The Cahn-Hilliard equation with dynamic boundary conditions

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The Cahn–Hilliard equation : recent advances and applications

First dynamic boundary conditions (G.R. Goldstein-A. Miranville-G. Schimperna) :

Total free energy : $\Psi = \Psi_\Omega + \Psi_\Gamma$

$$\Psi_\Omega(u, \nabla u) = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))dx$$

$$\Psi_\Gamma(u, \nabla_\Gamma u) = \int_\Gamma (\frac{\alpha_\Gamma}{2}|\nabla_\Gamma u|^2 + G(u))dx$$

Total mass conservation : $\frac{d}{dt}(\int_\Omega u dx + \int_\Gamma u d\sigma) = 0$

$\rightarrow \frac{\partial u}{\partial t} = \beta_\Gamma \Delta_\Gamma w - \kappa \frac{\partial w}{\partial \nu}$ on Γ, $\beta_\Gamma \geq 0$

Second boundary condition : w is a variational derivative of the total free energy Ψ w.r.t. u

$\rightarrow w = -\alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu}$ on Γ

Cahn-Hilliard type equation on the boundary

Equations :

$$\frac{\partial u}{\partial t} = \Delta\mu$$

$$\mu = -\Delta u + f(u)$$

$$\frac{\partial u}{\partial t} = \eta\Delta_\Gamma\mu - \frac{\partial\mu}{\partial\nu} \text{ on } \Gamma,\ \eta > 0$$

$$\mu = -\sigma\Delta_\Gamma u + \frac{\partial u}{\partial\nu} + g(u) \text{ on } \Gamma,\ \sigma > 0$$

$$u|_{t=0} = u_0$$

One space dimension : the Laplace–Beltrami operator does not appear in the equations

$\eta = 0$, $\sigma = 0$: no diffusion on the boundary

Integrate the first equation over Ω :

$$\frac{d}{dt}\int_{\Omega} u\,dx = \int_{\Gamma} \frac{\partial \mu}{\partial \nu}\,d\Sigma = -\int_{\Gamma} \frac{\partial u}{\partial t}\,d\Sigma$$

$\rightarrow$ Total mass conservation :

$$\frac{d}{dt}(\int_{\Omega} u\,dx + \int_{\Gamma} u\,d\Sigma) = 0$$

$f(s) = s^3 - s$, $g(s) = as + b$, $a > 0$

Thus :

$$f' \geq -1,\ g' = a > 0$$

$$f(s)(s-m) \geq c_1 F(s) - c_2(m) \geq -c_3(m),\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s,\ m \in \mathbb{R}$$

$$g(s)(s-m) \geq c_4 G(s) - c_5(m) \geq -c_6(m),\ c_4 > 0,\ c_5,\ c_6 \geq 0,\ s,\ m \in \mathbb{R}$$

$F(s) = \int_0^s f(\xi)\,d\xi$, $G(s) = \int_0^s g(\xi)\,d\xi$

c_i, $i = 1, \cdots, 6$, depend continuously on m

Linear operators :

Spaces :

• $H = L^2(\Omega)$, $H_\Gamma = L^2(\Gamma)$, $\mathcal{H} = H \times H_\Gamma$

Scalar products and associated norms, $((\cdot,\cdot))$, $\|\cdot\|$, $((\cdot,\cdot))_\Gamma$, $\|\cdot\|_\Gamma$, $((\cdot,\cdot))_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}}$

• $V = H^1(\Omega)$, $V_\Gamma = H^1(\Gamma)$, $\mathcal{V} = \{\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in V \times V_\Gamma,\ \varphi|_\Gamma = \psi\}$

Set, for $\phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in H \times H_\Gamma$:

$$\langle \phi \rangle = \frac{1}{\mathrm{Vol}(\Omega) + |\Gamma|}\left(\int_\Omega \varphi\, dx + \int_\Gamma \psi\, d\Sigma\right)$$

$$\dot{\mathcal{H}} = \{\phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathcal{H},\ \langle \phi \rangle = 0\}$$

$$\dot{\mathcal{V}} = \mathcal{V} \cap \dot{\mathcal{H}}$$

$\dot{\mathcal{V}} \subset \dot{\mathcal{H}} \subset \dot{\mathcal{V}}'$: dense, continuous, compact

Lemma : The norm $\| \cdot \|^2_{\dot{\mathcal{V}}} = \|\nabla \cdot \|^2 + \|\nabla_\Gamma \cdot \|^2$ is equivalent to the usual $H^1(\Omega) \times H^1(\Gamma)$-one on $\dot{\mathcal{V}}$.

Bilinear form :

$$a : \dot{\mathcal{V}} \times \dot{\mathcal{V}} \to \mathbb{R},\ (\phi, \Theta) \mapsto ((\nabla\varphi, \nabla\theta)) + ((\nabla_\Gamma\varphi, \nabla_\Gamma\theta))_\Gamma$$

$$\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$$

a : symmetric, continuous, coercive on $\dot{\mathcal{V}}$

Linear operator :

$$A : \dot{\mathcal{V}} \to \dot{\mathcal{V}}'$$

$$\langle A\phi, \Theta \rangle_{\dot{\mathcal{V}}', \dot{\mathcal{V}}} = a(\phi, \Theta),\ \phi,\ \Theta \in \dot{\mathcal{V}}$$

A : positive, selfadjoint, unbounded linear operator

Domain of A in $\dot{\mathcal{H}}$:

$$D(A) = \{\phi \in \dot{\mathcal{V}},\ \exists\, \Xi \in \dot{\mathcal{H}},\ ((\Xi, \Theta))_{\mathcal{H}} = a(\phi, \Theta),\ \forall\, \Theta \in \dot{\mathcal{V}}\}$$

A : isomorphism from $\dot{\mathcal{V}}$ onto $\dot{\mathcal{V}}'$ and from $D(A)$ onto $\dot{\mathcal{H}}$

A^{-1} : selfadjoint, compact operator on $\dot{\mathcal{H}}$

$\rightarrow$ We can define the powers of A : A^s, $s \in \mathbb{R}$

Proposition : The space $D(A)$ satisfies $D(A) = \dot{\mathcal{V}} \cap (H^2(\Omega) \times H^2(\Gamma))$ and the norm $\|A \cdot\|_{\mathcal{H}}$ is equivalent to the usual $H^2(\Omega) \times H^2(\Gamma)$-one on $D(A)$.

We have

$$A\phi = \begin{pmatrix} -\Delta\varphi \\ -\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} \end{pmatrix}$$

$$\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$$

Proposition : For $k \in \mathbb{N}$, the embedding $D(A^k) \subset H^{2k}(\Omega) \times H^{2k}(\Gamma)$ is continuous. Furthermore, the norm $\|A^k \cdot\|_{\mathcal{H}}$ is equivalent to the usual $H^{2k}(\Omega) \times H^{2k}(\Gamma)$-one on $D(A^k)$.

Proposition : For $k \in \mathbb{N} \cup \{0\}$, the embedding $D(A^{k+\frac{1}{2}}) \subset H^{2k+1}(\Omega) \times H^{2k+1}(\Gamma)$ is continuous and the norm $\|A^{k+\frac{1}{2}} \cdot \|_{\mathcal{H}}$ is equivalent to the usual $H^{2k+1}(\Omega) \times H^{2k+1}(\Gamma)$-one on $D(A^{k+\frac{1}{2}})$.

$D(A^{-\frac{1}{2}}) = \dot{\mathcal{V}}'$

$\dot{\mathcal{V}}' = \{\phi \in \mathcal{V}',\ \langle\phi\rangle = 0\}$, $\langle\phi\rangle = \frac{1}{\mathrm{Vol}(\Omega)+|\Gamma|}\langle\phi, 1\rangle_{\mathcal{V}',\mathcal{V}}$

$\| \cdot \|_{-1} = \|A^{-\frac{1}{2}} \cdot \|_{\mathcal{H}}$: equivalent to the usual $\mathcal{V}'$-norm on $\dot{\mathcal{V}}'$

Remark : a can also be defined on $\mathcal{V} \times \mathcal{V}$

$\rightarrow A$: operator from $\mathcal{V}$ onto $\mathcal{V}'$

Functional setting :

$\eta = \sigma = 1$

Equations in functional form :

$$\frac{dU}{dt} = -AW \text{ in } \mathcal{D}'(0,T;\mathcal{D}(A^{-1}))$$

$$W = AU + \mathcal{F}(U) \text{ in } \mathcal{D}'(0,T;\mathcal{V}')$$

$$U|_{t=0} = U_0 \text{ in } \mathcal{V}'$$

$$U = \begin{pmatrix} u \\ u \end{pmatrix},\ U_0 = \begin{pmatrix} u_0 \\ u_0 \end{pmatrix}$$

$$W = \begin{pmatrix} \mu \\ \mu \end{pmatrix},\ \mathcal{F}(U) = \begin{pmatrix} f(u) \\ g(u) \end{pmatrix}$$

Set

$$\overline{\phi} = \phi - \begin{pmatrix} \langle \phi \rangle \\ \langle \phi \rangle \end{pmatrix}$$

$\langle \overline{\phi} \rangle = 0$

Equivalent formulation :

$$A^{-1} \frac{dU}{dt} = -\overline{W} \text{ in } \mathcal{D}'(0, T; \dot{\mathcal{V}}')$$

Note that

$$\langle W \rangle = \langle \mathcal{F}(U) \rangle$$

$$A\phi = A\overline{\phi}$$

Equivalent formulations :

$$\frac{d\overline{U}}{dt} + A^2\overline{U} + A\overline{\mathcal{F}(U)} = 0 \text{ in } \mathcal{D}'(0,T;D(A^{-1}))$$

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \overline{\mathcal{F}(U)} = 0 \text{ in } \mathcal{D}'(0,T;\dot{\mathcal{V}}')$$

Existence and uniqueness of solutions ($\langle U_0 \rangle = 0$ for simplicity) :

$$A^{-1}\frac{dU}{dt} + AU + \overline{\mathcal{F}(U)} = 0 \text{ in } \mathcal{D}'(0,T;\dot{\mathcal{V}}')$$

$$U|_{t=0} = U_0 \text{ in } \dot{\mathcal{V}}'$$

A priori estimates :

Scalar product in $\dot{\mathcal{H}}$ by $\frac{dU}{dt}$:

$$\frac{d}{dt}(\|U\|^2_{\dot{\mathcal{V}}} + 2\int_\Omega F(u)\,dx + 2\int_\Gamma G(u)\,d\Sigma) + 2\|\frac{\partial U}{\partial t}\|^2_{-1} = 0$$

Scalar product by U :

$$\frac{d}{dt}\|U\|^2_{-1} + c(\|U\|^2_{\dot{\mathcal{V}}} + 2\int_\Omega F(u)\,dx + 2\int_\Gamma G(u)\,d\Sigma) \leq c',\ c > 0$$

Sum the two :

$$\frac{dE}{dt} + c(E + \|\frac{\partial U}{\partial t}\|^2_{-1}) \leq c',\ c > 0$$

$$E = \|U\|^2_{\dot{\mathcal{V}}} + \|U\|^2_{-1} + 2\int_\Omega F(u)\,dx + 2\int_\Gamma G(u)\,d\Sigma$$

$$E \geq c\|U\|^2_{\dot{\mathcal{V}}} - c',\ c > 0$$

Scalar product by AU :

$$\frac{d}{dt}\|U\|_{\mathcal{H}}^2 + \|AU\|_{\mathcal{H}}^2 \leq \|\mathcal{F}(U)\|_{\mathcal{H}}^2$$

$$\|\mathcal{F}(U)\|_{\mathcal{H}}^2 \leq c(\|u\|_{L^6(\Omega)}^6 + \|u\|_{\Gamma}^2 + 1)$$

$$\leq c(\|u\|_{H^1(\Omega)}^6 + \|u\|_{\Gamma}^2 + 1)$$

$$\leq c(\|U\|_{\dot{\mathcal{V}}}^6 + 1)$$

Thus :

$$\frac{d}{dt}\|U\|_{\mathcal{H}}^2 + \|AU\|_{\mathcal{H}}^2 \leq c(\|U\|_{\dot{\mathcal{V}}}^6 + 1)$$

Remark : No higher-order estimate

Theorem : We assume that $U_0 \in \mathcal{V}$. Then, the problem possesses a unique weak solution $U = \overline{U} + \kappa$ such that
$\overline{U} \in L^\infty(\mathbb{R}^+; \dot{\mathcal{V}}) \cap \mathcal{C}([0,T]; \dot{\mathcal{V}}') \cap L^2(0,T; D(A))$ and $\frac{\partial U}{\partial t} \in L^2(\mathbb{R}^+; \dot{\mathcal{V}}')$, $\forall T > 0$. Furthermore,

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \overline{\mathcal{F}(U)} = 0 \text{ in } L^2(0,T; \dot{\mathcal{H}}).$$

Remark : $U \in L^2(0,T; H^2(\Omega) \times H^2(\Gamma))$, $\forall T > 0$.

$A^{-1}\frac{d\overline{U}}{dt} \in L^2(0,T;H^1(\Omega) \times H^1(\Gamma)), T > 0$

g affine : $g(u) \in L^\infty(0,T;H^1(\Gamma))$

$H^1(\Omega) \subset L^6(\Omega) : f(u) \in L^\infty(0,T;L^2(\Omega))$

$\nabla f(u) = f'(u)\nabla u$ (Agmon's inequality) :

$$\begin{aligned}\|\nabla f(u)\|_{L^2(\Omega)} &\leq c(\|u\|^2_{L^\infty(\Omega)} + 1)\|\nabla u\| \\ &\leq c(\|u\|^2_{H^1(\Omega)}\|u\|_{H^2(\Omega)} + \|u\|_{H^1(\Omega)}),\end{aligned}$$

$\rightarrow f(u) \in L^2(0,T;H^1(\Omega)), \mathcal{F}(U) \in L^2(0,T;H^1(\Omega) \times H^1(\Gamma))$

Thus :

$$AU = \mathcal{G}(t),\ G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$G \in L^2(0,T;H^1(\Omega) \times H^1(\Gamma))$$

Elliptic problem :

$$-\Delta u = g_1$$

$$-\Delta_\Gamma u + u + \frac{\partial u}{\partial \nu} = g_2 + u \text{ on } \Gamma$$

$\to U \in L^2(0,T;H^3(\Omega) \times H^3(\Gamma))$

Differentiate

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \overline{\mathcal{F}(U)} = 0$$

with respect to time :

$$A^{-1}\frac{d}{dt}\frac{dU}{dt} + A\frac{dU}{dt} + \overline{\mathcal{F}'(U).\frac{dU}{dt}} = 0$$

$$\mathcal{F}'(U).\frac{dU}{dt} = \begin{pmatrix} f'(u)\frac{\partial u}{\partial t} \\ g'(u)\frac{\partial u}{\partial t} \end{pmatrix}$$

Scalar product of the above equation by $\frac{dU}{dt}$:

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial U}{\partial t}\|_{-1}^2 + \|\frac{\partial U}{\partial t}\|_{\dot{\mathcal{V}}}^2 \leq c\|\frac{\partial U}{\partial t}\|_{\mathcal{H}}^2$$

interpolation inequality $\|\frac{\partial U}{\partial t}\|_{\mathcal{H}}^2 \leq \|\frac{\partial U}{\partial t}\|_{-1}^2\|\frac{\partial U}{\partial t}\|_{\dot{\mathcal{V}}}^2$:

$$\frac{d}{dt}\|\frac{\partial U}{\partial t}\|_{-1}^2 + \|\frac{\partial U}{\partial t}\|_{\dot{\mathcal{V}}}^2 \leq c\|\frac{\partial U}{\partial t}\|_{-1}^2$$

$\rightarrow \frac{\partial U}{\partial t} \in L^\infty(r,T;\mathcal{V}') \cap L^2(r,T;H^1(\Omega)\times H^1(\Gamma)), T > r, r > 0$

$\|\Delta f(u)\| \leq c(\|u^2\Delta u\| + \|\Delta u\| + \|u|\nabla u|^2\|) : f(u) \in L^2(0,T;H^2(\Omega))$

Elliptic problem :

$$AU = \mathcal{G}(t)$$

$$G \in L^2(r,T;H^2(\Omega)\times H^2(\Gamma))$$

$\rightarrow U \in L^2(r,T;H^4(\Omega)\times H^4(\Gamma))$

$\rightarrow$ The solution is strong as soon as $t > 0$

We cannot write

$$\frac{dU}{dt} + A^2U + A\mathcal{F}(U) = 0$$

in $L^2(r, T; \mathcal{H})$ ($\mathcal{F}(U(t))$ does not belong to $D(A)$)

We can write

$$\frac{dU}{dt} + A(AU + \overline{\mathcal{F}(U)}) = 0 \text{ in } L^2(r, T; \mathcal{H})$$

$\rightarrow \frac{dU}{dt} + A(AU + \mathcal{F}(U)) = 0$ in $L^2(r, T; \mathcal{H})$
Thus :

$$\frac{dU}{dt} = -AW \text{ in } L^2(r, T; \mathcal{H})$$

$$W = AU + \mathcal{F}(U) \text{ in } L^2(r, T; \mathcal{H})$$

$\rightarrow$ We recover the original Cahn-Hilliard system for $t > 0$

Continuous (for the $\mathcal{V}'$-topology) semigroup

$$S(t) : \mathcal{V}_\kappa \to \mathcal{V}_\kappa,\ U_0 \mapsto U(t),\ t \geq 0$$

$$\mathcal{V}_\kappa = \{U \in \mathcal{V},\ \langle U \rangle = \kappa\}$$

Theorem : The semigroup $S(t)$ possesses the global attractor $\mathcal{A}_\kappa$ on $\mathcal{V}'_\kappa$ for the $\mathcal{V}'$-topology.

Logarithmic nonlinear terms :

First proof : duality arguments

Approximated problems :

$$A^{-1}\frac{d\overline{U}_N}{dt} + A\overline{U}_N + \overline{\mathcal{F}_N(U)} = 0$$

$$U_N(0) = U_0$$

$$\overline{\mathcal{F}_N(U)} = \begin{pmatrix} f_N(u) \\ g(u) \end{pmatrix}$$

A priori estimates :

$$\frac{dE_N}{dt} + c(E_N + \|\frac{\partial U_N}{\partial t}\|_{-1}^2 + \|f_N(u_N)\|_{L^1(\Omega)}) \leq c',\ c > 0$$

$$E_N = \|\overline{U}_N\|_{\dot{\mathcal{V}}}^2 + \|\overline{U}_N\|_{-1}^2 + 2\int_\Omega F_N(u_N)\,dx + 2\int_\Gamma G(u_N)\,d\Sigma$$

$$E_N \geq c\|\overline{U}_N\|_{\dot{\mathcal{V}}}^2 - c',\ c > 0$$

No estimate on $f_N(u_N)$ (in L^2)

We have :

$$\overline{\mathcal{F}_N(U_N)} = -A^{-1}\frac{d\overline{U}_N}{dt} - A\overline{U}_N$$

$\rightarrow \overline{\mathcal{F}_N(U_N)}$ is bounded, independently of N, in $L^2(0,T;\dot{\mathcal{V}}')$

$\rightarrow \overline{\mathcal{F}_N(U_N)} \rightarrow \xi$ in $L^2(0,T;\dot{\mathcal{V}}')$ weakly

At the limit :

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \xi = 0 \text{ in } L^2(0,T;\dot{\mathcal{V}}')$$

ξ can be related to $\overline{\mathcal{F}(U)}$ via subdifferentials

Different approach : variational solutions

Equation :

$$A^{-1}\frac{d\overline{U}}{dt}+A\overline{U}+\overline{\mathcal{F}(U)}=0$$

f : logarithmic nonlinear term ($f'\geq -c_0$)

Scalar product in $\mathcal{H}$ by $U-W$, $W=W(x)=\begin{pmatrix} w(x) \\ w(x) \end{pmatrix}\in\mathcal{V}$, $\langle W\rangle=\langle U_0\rangle$ ($f=f_1-c_0 s$) :

$$((A^{-1}\frac{d\overline{U}}{dt},U-W))_{\mathcal{H}}+((\overline{U},U-W))_{\dot{\mathcal{V}}}+((f_1(u),u-w))$$
$$-c_0((u,u-w))+((g(u),u-w))_{\Gamma}=0$$

f_1 is monotone increasing :

$$((A^{-1}\frac{d\overline{U}}{dt}, U - W))_{\mathcal{H}} + ((\overline{U}, U - W))_{\dot{\mathcal{V}}} + ((f_1(w), u - w))$$
$$\leq c_0((u, u - w)) - ((g(u), u - w))_{\Gamma}$$

Definition : Let U_0 belong to $\mathcal{V}$. Then, a function U is a variational solution, with initial datum U_0, if, $\forall T > 0$,

(i) $U \in \mathcal{C}([0,T];\mathcal{V}') \cap L^{\infty}(0,T;\mathcal{V})$.

(ii) $\frac{\partial U}{\partial t} \in L^2(0,T;\dot{\mathcal{V}}')$.

(iii) $f(u) \in L^1(\Omega \times (0,T))$.

(iv) $-1 < u(x,t) < 1$, a.e. (x,t).

(v) $U(0) = U_0$.

(vi) $\langle U(t) \rangle = \langle U_0 \rangle, \ \forall t \geq 0$.

(vii) The variational inequality is satisfied for almost every $t > 0$ and for every test function $W = W(x)$ such that $W \in \mathcal{V}, f(w) \in L^1(\Omega)$ and $\langle W \rangle = \langle W_0 \rangle$.

Theorem : We assume that $U_0 \in \mathcal{V}$. Then, the problem possesses at most one variational solution U such that $U(0) = U_0$.

We need to take as test functions the solutions themselves

We call admissible any function $W = W(x,t)$ such that W satisfies the regularity properties of a variational solution and $\langle W(t) \rangle = \langle U_0 \rangle$, $t \geq 0$ ($W \in \mathcal{C}([0,T]; \mathcal{V}_{\mathrm{w}})$, $\forall T > 0$)

Take $W = W(t)$:

$$\int_s^t (((A^{-1}\frac{d\overline{U}}{dt}, U - W))_{\mathcal{H}} + ((\overline{U}, U - W))_{\dot{\mathcal{V}}} + ((f_1(w), u - w)))\, d\tau$$

$$\leq \int_s^t (c_0((u, u - w)) - ((g(u), u - w))_{\Gamma})\, d\tau,\ \forall t > s > 0$$

Can be used in the definition

We need a second variational inequality

Set, for any admissible test function W :

$$Z_\alpha = (1-\alpha)U + \alpha W,\ \alpha \in (0,1]$$

$|f_1|$ is convex :

$$|f_1(z_\alpha)| \leq |f_1(u)| + |f_1(w)| \in L^1(\Omega \times (0,T)),\ T > 0$$

$\rightarrow Z_\alpha$ is an admissible test function

Take $W = Z_\alpha$, divide by α :

$$\int_s^t (((A^{-1}\frac{d\overline{U}}{dt}, U - W))_{\mathcal{H}} + ((\overline{U}, U - W))_{\dot{\mathcal{V}}} + ((f_1(z_\alpha), u - w)))\, d\tau$$

$$\leq \int_s^t (c_0((u, u - w)) - ((g(u), u - w))_\Gamma)\, d\tau,\ \forall t > s > 0$$

Pass to the limit $\alpha \to 0^+$ (Lebesgue's theorem) :

$$\int_s^t (((A^{-1}\frac{d\overline{U}}{dt}, U - W))_{\mathcal{H}} + ((\overline{U}, U - W))_{\dot{\mathcal{V}}} + ((f_1(u), u - w)))\, d\tau$$
$$\leq \int_s^t (c_0((u, u - w)) - ((g(u), u - w))_\Gamma)\, d\tau,\ \forall t > s > 0$$

Consider two variational solutions U_1, U_2, with initial data $U_{1,0}$, $U_{2,0}$ having the same total mass, take $U = U_1$, $W = U_2$ in the first VI, $U = U_2$ and $W = U_1$ in the second :

$$\int_s^t (((A^{-1}\frac{d\overline{U}_1}{dt}, U_1 - U_2))_{\mathcal{H}} + ((\overline{U}_1, U_1 - U_2))_{\dot{\mathcal{V}}} + ((f_1(u_2), u_1 - u_2)))\, d\tau$$

$$\leq \int_s^t (c_0((u_1, u_1 - u_2)) - ((g(u_1), u_1 - u_2))_\Gamma)\, d\tau,\ \forall t > s > 0$$

$$\int_s^t (((A^{-1}\frac{d\overline{U}_2}{dt}, U_2 - U_1))_{\mathcal{H}} + ((\overline{U}_2, U_2 - U_1))_{\dot{\mathcal{V}}} + ((f_1(u_2), u_2 - u_1)))\, d\tau$$

$$\leq \int_s^t (c_0((u_2, u_2 - u_1)) - ((g(u_2), u_2 - u_1))_\Gamma)\, d\tau,\ \forall t > s > 0$$

Sum (U_1 and U_2 have the same total mass) :

$$\frac{1}{2}(\|U_1(t) - U_2(t)\|_{-1}^2 - \|U_1(s) - U_2(s)\|_{-1}^2) + \int_s^t \|U_1 - U_2\|_{\dot{\mathcal{V}}}^2\, d\tau$$

$$\leq \int_s^t (c_0\|u_1 - u_2\|^2 - ((g(u_1) - g(u_2), u_1 - u_2))_\Gamma)\, d\tau$$

Thus :

$$\|U_1(t) - U_2(t)\|_{-1}^2 - \|U_1(s) - U_2(s)\|_{-1}^2 + 2\int_s^t \|U_1 - U_2\|_{\dot{\mathcal{V}}}^2 \, d\tau$$

$$\leq c \int_s^t \|U_1 - U_2\|_{\mathcal{H}}^2 \, d\tau$$

Interpolation inequality $\|U_1 - U_2\|_{\mathcal{H}} \leq \|U_1 - U_2\|_{-1}^{\frac{1}{2}} \|U_1 - U_2\|_{\dot{\mathcal{V}}}^{\frac{1}{2}}$:

$$\|U_1(t) - U_2(t)\|_{-1}^2 \leq \|U_1(s) - U_2(s)\|_{-1}^2 + c \int_s^t \|U_1 - U_2\|_{-1}^2 \, d\tau$$

Gronwall's lemma :

$$\|U_1(t) - U_2(t)\|_{-1} \leq e^{c(t-s)} \|U_1(s) - U_2(s)\|_{-1}, \ t \geq s > 0$$

Let s go to 0 :

$$\|U_1(t) - U_2(t)\|_{-1} \leq e^{ct} \|U_{1,0} - U_{2,0}\|_{-1}, \ t \geq 0$$

Remarks :

U_N satisfies a corresponding variational inequality

We cannot pass to the limit (not enough regularity on U_N)

We can consider more regular test functions : we lose the uniqueness

Second dynamic boundary conditions :

First boundary condition : no mass flux at the boundary :

$$\frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma$$

Second boundary condition : we consider, in addition to the Ginzburg-Landau free energy

$$\Psi_\Omega(u, \nabla u) = \int_\Omega (\frac{\alpha}{2}|\nabla u|^2 + F(u))dx$$

the surface free energy

$$\Psi_\Gamma(u, \nabla u) = \int_\Gamma (\frac{\alpha_\Gamma}{2}|\nabla_\Gamma u|^2 + G(u))dx$$

$\alpha_\Gamma > 0$
∇_Γ : surface gradient

Original surface potential : $G(s) = \frac{1}{2}a_\Gamma s^2 + b_\Gamma s$, $a_\Gamma > 0$

Total energy : $\Psi = \Psi_\Omega + \Psi_\Gamma$

The system tends to minimize the excess surface energy :

$$\frac{1}{d}\frac{\partial u}{\partial t} - \alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

$d > 0$: relaxation parameter
Δ_Γ : Laplace-Beltrami operator
$g = G'$

$\rightarrow$ Dynamic boundary condition

Regular potentials : the system is well understood

Contributors : R. Chill, C.G. Gal, E. Fašangová, A. Miranville, J. Pruess, R. Racke, H. Wu, S. Zelik, S. Zheng, ...

Singular potentials : more complicated

First existence and uniqueness result : G. Gilardi-A. Miranville-G. Schimperna

For f singular and g regular : sign assumptions on g near the singular points of f :

$$g(1) > 0, \; g(-1) < 0$$

Forces the order parameter to stay away from ± 1 on Γ

Question :

• What happens when the sign conditions are not satisfied ?

Nonexistence of classical solutions :

When the sign conditions are not satisfied, we can have nonexistence of classical solutions

We consider the scalar ODE

$$\begin{aligned} &y'' - f(y) = 0, \ x \in (-1,1) \\ &y'(\pm 1) = K > 0 \end{aligned}$$

Assumptions :

• f is singular at ± 1
• $F(\pm 1) < +\infty$ ($F' = f$)
• f is odd

Satisfied by the usual logarithmic potentials

When K is small : existence and uniqueness of a solution which is separated from the singular values ($\|y\|_{L^\infty(-1,1)} < 1$) and is odd

Standard interior regularity estimates yield

$$|y'(x)| \leq c_0, \ |y(x)| \leq 1 - \delta$$

$x \in (-\frac{1}{2}, \frac{1}{2})$, $\delta > 0$, c_0 independent of K

Multiply the equation by y' and integrate over $(0, 1)$:

$$|\frac{1}{2}K^2 - F(y(1))| \leq c_1$$

c_1 (and $F(\pm 1)$) independent of K

This inequality cannot hold when K is large

$\rightarrow$ We do not have a classical solution

Since y is odd, the ODE can be rewritten as

$$y'' - f(y) =< y'' - f(y) >$$

$< . >= \frac{1}{\mathrm{Vol}(\cdot)} \int_\Omega .dx$

$\rightarrow$ 1D stationary Cahn-Hilliard system with dynamic BCs

Convergence of a sequence of solutions to regularized problems :

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta w \\
&w = -\Delta u + f_0(u) + \lambda u,\ \lambda \in R \\
&\frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \\
&\frac{\partial \psi}{\partial t} - \Delta_\Gamma \psi + g_0(\psi) + \psi + \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \\
&\psi = u|_\Gamma
\end{aligned}
$$

$f(s) = f_0(s) + \lambda s,\ g(s) = g_0(s) + s$

Assumptions :

- $f_0 \in \mathcal{C}^2(-1,1),\ f_0(0) = 0$
- $\lim_{s\to\pm 1} f_0(s) = \pm\infty,\ \lim_{s\to\pm 1} f_0'(s) = +\infty$
- $f_0' \geq 0,\ \mathrm{sgn}(s) f_0''(s) \geq 0$
- $g_0 \in \mathcal{C}^2(R),\ \|g_0\|_{\mathcal{C}^2(R)} \leq c$

Regularized potential :

$$
\begin{aligned}
&f_{0,N}(s) = f_0(s),\ |s| \leq 1 - \tfrac{1}{N} \\
&f_{0,N}(s) = f_0(1 - \tfrac{1}{N}) + f_0'(1 - \tfrac{1}{N})(s - 1 + \tfrac{1}{N}) \\
&s > 1 - \tfrac{1}{N} \\
&f_{0,N}(s) = f_0(-1 + \tfrac{1}{N}) + f_0'(-1 + \tfrac{1}{N})(s + 1 - \tfrac{1}{N}) \\
&s < -1 + \tfrac{1}{N}
\end{aligned}
$$

Regularized problem : f_0 replaced by $f_{0,N}$

Existence and uniqueness of the solution u_N to the regularized problem

Satisfies, for N large enough

$$\begin{aligned}
&\|u_N(t)\|^2_{\mathcal{C}^\alpha(\Omega)} + \|u_N(t)\|^2_{H^2(\Gamma)} + \|u_N(t)\|^2_{H^2(\Omega_\epsilon)} + \|u_N(t)\|^2_{H^1(\Omega)} + \\
&\|\tfrac{\partial u_N}{\partial t}(t)\|^2_{H^{-1}(\Omega)} + \|\tfrac{\partial u_N}{\partial t}(t)\|^2_{L^2(\Gamma)} + \\
&\|\nabla D_\tau u_N(t)\|^2_{L^2(\Omega)^{(n-1)n}} + \|f_{0,N}(u_N(t))\|_{L^1(\Omega)} + \\
&\int_t^{t+1}(\|\tfrac{\partial u_N}{\partial t}(s)\|^2_{H^{-1}(\Omega)} + \|\tfrac{\partial u_N}{\partial t}(s)\|^2_{L^2(\Gamma)})ds \leq \\
&ce^{-\beta t}(1 + \|u_N(0)\|^2_{H^1(\Omega)} + \|u_N(0)\|^2_{H^1(\Gamma)} + \\
&\|\tfrac{\partial u_N}{\partial t}(0)\|^2_{H^{-1}(\Omega)} + \|\tfrac{\partial u_N}{\partial t}(0)\|^2_{L^2(\Gamma)})^2 + c'
\end{aligned}$$

$\Omega_\epsilon = \{x \in \Omega,\ d(x,\Gamma) > \epsilon\},\ \epsilon > 0$
$D_\tau u_N = \nabla u_n - \frac{\partial u_N}{\partial \nu}\nu$
$\alpha > 0,\ \beta > 0,\ c,\ c'$ independent of N

Remark : Actually, $u_N(t) \in H^2(\Omega)$, but this regularity does not pass to the limit

Smoothing property :

$$\|\frac{\partial u_N}{\partial t}(t)\|^2_{H^{-1}(\Omega)} + \|\frac{\partial u_N}{\partial t}(t)\|^2_{L^2(\Gamma)} \leq$$
$$\frac{c}{t}(1 + \|u_N(0) - \langle u_N(0)\rangle\|^2_{H^{-1}(\Omega)} + \|u_N(0)\|^2_{L^2(\Gamma)})$$

$t \in (0,1],\ c$ independent of N

Lipschitz estimate :

$$\|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} +$$
$$\|u_1(t) - u_2(t)\|_{L^2(\Gamma)} \leq$$
$$ce^{c't}(\|u_1(0) - u_2(0)\|_{H^{-1}(\Omega)} +$$
$$\|u_1(0) - u_2(0)\|_{L^2(\Gamma)})$$
$$\langle u_1(0)\rangle = \langle u_2(0)\ = m,\ t \geq 0$$

$c,\ c'$ independent of $t,\ N,\ u_1,\ u_2$

u_N converges to some function u

We wish to call u the "generalized" solution to the singular problem

Variational solutions :

We set

$$B(u,v) = (\nabla u, \nabla v)_\Omega + \lambda(u,v)_\Omega + \\ +L((-\Delta)^{-1}\overline{u}, \overline{v})_\Omega + (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma$$

$u,\ v \in H^1(\Omega) \otimes H^1(\Gamma) = \{w,\ w \in H^1(\Omega), w|_\Gamma \in H^1(\Gamma)\}$

$L > 0$ chosen s.t.

$$\|\nabla u\|^2_{L^2(\Omega)^n} + \lambda\|u\|^2_{L^2(\Omega)} + L\|u\|^2_{H^{-1}(\Omega)} \geq \\ \frac{1}{2}\|u\|^2_{H^1(\Omega)},\ u \in H^1(\Omega),\ \langle u \rangle = 0$$

$\overline{u} = u - \langle u \rangle$
$(.,.)_\Omega$, $(.,.)_\Gamma$: scalar products in $L^2(\Omega)$ and $L^2(\Gamma)$

We rewrite the problem as

$$
\begin{array}{l}
(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + \\
f_0(u) + \lambda u - \langle w \rangle = 0 \\
w = -\Delta u + f_0(u) + \lambda u \\
\frac{\partial \psi}{\partial t} - \Delta_\Gamma \psi + g(\psi) + \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \\
\psi = u|_\Gamma \\
u|_{t=0} = u_0,\ \psi|_{t=0} = \psi_0
\end{array}
$$

We multiply the first equation by $u - v$, $v = v(x)$ s.t.

$$\langle u(t) - v \rangle = 0,\ t \geq 0 :$$

$$
\begin{array}{l}
((-\Delta)^{-1}\frac{\partial u}{\partial t}, u - v)_\Omega + (\frac{\partial u}{\partial t}, u - v)_\Gamma + \\
B(u, u - v) + (f_0(u), u - v)_\Omega = \\
L(u, (-\Delta)^{-1}(u - v))_\Omega - (g(u), u - v)_\Gamma
\end{array}
$$

Positivity of B and monotonicity of f_0 :

$$\begin{aligned}
&((-\Delta)^{-1}\tfrac{\partial u}{\partial t}, u-v)_\Omega + (\tfrac{\partial u}{\partial t}, u-v)_\Gamma + \\
&B(v, u-v) + (f_0(v), u-v)_\Omega \leq \\
&L(u, (-\Delta)^{-1}(u-v))_\Omega - (g(u), u-v)_\Gamma
\end{aligned}$$

Variational inequality (VI)

We set

$$\begin{aligned}
&\Phi = \{(u,\psi) \in L^\infty(\Omega) \times L^\infty(\Gamma), \\
&\|u\|_{L^\infty(\Omega)} \leq 1,\ \|\psi\|_{L^\infty(\Gamma)} \leq 1\}
\end{aligned}$$

Definition : Let $(u_0, \psi_0) \in \Phi$. Then, (u, ψ) is a variational solution if

(i) $u(t)|_\Gamma = \psi(t)$ a.e. $t > 0$, $u(0) = u_0$, $\psi(0) = \psi_0$;

(ii) $-1 < u(t,x) < 1$ a.e. $(t,x) \in R^+ \times \Omega$;

(iii) $(u, \psi) \in \mathcal{C}([0, +\infty); H^{-1}(\Omega) \times L^2(\Gamma)) \cap L^2(0, T; H^1(\Omega) \times H^1(\Gamma))$, $T > 0$;

(iv) $f(u) \in L^1((0,T) \times \Omega), \ T > 0$;

(v) $(\frac{\partial u}{\partial t}, \frac{\partial \psi}{\partial t}) \in L^2(\tau, T; H^{-1}(\Omega) \times L^2(\Gamma)), T > \tau > 0$;

(vi) $\langle u(t) \rangle = \langle u_0 \rangle, \ t \geq 0$;

(vii) the variational inequality (VI) is satisfied for a.e. $t > 0$ and every test function $v = v(x)$ s.t. $v \in H^1(\Omega) \otimes H^1(\Gamma), f(v) \in L^1(\Omega), \langle v \rangle = \langle u_0 \rangle$.

Remark : $u(t)|_\Gamma = \psi(t)$ only for $t > 0$

• A variational solution, if it exists, is unique

• $\forall (u_0, \psi_0) \in \Phi$, $\exists$ a variational solution and $(u_N, \psi_N = u_n|_\Gamma)$ converges (for a subsequence) to a variational solution

• The variational solutions satisfy the a priori estimates mentioned earlier

• The variational solutions satisfy the smoothing and Lipschitz properties

A variational solution does not necessarily solve the equations in the usual sense

True if $u(t) \in H^2(\Omega)$

A variational solution solves the first equation

$$\begin{gathered}(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + \\ f_0(u) + \lambda u - \langle w \rangle = 0 \text{ in } \mathcal{D}'\end{gathered}$$

Does not necessarily satisfy the dynamic boundary condition

More precisely, the trace

$$\frac{\partial u}{\partial \nu} = [\frac{\partial u}{\partial \nu}]_{\text{int}}$$

exists in $L^\infty(\tau, T; L^1(\Gamma))$, $0 < \tau < T$

(u_N, ψ_N) satisfies

$$\frac{\partial \psi_N}{\partial t} - \Delta_\Gamma \psi_N + g(\psi_N) + \frac{\partial u_N}{\partial \nu} = 0 \text{ on } \Gamma$$

in $L^\infty(\tau, T; L^2(\Gamma)),\ T > \tau > 0$, and the limit

$$[\frac{\partial u}{\partial \nu}]_{\text{ext}} = \lim_{N \to +\infty} \frac{\partial u_N}{\partial \nu}$$

exists in $L^\infty(\tau, T; L^2(\Gamma))$ weak star

$$\to \frac{\partial \psi}{\partial t} - \Delta_\Gamma \psi + g(\psi) + [\frac{\partial u}{\partial \nu}]_{\text{ext}} = 0 \text{ on } \Gamma$$

$\to$ A variational solution is a classical one if

$$[\frac{\partial u}{\partial \nu}]_{\text{int}} = [\frac{\partial u}{\partial \nu}]_{\text{ext}} \text{ a.e. } (t, x) \in R^+ \times \Gamma$$

Remark : Scalar ODE

$$\begin{aligned} &y'' - f(y) = 0,\ x \in (-1, 1) \\ &y'(\pm 1) = K > 0 \end{aligned}$$

There exists a critical value K_0 s.t., if $K > K_0$, there is no classical solution

However, there exists a variational solution which is solution to

$$\begin{aligned} &y'' - f(y) = 0,\ x \in (-1, 1) \\ &y(\pm 1) = \pm 1 \end{aligned}$$

$$y'|_{x=\pm 1} \neq K$$

Existence of classical solutions :

Related to the H^2-regularity and the separation from the singularities of f_0

Theorem : Let (u, ψ) be a variational solution and set, for $\delta > 0$ and $T > 0$,

$$\Omega_\delta(T) = \{x \in \Omega,\ |u(T,x)| < 1 - \delta\}.$$

Then, $u(T) \in H^2(\Omega_\delta(T))$ and

$$\|u(T)\|_{H^2(\Omega_\delta(T))} \leq Q_{\delta,T},$$

where $Q_{\delta,T}$ is independent of u.

Consequence : if

$$|u(t,x)| < 1 \text{ a.e. } (t,x) \in R^+ \times \Gamma$$

then

$$[\frac{\partial u}{\partial \nu}]_{\mathrm{int}} = [\frac{\partial u}{\partial \nu}]_{\mathrm{ext}} \text{ a.e. } (t,x) \in R^+ \times \Gamma$$

and u is a classical solution

$\rightarrow$ The existence of classical solutions is related to the separation property on the boundary

True if f_0 has sufficiently strong singularities

Theorem : We assume that

$$\lim_{s \to \pm 1} F_0(s) = +\infty, \; F_0' = f_0.$$

Then, the separation property on the boundary holds and a variational solution is a classical one.

True if f_0 behaves like $\frac{s}{(1-s^2)^p}$, $p > 1$

Not true for logarithmic potentials

In that case, we can have $|u(t,x)| = 1$ on a set with nonzero measure on the boundary (possibly, on the whole boundary)

Theorem : We assume that

$$\pm g(\pm 1) > 0.$$

Then, a variational solution is a classical one.