm{dist}_E(S(t)B, X) \to 0 \text{ as } t \to +\infty,$$

where dist_E denotes the Hausdorff semidistance between sets, defined as

$$\mathrm{dist}_E(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_E.$$

Theorem : We assume that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0$ and that, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B) > 0$ such that $\cup_{t \geq t_0} S(t)B$ is relatively compact in E. Then, $S(t)$ possesses the global attractor $\mathcal{A}$.

Example : $S(t)$ possesses a bounded absorbing set $\mathcal{B}_1$ such that $\mathcal{B}_1$ is relatively compact in E

Theorem : We assume that $S(t)$ possesses a compact attracting set K. Then, $S(t)$ possesses the global attractor $\mathcal{A}$.

Definition : Let $X \subset E$ be a (relatively) compact set. For $\epsilon > 0$, let $N_\epsilon(X)$ (if it is necessary to precise the topology, we will also use the notation $N_\epsilon(X, E)$) be the minimal number of balls of radius ϵ which are necessary to cover X. Then, the fractal dimension of X is the quantity (which belongs to $[0, +\infty]$)

$$\dim_{\mathrm{F}} X = \limsup_{\epsilon \to 0^+} \frac{\log_2 N_\epsilon(X)}{\log_2 \frac{1}{\epsilon}} \left(= \limsup_{\epsilon \to 0^+} \frac{\ln N_\epsilon(X)}{\ln \frac{1}{\epsilon}}\right).$$

Furthermore, the quantity $\mathcal{H}_\epsilon(X) (= \mathcal{H}_\epsilon(X, E)) = \log_2 N_\epsilon(X)$ is called the Kolmogorov ϵ-entropy of X.

If X is a smooth m-dimensional manifold, then $\dim_{\mathrm{F}} X = m$

If the minimal number of balls of radius ϵ which are necessary to cover X satisfies

$$N_\epsilon(X) \leq c(\frac{1}{\epsilon})^d \text{ (i.e., } \mathcal{H}_\epsilon(X) \leq d \log_2 \frac{1}{\epsilon} + c',\ c' = \log_2 c)$$

c and d independent of ϵ, then

$$\mathrm{dim}_{\mathrm{F}} X \leq d$$

Theorem : Let X be a compact subset of E. We assume that there exist a Banach space E_1, with norm $\|.\|_{E_1}$, such that E_1 is compactly embedded into E and a mapping $L : X \to X$ such that $L(X) = X$ and

$$\|Lx_1 - Lx_2\|_{E_1} \leq c\|x_1 - x_2\|_E,\ \forall x_1,\ x_2 \in X,\ c > 0.$$

Then, the fractal dimension of X is finite and satisfies

$$\dim_{\mathrm{F}} X \leq \mathcal{H}_{\frac{1}{4c}}(B_{E_1}(0,1), E),$$

where $B_{E_1}(0,1)$ is the unit ball in E_1.

Assume that $\kappa = 0$ ($\langle u_0 \rangle = 0$)

Theorem : The semigroup $S(t)$ associated with the problem possesses the global attractor $\mathcal{A}_0$ on V which is bounded in $H^2(\Omega)$.

We have :

$$S(t) : V \to V, \; u_0 \mapsto u(t)$$

$x \mapsto S(t)x$ is continuous with respect to the H^{-1}-topology

We have :

$$\frac{d}{dt}\|u\|_{-1}^2 + c_0^2\|u\|_{-1}^2 \leq \mathrm{Vol}(\Omega)$$

Gonwall's lemma :

$$\|u(t)\|_{-1}^2 \leq e^{-c_0^2 t}\|u_0\|_{-1}^2 + c_1$$

$\to$ Bounded absorbing set in V

Uniform Gronwall's lemma :

Proposition : Let g, h and y be three locally integrable and nonnegative functions such that y' is locally integrable and, for $t \geq t_0$, $t_0 \in \mathbb{R}$,

$$y' \leq gy + h.$$

We further assume that, for $r > 0$ given,

$$\int_t^{t+r} g(s)\, ds \leq a_1,\ \int_t^{t+r} h(s)\, ds \leq a_2,\ \int_t^{t+r} y(s)\, ds \leq a_3$$

(here, a_1, a_2 and a_3 depend on r). Then, there holds, for $t \geq t_0$,

$$y(t+r) \leq (\frac{a_3}{r} + a_2)e^{a_1}.$$

Remark :

a) Uniform bound on y for $t \geq t_0 + r$

b) Explodes as $r \to 0$

We have :

$$\int_t^{t+1} \|u\|_V^2 \, ds \leq c, \; \int_t^{t+1} \int_\Omega F(u) \, dx \, ds \leq c$$

Energy dissipation :

$$\frac{d}{dt}(\|u\|_V^2 + 2 \int_\Omega F(u) \, dx) + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 \leq 0$$

Uniform Gronwall's lemma : bounded absorbing set in V

We have :

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^2(\Omega)} \leq \|u\|^2,\ c > 0$$

$\rightarrow \int_t^{t+1} \|u\|^2_{H^2(\Omega)}\, ds \leq c$

We have :

$$\frac{d}{dt}\|Au\|^2 + \|A^2u\|^2 \leq c(1 + \|u\|_V^{14})$$

Uniform Gronwall's lemma : bounded absorbing set in V bounded in $H^2(\Omega)$

Theorem : The global attractor $\mathcal{A}_0$ has finite fractal dimension for the topology of V'.

Let u_1 and u_2 be solutions to

$$(-\Delta)^{-1}\frac{\partial u_1}{\partial t} - \Delta u_1 + f(u_1) - \langle f(u_1)\rangle = 0$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial \Delta u_1}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u_1|_{t=0} = u_{1,0}$$

and

$$(-\Delta)^{-1}\frac{\partial u_2}{\partial t} - \Delta u_2 + f(u_2) - \langle f(u_2)\rangle = 0$$

$$\frac{\partial u_2}{\partial \nu} = \frac{\partial \Delta u_2}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u_2|_{t=0} = u_{2,0}$$

$u = u_1 - u_2$ and $u_0 = u_{1,0} - u_{2,0}$ satisty

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) - (\langle f(u_1)\rangle - \langle f(u_2)\rangle) = 0$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$

$$u|_{t=0} = u_0$$

Multiply the equation by u :

$$\frac{d}{dt}\|u\|_{-1}^2 + \|u\|_V^2 \leq \|u\|_{-1}^2$$

Gronwall's lemma :

$$\|u(t)\|_{-1}^2 \leq e^t\|u_0\|_{-1}^2,\ t \geq 0,$$

Thus :

$$\int_0^1 \|u\|_V^2\, dt \leq c\|u_0\|_{-1}^2$$

Multiply the equation by $t\frac{\partial u}{\partial t}$:

$$\frac{1}{2}\frac{d}{dt}(t\|u\|_V^2) + t\|\frac{\partial u}{\partial t}\|_{-1}^2 + t((f(u_1) - f(u_2), \frac{\partial u}{\partial t})) = \frac{1}{2}\|u\|_V^2$$

We have :

$$|((f(u_1) - f(u_2), \frac{\partial u}{\partial t}))| \leq \|\frac{\partial u}{\partial t}\|_{-1}^2 \|\nabla[(u_1^2 + u_1 u_2 + u_2^2 - 1)u]\|$$

$$\nabla[(u_1^2 + u_1 u_2 + u_2^2 - 1)u] = (u_1^2 + u_1 u_2 + u_2^2 - 1)\nabla u$$

$$+u(2u_1\nabla u_1 + 2u_2\nabla u_2 + u_1\nabla u_2 + u_2\nabla u_1)$$

$\mathcal{A}_0 \subset H^2(\Omega)$ and is invariant : $u_i \in L^\infty(\mathbb{R}^+; H^2(\Omega))$, $i = 1, 2$

We have :

$$\|\nabla[(u_1^2+u_1u_2+u_2^2-1)u]\| \leq \|(u_1^2+u_1u_2+u_2^2-1)\nabla u\|$$

$$+\|u(2u_1\nabla u_1+2u_2\nabla u_2+u_1\nabla u_2+u_2\nabla u_1)\|$$

$$\leq 2(\|u_1\|^2_{L^\infty(\Omega)}+\|u_2\|^2_{L^\infty(\Omega)}+1)\|\nabla u\|$$

$$+2(\|u_1\|_{L^\infty(\Omega)}+\|u_2\|_{L^\infty(\Omega)})(\|\nabla u_1\|_{L^4(\Omega)}+\|\nabla u_2\|_{L^4(\Omega)})\|u\|_{L^4(\Omega)}$$

Thus :

$$\|\nabla[(u_1^2+u_1u_2+u_2^2-1)u]\| \leq c((\|u_1\|^2_{H^2(\Omega)}+\|u_2\|^2_{H^2(\Omega)}+1)\|u\|_V$$

$$\leq c\|u\|_V$$

and

$$\frac{d}{dt}(t\|u\|_V^2)+t\|\frac{\partial u}{\partial t}\|^2_{-1} \leq c(t+1)\|u\|_V^2$$

Gonwall's lemma :

$$\|u(1)\|_V \leq c\|u_0\|_{-1}$$

$\rightarrow$ Apply the theorem for $E = V'$, $E_1 = V$ and $L = S(1)$

Remark : We can also prove the finite fractal dimensionality of the global attractor with respect to the $H^1(\Omega)$-topology

Assume that $\kappa \neq 0$: set

$$V_\kappa = \{u \in H^1(\Omega),\ \langle u \rangle = \kappa\} = V + \kappa$$

$$S_\kappa : V_\kappa \rightarrow V_\kappa,\ t \geq 0$$

$$S_\kappa(t)u = S(t)\overline{u} + \kappa$$

$S_\kappa(t)$ possesses the finite-dimensional global attractor $\mathcal{A}_\kappa$ on V_κ, $\mathcal{A}_\kappa = \mathcal{A} + \kappa$

Remark :

a) Take $\kappa \in [-\kappa_1, \kappa_1]$: we can construct the global attractor $\tilde{\mathcal{A}}_{\kappa_1} = \cup_{|\kappa| \leq \kappa_1} \mathcal{A}_\kappa$ for the corresponding semigroup on

$$\mathcal{V}_{\kappa_1} = \{u \in H^1(\Omega),\ |\langle u \rangle| \leq \kappa_1\}$$

Finite fractal dimensionality : more involved

b) The set $\cup_{\kappa \in \mathbb{R}} \mathcal{A}_\kappa$ is not compact

Remark : The order parameter does not remain in $[-1, 1]$.

Counterexample : Consider the one-dimensional Cahn-Hilliard equation with $\kappa = \alpha = 1$ and the cubic nonlinearity $f(s) = s^3 - s$ in $\Omega = (-1, 1)$

Take $u_0(x) = 1 - x^4$ in the neighborhood of 0 and extend this function by a smooth function with a prescribed average over Ω and with values in $[-1, 1]$

Note that $u_0'(0) = u_0''(0) = 0$, so that $[f(u_0)]'' = 0$ at $x = 0$

Furthermore, $u_0^{(4)} \equiv -24$: $\frac{\partial u}{\partial t}(0, 0) = 24 > 0$

$\rightarrow u(0, t) = u(0, 0) + t\frac{\partial u}{\partial t}(0, 0) + o(t) = 1 + 24t + o(t) > 1$, for $t > 0$ small