The Cahn-Hilliard equation with logarithmic nonlinear terms (II)

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NSF-CBMS Conference
The Cahn–Hilliard equation : recent advances and applications

Viscous Cahn-Hilliard equation : A. Miranville-S. Zelik

Cahn-Hilliard equation :

$$\begin{aligned}&(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u) - \langle f(u)\rangle = 0\\ &\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\\ &u\big|_{t=0} = u_0\end{aligned}$$

$(-\Delta)^{-1}$: inverse minus Laplace operator with Neumann BCs on functions with vanishing average
$\langle . \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega .dx$

Conservation of mass : $\langle u(t)\rangle = \langle u_0\rangle,\ t \geq 0$

Assumption : $\langle u_0 \rangle = m_0$, $|m_0| \leq 1 - \kappa$, $\kappa \in (0, 1)$

Set, for $m_0 \in [-1 + \kappa, 1 - \kappa]$

$$\begin{aligned}
&D_{m_0} = \{q \in H^2(\Omega),\ \tfrac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma,\\
&\|q\|_{L^\infty(\Omega)} \leq 1,\ \langle q \rangle = m_0,\ f(q) \in L^2(\Omega),\\
&\Delta^2 q - \Delta f(q) \in H^{-1}(\Omega)\}
\end{aligned}$$

$$\begin{aligned}
&\|q\|^2_{D_{m_0}} = \|q\|^2_{H^2(\Omega)} + \|f(q)\|^2_{L^2(\Omega)} +\\
&\|\Delta^2 q - \Delta f(q)\|^2_{H^{-1}(\Omega)}
\end{aligned}$$

A function u is solution if $u \in L^\infty(0, T; D_{m_0}) \cap \mathcal{C}([0, T]; H^{-1}(\Omega))$, $\forall T > 0$, and the equation is satisfied in the sense of distributions

Viscous Cahn-Hilliard equation :

$$\beta\frac{\partial u}{\partial t} + (-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u) - \langle f(u)\rangle = 0,\ \beta > 0$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$
$$u\big|_{t=0} = u_0$$

Set

$$D^{\beta}_{m_0} = \{q \in H^2(\Omega),\ \frac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$\|q\|_{L^\infty(\Omega)} \leq 1,\ \langle q\rangle = m_0,\ f(q) \in L^2(\Omega),$$
$$\sqrt{\beta}\phi \in L^2(\Omega),\ \phi \in H^{-1}(\Omega),$$
$$\phi = (\beta + (-\Delta)^{-1})^{-1}(\Delta q - f(q) + \langle f(q)\rangle)\}$$

$$\|q\|^2_{D^{\beta}_{m_0}} = \|q\|^2_{H^2(\Omega)} + \|f(q)\|^2_{L^2(\Omega)} +$$
$$\beta\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{H^{-1}(\Omega)}$$

Difficulty : prove that u is a priori separated from ± 1

• Dissipative estimate :

$$\|u(t)\|^2_{D^\beta_{m_0}} + \int_t^{t+1} \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} ds \leq$$
$$c_\kappa(\|u_0\|_{D^\beta_{m_0}})\chi(1-t) + c'_\kappa,\ t \geq 0,\ \beta \geq 0$$

χ : Heaviside function
$c_\kappa,\ c'_\kappa$: independent of u and β

• Consequence : $\forall \mu > 0$

$$\int_t^{t+1} \|f(u)\|^2_{L^\infty(\Omega)} ds \leq c_{\kappa,\mu},\ t \geq \mu$$

$c_{\kappa,\mu}$: independent of $\beta \geq 0$, t, u

$\rightarrow |u(t,x)| < 1,$ a.e. (t,x)

Do we have

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,\ \delta \in (0,1)?$$

Can be proved in general only for $\beta > 0$:

Theorem : We assume that $\beta > 0$. Then, $\forall \mu > 0$,

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_{\beta,\kappa,\mu},\ t \geq \mu,$$

where $\delta_{\beta,\kappa,\mu} \in (0,1)$ is independent of u. Furthermore, if $\|u_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0$, $\delta_0 \in (0,1)$, then

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta'_{\beta,\kappa},\ t \geq 0,$$

where $\delta'_{\beta,\kappa} \in (0,1)$ is independent of u.

Remarks :

(i) We can rewrite the equation in the form

$$\beta\frac{\partial u}{\partial t} - \Delta u + f(u) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + \langle f(u)\rangle$$

(ii) If $\|u_0\|_{L^\infty(\Omega)} \leq 1$, the solution $u(t)$ of the viscous Cahn-Hilliard equation is strictly separated from ± 1 for $t > 0$

(iii) Both δ and δ' tend to 0 as $\beta \to 0$

$\to$ We cannot say anything for the Cahn-Hilliard equation

(iv) This is true for the Cahn-Hilliard equation under the additional assumption

$$|f'(s)| \leq c(|f(s)|^2 + 1),\ s \in (-1, 1)$$

Not satisfied by logarithmic potentials

(v) True in 1D, due to the continuous embedding $H^1 \subset \mathcal{C}$

(iv) In 2D, using the embedding of H^1 into an Orlicz space : true if

$$|f'(s)| \leq e^{c_1|f(s)|+c_2},\ s \in (-1,1)$$

Satisfied by logarithmic potentials

Idea of the proof :

We consider the equation

$$\beta \frac{\partial u}{\partial t} - \Delta u + f(u) = h,\ h \in L^\infty(0,T;H^1(\Omega)),\ f' \geq 0$$

It suffices to obtain an estimate of the form

$$\|f'(u)\|_{L^p(\Omega\times(0,T))} \leq c(p,T),\ p \geq 1,\ T > 0$$

($p = 4$ is sufficient)

Lemma : We have

$$\int_{\Omega\times(0,T)} e^{L|f(u)|}\, dx\, dt \leq c(T),\ L>0,\ T>0.$$

Multiply the equation by $f(u)e^{L|f(u)|}$

Use the young's inequality

$$ab \leq \phi(a) + \psi(b),\ a,\ b \geq 0$$

$$\phi(s) = e^s - s - 1,\ \psi(s) = (1+s)\ln(1+s) - s,\ s \geq 0$$

$\rightarrow$ We obtain

$$\int_{\Omega\times(0,T)} |f(u)|^2 e^{L|f(u)|}\, dx\, dt \leq c \int_{\Omega\times(0,T)} e^{c'|h|}\, dx\, dt$$

We conclude by using the Orlicz embedding

$$\int_{\Omega} e^{c|v|}\,dx \leq e^{c'(\|v\|^2_{H^1(\Omega)}+1)},\ v \in H^1(\Omega)$$

We use the inequality

$$|f'| \leq e^{c_1|f|+c_2}$$

$\to f'(u) \in L^p(\Omega \times (0,T)),\ T > 0,\ p \geq 1$

Remark : Degenerate mobility of the form $\kappa(s) = 1 - s^2$ and logarithmic nonlinear term :

One regularizes the mobility and the nonlinear term

$\rightarrow$ Existence of a weak solution

Simplification : $\kappa(s)f'(s)$ is not singular

Separation (not strict one) from the pure states

Degenerate mobility and regular nonlinear term : existence of a generalized solution, separation from the pure states

Allen-Cahn equation :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = 0$$

Comparison principle : strict separation property

More generally :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t)$$

$$g \in L^{\infty}(\Omega \times (0,T)),\ \forall T > 0$$

Caginalp phase-field system :

$$\beta \frac{\partial u}{\partial t} - \Delta u + f(u) = \theta,\ \beta > 0$$

$$\delta \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t},\ \delta > 0$$

u : order parameter
θ : relative temperature

Models phase transition phenomena (e.g., ice)

$\beta = \delta = 0$:

$$-\Delta u + f(u) = \theta$$

$$\Delta\theta = \frac{\partial u}{\partial t}$$

Laplacian of the first equation :

$$-\Delta^2 u + \Delta f(u) = \Delta\theta$$

$\rightarrow$ Cahn–Hilliard equation

$\beta > 0, \delta = 0:$

$$\beta \frac{\partial u}{\partial t} - \Delta u + f(u) = \theta$$

$$\Delta \theta = \frac{\partial u}{\partial t}$$

$\rightarrow$ Viscous Cahn-Hilliard equation

$\rightarrow$ "Contains" both equations

We can derive an $L^\infty(\Omega \times (0,T))$ estimate on θ

$\rightarrow$ Strict separation property

Generalization :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}$$
$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t}$$

$\alpha = \int_0^t \theta \, ds + \alpha_0$: thermal displacement variable

Based on the Maxwell-Cattaneo law

By approximating f as above : existence of a solution such that

$$|u(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,T)$$

Strict separation property : more involved

One possibility : prove an $L^\infty(\Omega)$-estimate on $\frac{\partial \alpha}{\partial t}$

The best we can have in general :

$$\|\frac{\partial \alpha}{\partial t}\|_{L^\infty(0,T;H_0^1(\Omega))} \leq c(T),\ T > 0$$

Here : $u_0 \in H_0^1(\Omega) \times H^3(\Omega)$, $\alpha_0 \in H_0^1(\Omega) \times H^3(\Omega)$, $\alpha_1 \in H_0^1(\Omega) \times H^2(\Omega)$

In one space dimension : we can conclude with the continuous embedding $H^1(\Omega) \subset \mathcal{C}(\overline{\Omega})$

We can also prove the strict separation in two space dimensions

We need ans estimate of the form

$$\|f'(u)\|_{L^p(\Omega\times(0,T))} \leq c(p,T),\ p \geq 1,\ T > 0$$

($p = 4$ is sufficient)

Lemma : We have

$$\int_{\Omega\times(0,T)} e^{L|f(u)|}\,dx\,dt \leq c(T),\ L > 0,\ T > 0.$$

Multiply the equation by $f(u)e^{L|f(u)|}$

Use the young's inequality

$$ab \leq \phi(a) + \psi(b),\ a,\ b \geq 0$$

$$\phi(s) = e^s - s - 1,\ \psi(s) = (1+s)\ln(1+s) - s,\ s \geq 0$$

$\rightarrow$ We obtain

$$\int_{\Omega\times(0,T)} |f(u)|^2 e^{L|f(u)|}\, dx\, dt \leq c$$
$$+2\int_{\Omega\times(0,T)} e^{c'|\frac{\partial\alpha}{\partial t}|}\, dx\, dt$$

We conclude by using the Orlicz embedding

$$\int_\Omega e^{c|v|}\, dx \leq e^{c'(\|v\|^2_{H^1(\Omega)}+1)},\ v \in H^1(\Omega)$$

We assume that

$$|f'| \leq e^{c|f|+c'}$$

(True for the logarithmic nonlinear terms)

$\rightarrow f'(u) \in L^p(\Omega \times (0,T)),\ T > 0,\ p \geq 1$

Differentiating the equation for u with respect to t :

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))$$

Inject in the equation for α :

$\rightarrow \frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega))$

In three space dimensions : we need

$$f'(u) \in L^6(\Omega \times (0,T)),\ T > 0$$

We can conclude when $|f'| \leq c|f|^{\frac{6}{5}} + c'$

$\rightarrow$ Not satisfied by the logarithmic nonlinear terms

Satisfied when f has a growth of the form

$$\frac{c}{(1-s^2)^r},\ r \geq 5,\ c > 0$$

close to ± 1

Higher-order Cahn-Hilliard equations :

We are not able to prove the existence of classical solutions

Example : Phase-field crystal equation

$$\frac{\partial u}{\partial t} - \Delta^3 u - 2\Delta^2 u - \Delta f(u) = 0$$

Atomistic models of crystal growth (K. Elder et al.)

Simulation methodology for problems in materials science where atomic- and microscales are tightly coupled

Operates on atomic length and diffusive time scales

Constitutes a computationally efficient alternative to molecular simulation methods

Associated free energy :

$$\Psi = \int_{\Omega} (\frac{1}{2}|\Delta u|^2 - |\nabla u|^2 + F(u))\, dx,\ F' = f$$

Ω : domain occupied by the system

Evolution equation :

$$\frac{\partial u}{\partial t} = \Delta \frac{\delta \Psi}{\delta u}$$

$\frac{\delta}{\delta u}$: variational derivative

Regular nonlinear terms :

Typically : $f(s) = \frac{s^3}{3} - \frac{s^2}{2} + as$

Mathematical analysis : M. Grasselli, H. Wu

Well-posedness, regularity of solutions

Existence of finite-dimensional attractors

Convergence of trajectories to steady states

Numerical analysis, simulations : S. Wise et al., M. Grasselli, M. Pierre

Logarithmic nonlinear terms :

Approximated problems :

$$\frac{\partial u_N}{\partial t} - \Delta^3 u_N - 2\Delta^2 u_N - \Delta f_N(u_N) = 0$$

$$u_N|_{t=0} = u_0$$

Well-posedness, regularity : standard

A priori estimates :

$$\frac{dE_N}{dt} + c(E_N + \|u_N\|^2_{H^3(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\frac{\partial u_N}{\partial t}\|^2_{H^{-1}(\Omega)})$$
$$\leq c'\|u_N\|^2_{H^{-1}(\Omega)} + c'',\ c > 0$$

$$E_N =< u_N >^2 + \|v_N\|^2_{-1} + \|u_N\|^2 + \zeta(\|\Delta v_N\|^2 - 2\|\nabla v_N\|^2 + 2\int_\Omega F_N(u_N)\,dx)$$

$$\zeta > 0 \text{ small},\ v_N = u_N - \langle u_0 \rangle$$

$$c(\|u_N\|^2_{H^2(\Omega)} + \int_\Omega F_N(u_N)\,dx) \leq E_N \leq c'(\|u_N\|^2_{H^2(\Omega)} + \int_\Omega F_N(u_N)\,dx),\ c > 0$$

No estimate on $f_N(u_N)$ in L^2

$\rightarrow$ We cannot pass to the limit in the nonlinear term

$\rightarrow$ We cannot prove the existence of a classical solution

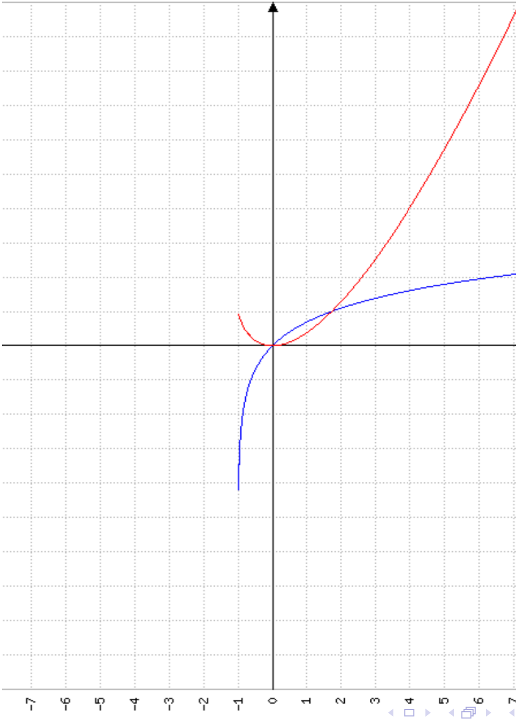
We can prove the existence of weaker solutions, based on a variational inequality

Relevant logarithmic nonlinear term : $f(s) = \ln(1+s)$, $s > -1$

(More generally : $f(s) = \ln(1+s) + (a-1)s$)

Follows from the potential $F(s) = (1+s)\ln(1+s) - s$, $f = F'$

$\rightarrow$ Not a double-well potential, contrary to the classical Cahn-Hilliard theory



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Properties of f :

• f is monotone increasing

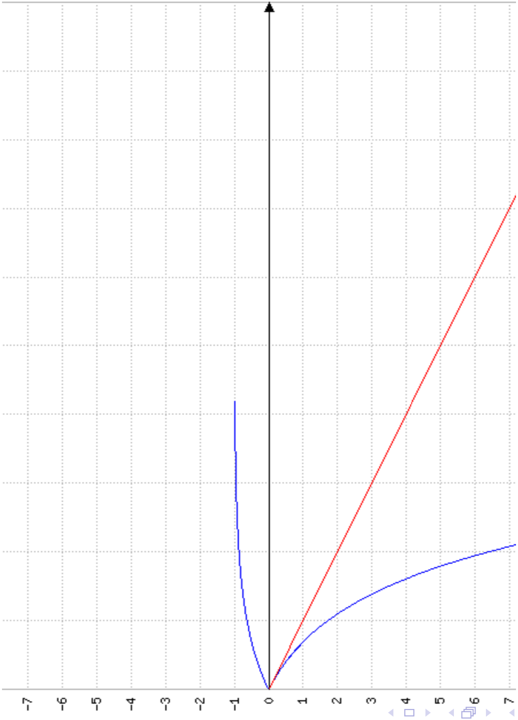
$\rightarrow$ Essential to introduce the variational inequality

• There exists a nonnegative convex function φ such that

(i) $|f(s)| \leq \varphi(s)$, $s > -1$

(ii) $\varphi(w) \in L^1((0,T) \times \Omega)$, whenever $w \in L^1((0,T) \times \Omega)$ and $f(w) \in L^1((0,T) \times \Omega)$, $T > 0$

$\rightarrow$ Essential for the uniqueness of variational solutions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Approximated problems :

We introduce the $\mathcal{C}^1-$ functions f_N, $N \in \mathbb{N}$, defined by

$$f_N(s) = \begin{cases} f(s), \ s \geq -1 + \frac{1}{N} \\ f(-1 + \frac{1}{N}) + f'(-1 + \frac{1}{N})(s + 1 - \frac{1}{N}), \ s < -1 + \frac{1}{N} \end{cases}$$

Properties of f_N :

- f_N is monotone increasing
- $f_N(s)s \geq F_N(s) \geq 0, \ s \in \mathbb{R}$, $F_N(s) = \int_0^s f_N(\xi)\, d\xi$

• For every $m > -1$, there exist two constants $\kappa_1 = \kappa_1(m) > 0$ and $\kappa_2 = \kappa_2(m) \geq 0$ such that, at least for $N \geq N_0 = N_0(m)$ large enough,

$$f_N(s)(s-m) \geq \kappa_1 |f_N(s)| - \kappa_2, \; s \in \mathbb{R}$$

• For every $m > -1$, there exist two constants $\kappa_3 = \kappa_3(m) > 0$ and $\kappa_4 = \kappa_4(m) \geq 0$ such that, at least for $N \geq N_0 = N_0(m)$ large enough,

$$f_N(s)(s-m) \geq \kappa_3 F_N(s) - \kappa_4, \; s \in \mathbb{R}$$

Remark : If $-1 + m_1 \leq m \leq m_2$, m_1, $m_2 > 0$, then κ_1, ..., κ_4 can be chosen so that they only depend on m_1 and m_2

Approximated problems :

$$\frac{\partial u_N}{\partial t} - \Delta^3 u_N - 2\Delta^2 u_N - \Delta f_N(u_N) = 0$$

$$u_N|_{t=0} = u_0$$

Well-posedness, regularity : standard

Uniform a priori estimates :

We assume that

• $u_0(x) > -1$ a.e. $x \in \Omega$

• $-1 + m_1 \leq \langle u_0 \rangle \leq m_2$, $m_1, m_2 > 0$ fixed (independently of u_0)

We have

$$\langle u_N(t) \rangle = \langle u_0 \rangle, \ t \geq 0$$

Furthermore

$$\frac{dE_N}{dt} + c(E_N + \|u_N\|^2_{H^3_(\Omega)} + \|f_N(u_N)\|_{L^1(\Omega)} + \|\frac{\partial u_N}{\partial t}\|^2_{H^{-1}(\Omega)})$$
$$\leq c'\|u_N\|^2_{H^{-1}(\Omega)} + c'',\ c > 0$$

$$E_N =< u_N >^2 + \|v_N\|^2_{-1} + \|u_N\|^2 + \zeta(\|\Delta v_N\|^2 - 2\|\nabla v_N\|^2 + 2\int_\Omega F_N(u_N)\,dx)$$

$$\zeta > 0 \text{ small},\ v_N = u_N - \langle u_0 \rangle$$

$$c(\|u_N\|^2_{H^2(\Omega)} + \int_\Omega F_N(u_N)\,dx) \leq E_N \leq c'(\|u_N\|^2_{H^2(\Omega)} + \int_\Omega F_N(u_N)\,dx),\ c > 0$$

Remark : Dissipative estimate : we are not able to absorb the right-hand side

Usual Cahn-Hilliard logarithmic nonlinear term : we can construct f_N such that

$$F_N(s) \geq cs^4 - c',\ c > 0$$

c, c' independent of N

$\rightarrow$ We can derive a dissipative estimate

Variational solutions :

We rewrite the equation in the form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t}+\Delta^2 u+2\Delta u+f(u)-\langle f(u)\rangle=0$$

We multiply by $u-w$, $w=w(x)$ smooth, $\langle w\rangle=\langle u_0\rangle$:

$$(((-\Delta)^{-1}\frac{\partial u}{\partial t},u-w))+((\Delta u,\Delta(u-w)))-2((\nabla u,\nabla(u-w)))$$

$$+((f(u),u-w))=0$$

f is monotone increasing :

$$(((-\Delta)^{-1}\frac{\partial u}{\partial t}, u-w)) + ((\Delta u, \Delta(u-w))) - 2((\nabla u, \nabla(u-w)))$$

$$+((f(w), u-w)) \leq 0$$

(variational inequality (VI))

Definition : We assume that $u_0 \in H^2(\Omega)$, with $u_0(x) > -1$ a.e. $x \in \Omega$. Then, $u = u(t,x)$ is a variational solution if

(i) $u(t,x) > -1$ a.e. (t,x)

(ii) $u \in \mathcal{C}([0,T]; H^{-1}(\Omega)) \cap L^\infty(0,T; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$, $\forall T > 0$

(iii) $\frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega))$, $\forall T > 0$

(iv) $f(u) \in L^1((0,T) \times \Omega)$, $\forall T > 0$

(v) $u(0) = u_0$

(vi) $\langle u(t) \rangle = \langle u_0 \rangle$, $t \geq 0$

(vii) the variational inequality (VI) is satisfied for every $t > 0$ and every test function $w = w(x)$ such that $w \in H^2(\Omega), f(w) \in L^1(\Omega)$ and $\langle w \rangle = \langle u_0 \rangle$

Uniqueness of variational solutions : we need to define time-dependent test functions

We call admissible any function $w = w(t,x)$ such that $w \in \mathcal{C}([0,T]; H^{-1}(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), f(w) \in L^1((0,T) \times \Omega)$, $\frac{\partial w}{\partial t} \in L^2(0,T; H^{-1}(\Omega))$, $\forall T > 0$, and $\langle w(t) \rangle = \langle u_0 \rangle$, $t \geq 0$

We take $w = w(t,.)$, for almost every $t > 0$: (vii) can be replaced by

$$\int_s^t [(((-\Delta)^{-1} \frac{\partial u}{\partial t}, u - w)) + ((\Delta u, \Delta(u - w))) - 2((\nabla u, \nabla(u - w)))$$

$$+((f(w), u - w))] \, d\xi \leq 0$$

for all $0 < s < t$ and for every admissible test function $w = w(t,x)$ (all terms are L^1 with respect to time)

We need a second variational inequality : we set

$$w_\eta = (1-\eta)u + \eta z,\ \eta \in (0,1]$$

We have

$$|f(w_\eta)| \leq \varphi(u) + \varphi(z)$$

$\rightarrow w_\eta$ is an admissible test function

Take $w = w_\eta$ and divide by η :

$$\int_s^t [(((-\Delta)^{-1}\frac{\partial u}{\partial t}, u-z)) + ((\Delta u, \Delta(u-z))) - 2((\nabla u, \nabla(u-z)))$$

$$+((f(w_\eta), u-z))]\, d\xi \leq 0$$

Pass to the limit $\eta \to 0$ (Lebesgue's dominated convergence theorem) :

$$\int_s^t [(((-\Delta)^{-1}\frac{\partial u}{\partial t}, u - z)) + ((\Delta u, \Delta(u - z))) - 2((\nabla u, \nabla(u - z)))$$

$$+((f(u), u - z))]\, d\xi \leq 0$$

for all $0 < s < t$ and for every admissible test function $z = z(t, x)$

Combine the two variational inequalities (all terms are absolutely continuous) : if u_1 and u_2 are two solutions such that $\langle u_1(0\rangle = \langle u_2(0)\rangle$

$$\frac{1}{2}\|u_1(t) - u_2(t)\|_{-1}^2 - \frac{1}{2}\|u_1(s) - u_2(s)\|_{-1}^2$$

$$+\int_s^t (\|\Delta(u_1 - u_2)\|^2 - 2\|\nabla(u_1 - u_2)\|^2)\, d\xi \leq 0$$

This yields

$$\|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} \leq ce^{c'(t-s)}\|u_1(s) - u_2(s)\|_{H^{-1}(\Omega)}$$

Pass to the limit $s \to 0$:

$$\|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} \leq ce^{c't}\|u_1(0) - u_2(0)\|_{H^{-1}(\Omega)},\ t \geq 0$$

Theorem : We assume that $u_0 \in H^2(\Omega)$, $u_0(x) > -1$ a.e. $x \in \Omega$, and $-1 + m_1 \leq \langle u_0 \rangle \leq m_2$, with $m_1, m_2 > 0$ fixed. Then, there exists a unique variational solution u

Note that u_N satisfies

$$\int_s^t [(((-\Delta)^{-1} \frac{\partial u_N}{\partial t}, u_N - w)) + ((\Delta u_N, \Delta(u_N - w))) - 2((\nabla u_N, \nabla(u_N - w)))$$

$$+((f_N(w), u_N - w))]\, d\xi \leq 0$$

for all $0 < s < t$ and for every admissible test function $w = w(t, x)$

u_N converges to a limit function u in the following sense :

$$u_N \to u \text{ in } L^\infty(0,T;H^2(\Omega)) \text{ weak} - \star \text{ and } L^2(0,T;H^3(\Omega)) \text{ weak}$$

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(0,T;H^{-1}(\Omega)) \text{ weak}$$

$$u_N \to u \text{ in } \mathcal{C}([0,T];H^2(\Omega)),\ L^2(0,T;H^2(\Omega)) \text{ and a.e. in } (0,T)\times\Omega$$

Only difficulty : passage to the limit in $\int_s^t ((f_N(w), u_N - w))\,d\xi$

By construction :

$$|f_N(w)| \leq |f(w)|$$

Lebesgue's dominated convergence theorem ($f(w) \in L^1((0,T)\times\Omega)$)

Separation property :

$f_N(u_N)$ is uniformly bounded in $L^1((0,T)\times\Omega)$ and explicit expression of f_N :

$$\text{meas}\{(t,x)\in(0,T)\times\Omega,\ u_M(t,x)<-1+\frac{1}{N}\}\leq\varphi(\frac{1}{N}),\ M\geq N$$

$$\varphi(s)=\frac{1}{|f(s-1)|}$$

c independent of N and M

Pass to the limit $M\to+\infty$ (Fatou's Lemma) and then $N\to+\infty$ ($\varphi(s)\to 0$ as $s\to 0$) :

$$\text{meas}\{(t,x)\in(0,T)\times\Omega,\ u(t,x)\leq -1\}=0$$

$f(u) \in L^1((0,T) \times \Omega)$:

Almost everywhere convergence of u_N to u and explicit expression of f_N :

$$f_N(u_N) \to f(u) \text{ a.e. in } (0,T) \times \Omega$$

Fatou's lemma :

$$\|f(u)\|_{L^1((0,T)\times\Omega)} \leq \liminf \|f_N(u_N)\|_{L^1((0,T)\times\Omega)} < +\infty$$

Remark : We can prove the existence (and uniqueness) of variational solutions in

$$\Phi_{m_1,m_2} = \{v \in L^\infty(\Omega),\ w(x) \geq -1 \text{ a.e. } x \in \Omega,$$

$$< w >= m,\ -1 + m_1 \leq m \leq m_2\}$$

$m_1, m_2 > 0$

These solutions regularize instantaneously

Remark : We have similar results for the usual Cahn-Hilliard nonlinear term

$$\begin{aligned}
&F(s) = -\theta_0 s^2 + \theta_1((1+s)\ln(1+s) \\
&+(1-s)\ln(1-s)) \\
&f(s) = F'(s) = -2\theta_0 s + \theta_1 \ln \tfrac{1+s}{1-s} \\
&s \in (-1,1),\ 0 < \theta_1 < \theta_0
\end{aligned}$$

In that case, we also have a dissipative estimate

$\rightarrow$ We can prove the existence of finite-dimensional attractors