POLARITY OF POINTS FOR GAUSSIAN RANDOM FIELDS

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We show that for a wide class of Gaussian random fields, points are polar in the critical dimension. Examples of such random fields include solutions of systems of linear stochastic partial differential equations with deterministic coefficients, such as the stochastic heat equation or wave equation with space–time white noise, or colored noise in spatial dimensions $k \geq 1$. Our approach builds on a delicate covering argument developed by M. Talagrand [*Ann. Probab.* **23** (1995) 767–775; *Probab. Theory Related Fields* **112** (1998) 545–563] for the study of fractional Brownian motion, and uses a harmonizable representation of the solutions of these stochastic PDEs.

CONTENTS

1. Introduction. Hitting probabilities are one of the most studied features of stochastic processes. Given a process $X = (X_t)$ with values in $\mathbb{R}^d$ and a subset A of $\mathbb{R}^d$, we say that X hits A if

$$P\{X_t \in A \text{ for some } t\} > 0.$$

The set A is *polar* for X if $P\{X_t \in A \text{ for some } t\} = 0$. When X is a Markov process, potential theory gives a necessary and sufficient condition for a set to be

polar; see [3] for an extensive discussion. One first constructs a potential theory associated to X, after which it follows that X hits A with positive probability if and only if $\text{cap}(A) > 0$, where $\text{cap}(A)$ is the capacity of A with respect to the potential theory associated to X.

For processes other than Markov processes, and even for Gaussian random fields, results on hitting probabilities are much less complete. One exception is the Brownian sheet, which has specific properties such as independence of increments. Using these properties, Khoshnevisan and Shi [21] have given essentially complete answers about hitting probabilities for the sheet, and the recent work of Dalang, Khoshnevisan, Nualart, Wu and Xiao [12] and Dalang and Mueller [13] has even settled the issue of multiple points of the Brownian sheet in critical dimensions.

Other interesting Gaussian random fields are for instance those obtained as solutions of linear systems of stochastic partial differential equations (SPDEs). Mueller and Tribe [25] considered systems of d stochastic heat equations

$$
(1.1) \qquad \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \dot{W}(t, x),
$$

where $t > 0$, $x \in \mathbb{R}$, $\dot{W} = \dot{W}(t, x)$ is an $\mathbb{R}^d$-valued two-parameter white noise, and the function $u(0, \cdot)$ takes values in $\mathbb{R}^d$ and is suitably specified. This system of SPDEs is interpreted in integral form in the framework of Walsh [31]. They showed (among other things) that points are polar if and only if $d \geq 6$, so that the critical dimension for hitting points is $d = 6$ for the random field u and points are polar in this critical dimension. It turns out that the method of [25] is quite specific and cannot be extended, for instance, even to the case where the system has deterministic but nonconstant coefficients.

Another case in which the issue of polarity in the critical dimension has been resolved concerns systems of reduced stochastic wave equations (in one spatial dimension) studied by Dalang and Nualart in [14]. In this case, the critical dimension is $d = 4$ and points are polar in this dimension (for linear and nonlinear systems of such equations). This situation is again special, because the natural filtration of the process has the commutation property F4 of Cairoli and Walsh [4], which makes it possible to use Cairoli's maximal inequality for multiparameter martingales [19], Chapter 7.2.

For linear and nonlinear systems of stochastic heat and wave equations, there has been much progress in recent years for all dimensions except the critical dimension. A typical result for nonlinear systems of stochastic heat equations in spatial dimension 1 is given in [9, 10]. In these papers, the authors establish upper and lower bounds on hitting probabilities of the following type:

$$
c^{-1} \operatorname{Cap}_{d-6+\eta}(A) \leq P\{u(t, x) \in A \text{ for some } (t, x) \in [1, 2]^2\} \leq c \mathcal{H}_{d-6-\eta}(A),
$$

where Cap denotes Bessel–Riesz capacity, $\mathcal{H}$ denotes Hausdorff measure, and $\eta > 0$. This type of upper and lower bound is also available for systems of heat

and wave equations in spatial dimensions $k \geq 1$ (see [11]), for linear systems of stochastic wave equations in spatial dimensions $k \geq 1$ (see [15]) and for nonlinear systems of stochastic wave equations in spatial dimensions $k \in \{1, 2, 3\}$ (see [16]). For a wide class of so-called anisotropic Gaussian random fields $v = (v(x), x \in \mathbb{R}^k)$, Biermé, Lacaux and Xiao [2] identified the critical dimension and obtained the following result. Let α_i be the Hölder exponent of the random field when the ith coordinate varies and the others are fixed, and set $Q = \alpha_1^{-1} + \cdots + \alpha_k^{-1}$. Under certain assumptions, they established the following upper and lower bounds on hitting probabilities: Fix $M > 0$ and a compact set $I \subset \mathbb{R}^k$. Then there is $0 < C < \infty$ such that for every compact set $A \subset B(0, M)$ (the open ball in $\mathbb{R}^d$ centered at 0 with radius M),

$$C^{-1} \operatorname{Cap}_{d-Q}(A) \leq P\{\exists x \in I : v(x) \in A\} \leq C \mathcal{H}_{d-Q}(A).$$

This result provides lots of information about hitting probabilities when $d \neq Q$ (see also [34]). However, in the critical case where $d = Q$ and $A = \{z_0\}$ is a single point, these two inequalities essentially reduce to $0 \leq P\{\exists x \in I : v(x) = z_0\} \leq 1$, which is uninformative. Some other references on hitting probabilities for linear systems of SPDEs include [6, 27, 32].

In order to prove that a set is polar, one typically estimates the probability that the random field visits a small ball, and then one uses a covering argument. When the dimension is strictly larger than the critical dimension, rather simple coverings do the job (typically, the covering is obtained via a deterministic partition of the parameter space). For instance, it is rather straightforward to establish that points are polar for standard Brownian motion in dimensions $d \geq 3$, but the critical dimension $d = 2$ is more difficult to handle (see [20], for instance).

In order to address the issues of exact Hausdorff measure functions and existence of multiple points for a non-Markovian random field such as fractional Brownian motion, Talagrand introduced a new kind of covering argument in the two important papers [29, 30]. His idea was to consider balls of different (random) sizes that cover a given point in the parameter space. Having noticed that at a typical point, the local (Hölder-type) regularity is better, with high probability, than what one would expect, he chooses "good balls" that give a sharp cover of the range of the process, allowing the method to succeed even in the critical dimension. His argument relies on properties of Gaussian processes as well as on certain specific properties of fractional Brownian motion. However, it seems that one of his goals was to develop a method that would extend to other situations, since he states, as one reason for studying fractional Brownian motion, that (ordinary) "Brownian motion suffers from an over abundance of special properties; and that moving away from these forces to find proofs that rely on general principles, and arguably lie at a more fundamental level."

This paper shows that Talagrand's intuition was correct. Indeed, we have isolated sufficient conditions on an anisotropic Gaussian random field $v = (v(x), x \in$

$\mathbb{R}^k$), as considered in [2, 34], under which it is possible to extend Talagrand's argument and establish polarity of points in the critical dimension; see Assumptions 2.1 and 2.4. These assumptions are satisfied by many multiparameter Gaussian random fields, for which the Hölder exponents in each parameter may be different. The random fields that we consider are typically nowhere-differentiable (see, e.g., Theorem 3.1 in [33] and Theorem 8.1 in [34]), and this assumption states the existence of particular approximations that are Lipschitz continuous but whose Lipschitz constants have a certain asymptotic growth rate. The main assumption (Assumption 2.1) is discussed in more detail at the beginning of Section 2. This assumption also leads to an upper bound on the canonical metric associated with the Gaussian random field (see Proposition 2.2).

The first technical effort is to establish Proposition 2.3, which extends an analogous result of Talagrand ([30], Proposition 3.4) and makes precise the idea that for any $x \in \mathbb{R}^k$, with high probability, there is a (random) neighborhood of x in which the increments $v(y) - v(x)$ are smaller than expected. With this result in hand, and under the assumption that the process has covariances that have better Hölder regularity than its sample paths (see Assumption 2.4), which is the case in the examples that we are interested in, we extend the method of Talagrand [30] and establish polarity of points in the critical dimension Q (see Theorem 2.6). These results are proved in Sections 3–5.

The next step is to show that the two main assumptions are satisfied in a wide class of important examples. As a warm-up, we begin in Section 6 with the case of fractional Brownian motion. Then we turn to linear systems of stochastic heat equations. In Section 7, we consider first the case of constant coefficients, in spatial dimension 1, with space–time white noise as in (1.1), and recover the result of Mueller and Tribe [25]: points are polar for this process in dimension $d = 6$. However, essentially the same calculations apply to the case of higher spatial dimensions, with spatially homogeneous noise with covariance given by a Riesz kernel with exponent $\beta \in]0, 2[$, so we also obtain polarity of points in the critical dimension $d = (4 + 2k)/(2 - \beta)$ for this case (when this fraction is an integer). The verification of Assumption 2.1 relies on a *harmonizable representation* of the solution $u(t, x)$ of the stochastic heat equation; see (7.3): this representation is analogous to the spectral representation of stationary processes (see [17, 18, 35]). It also appears in [1] and is of independent interest.

As we mentioned above, the method of Mueller and Tribe was not robust enough to extend to systems of heat equations with deterministic but nonconstant coefficients. We examine this situation in Section 8, and we obtain, under the assumption that these coefficients have some smoothness properties (expressed in terms of their Fourier transform: see Assumption 8.1), polarity of points in the critical dimension. This applies in particular to the case of spatial dimension 1 with space–time white noise, and the critical dimension remains $d = 6$.

In Section 9, we turn to linear systems of stochastic wave equations with constant coefficients. Here, we consider both the cases of spatial dimension $k = 1$

with space–time white noise, and higher spatial dimensions with spatially homogeneous noise with covariance given by a Riesz kernel with exponent $\beta \in {]}0, 2[$. The stochastic wave equation presents additional difficulties because the fundamental solution is irregular (it is not even a function when $k \geq 3$). This means that Walsh formalism does not apply directly and we use the extension of this theory developed by Dalang [8]. For the spatial dimension $k = 1$ with space–time white noise, we show that points are polar in the critical dimension $d = 4$, and in higher spatial dimensions, under the assumption $\beta \in [1, 2[$, we obtain polarity of points in the critical dimension $d = 2(k + 1)/(2 - \beta)$ (when this fraction is an integer).

The method developed by Talagrand and the extensions presented in this paper can also be applied to the issue of multiple points of Gaussian random fields in critical dimensions, and can also be used to study the same type of questions for nonlinear systems of SPDEs. These topics are the subject of research in progress and we expect to present them in future papers.

2. Main assumptions and results.

Recall that a *white noise based on a measure ν* is a set function $A \mapsto W(A)$ defined on $\mathcal{B}(\mathbb{R}^k)$ with values in $L^2(\Omega, \mathcal{F}, P)$ such that for each A, $W(A)$ is a centered normal random variable with variance $\nu(A)$, and when $A \cap B = \varnothing$, then $W(A \cup B) = W(A) + W(B)$ and $W(A)$ and $W(B)$ are independent. If $W(A)$ is a centered normal random vector with values in $\mathbb{R}^d$ instead of $\mathbb{R}$ and covariance matrix $\nu(A) \cdot I_d$ (where I_d denotes the $d \times d$ identity matrix), then we say that $A \mapsto W(A)$ is an $\mathbb{R}^d$-*valued white noise*.

In order to motivate Assumption 2.1 below, recall that many stationary Gaussian process $(v(t), t \in \mathbb{R})$ admit a "moving average" representation of the form $v(t) = \int_{\mathbb{R}} f(t - s) \, dW_s$, where f is a function and (W_s) is a Brownian motion (see [17], Chapter XI, Section 8). For fixed $t \in \mathbb{R}$, we can define a white noise by setting $v(A, t) = \int_A f(t - s) \, dW_s$. In many cases, when $f(s)$ is smooth and has appropriate decay as $s \to \pm\infty$, it happens that if $|t - s| \sim 2^{-n/\alpha}$, for some $\alpha > 0$, then $v(t) - v(s)$ is well approximated by $v([2^n, 2^{n+1}[, t) - v([2^n, 2^{n+1}[, s)$. Even though we will not be dealing with stationary processes, but with nonstationary random fields, it is often possible to construct a process that plays the same role as $v(A, t)$. This is the motivation for Assumption 2.1 below, and this assumption will be verified for the solutions to the SPDEs that we will consider in Sections 7–9, as we explain just below.

Let $v = (v(x), x \in \mathbb{R}^k)$ be a centered continuous $\mathbb{R}^d$-valued Gaussian random field with i.i.d. components. We write $v(x) = (v_1(x), \ldots, v_d(x))$.

ASSUMPTION 2.1. Let $I \subset \mathbb{R}^k$ be a closed box: $I = \prod_{j=1}^k [c_j, d_j]$, where $c_j < d_j$. Let $I^{(\varepsilon)}$ denote an ε-enlargement of I, in Euclidean norm. There is a Gaussian random field $(v(A, x), A \in \mathcal{B}(\mathbb{R}_+), x \in \mathbb{R}^k)$ and $\varepsilon_0 > 0$ such that:

(a) for all $x \in I^{(\varepsilon_0)}$, $A \mapsto v(A, x)$ is an $\mathbb{R}^d$-valued white noise with i.i.d. components, $v(\mathbb{R}_+, x) = v(x)$ and when A and B are disjoint, $v(A, \cdot)$ and $v(B, \cdot)$ are independent;

(b) there are constants $c_0 \in \mathbb{R}_+$, $a_0 \in \mathbb{R}_+$ and $\gamma_j > 0$, $j = 1, \ldots, k$, such that for all $a_0 \le a \le b \le +\infty$, $x, y \in I^{(\varepsilon_0)}$,

$$(2.1) \quad \left\| v([a, b[, x) - v(x) - v([a, b[, y) + v(y) \right\|_{L^2} \le c_0 \left[\sum_{j=1}^{k} a^{\gamma_j} |x_j - y_j| + b^{-1} \right]$$

and

$$(2.2) \qquad \left\| v([0, a_0], x) - v([0, a_0], y) \right\|_{L^2} \le c_0 \sum_{j=1}^{k} |x_j - y_j|.$$

In order to see that the above assumption is satisfied by many solutions of SPDEs, it is necessary in each case to construct the random field $v(A, x)$. Let us consider, for example, the solution $v(x)$ of the linear one-dimensional heat equation driven by space–time white noise. Then $\mathbb{R}^k$ will be replaced by $\mathbb{R}_+ \times \mathbb{R}$, and the generic variable x above becomes (t, x). We define

$$v(A, t, x) = \iint_{\max(|\tau|^{\frac{1}{4}}, |\xi|^{\frac{1}{2}}) \in A} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{\xi^2 - i\tau} W(d\tau, d\xi).$$

Then we will see in Section 7 that Assumption 2.1 is satisfied (with the exponents $\gamma_1 = 3$, $\gamma_2 = 1$, that is, $\alpha_1 = 1/4$ and $\alpha_2 = 1/2$, where the α_j are defined in the next lines), as is Assumption 2.4 below.

Define $\alpha_j \in]0, 1[$ by the relation

$$\gamma_j = \alpha_j^{-1} - 1, \qquad \text{that is,} \quad \alpha_j = (\gamma_j + 1)^{-1},$$

and define a metric

$$\Delta(x, y) = \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j}.$$

Consider also the canonical metric associated with v:

$$d(x, y) = \left\| v(x) - v(y) \right\|_{L^2}.$$

It turns out that under Assumption 2.1, the metric Δ provides an upper bound on the canonical metric.

PROPOSITION 2.2. *Under Assumption 2.1, for all $x, y \in I^{(\varepsilon_0)}$ with $\Delta(x, y) \le \min(a_0^{-1}, 1)$, we have $d(x, y) \le 4c_0 \Delta(x, y)$.*

PROOF. Fix $x, y \in I^{(\varepsilon_0)}$. Observe that for any $a \ge a_0$

$$d(x, y) \le \left\| v(x) - v([a_0, a[, x) - v(y) + v([a_0, a[, y) \right\|_{L^2}$$
$$+ \left\| v([a_0, a[, x) - v([a_0, a[, y) \right\|_{L^2}$$

and by Assumption 2.1(a),

$$\|v([a_0, a[, x) - v([a_0, a[, y)\|_{L^2}$$
$$\leq \|v(x) - v([a, \infty[, x) - v(y) + v([a, \infty[, y)\|_{L^2}$$
$$+ \| - v([0, a_0[, x) + v([0, a_0[, y)\|_{L^2}.$$

Applying Assumption 2.1(b), we see that

$$(2.3) \quad d(x, y) \leq c_0 \left[\sum_{j=1}^{k} (a_0^{\alpha_j^{-1}-1} + a^{\alpha_j^{-1}-1}) |x_j - y_j| + a^{-1} + \sum_{j=1}^{k} |x_j - y_j| \right].$$

By hypothesis, $\max_{j=1,\ldots,k} |x_j - y_j|^{\alpha_j} \leq \Delta(x, y) \leq a_0^{-1}$, so we choose $a \geq a_0$ such that $\max_{j=1,\ldots,k} |x_j - y_j|^{\alpha_j} = a^{-1}$. Notice that

$$(a_0^{\alpha_j^{-1}-1} + a^{\alpha_j^{-1}-1}) |x_j - y_j|$$

$$(2.4) \qquad = \left[(a_0 |x_j - y_j|^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}} + (a |x_j - y_j|^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}} \right] |x_j - y_j|^{\alpha_j}$$

$$\leq 2 (a |x_j - y_j|^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}} |x_j - y_j|^{\alpha_j}$$

$$\leq 2 |x_j - y_j|^{\alpha_j}$$

by the choice of a. Now (2.3) and (2.4) imply that

$$d(x, y) \leq c_0 \left[2 \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j} + \max_{j=1,\ldots,k} |x_j - y_j|^{\alpha_j} + \sum_{j=1}^{k} |x_j - y_j| \right].$$

For $\Delta(x, y) \leq 1$, since $0 < \alpha_j < 1$, we conclude that $d(x, y) \leq 4c_0 \Delta(x, y)$. □

A first objective is to prove the following analogue for v of Proposition 3.4 of Talagrand [30].

PROPOSITION 2.3. *Let Assumption 2.1 hold, and let*

$$(2.5) \qquad Q = \sum_{j=1}^{k} (\gamma_j + 1) = \sum_{j=1}^{k} \frac{1}{\alpha_j}.$$

Then there are constants $\tilde{K} < \infty$ and $\rho_0 \in]0, 1]$ with the following property. Given $0 < r_0 < \rho_0$, for all $x_0 \in I$, we have

$$(2.6) \qquad P \left\{ \exists r \in [r_0^2, r_0] : \sup_{y : \Delta(y, x_0) < r} |v(y) - v(x_0)| \leq \tilde{K} \frac{r}{(\log \log \frac{1}{r})^{1/Q}} \right\}$$

$$\geq 1 - \exp \left[- \left[\log \frac{1}{r_0} \right]^{\frac{1}{2}} \right].$$

When $d(y, x_0) \leq 4c_0 \Delta(y, x_0) \leq 4c_0 r$, one expects that $v(y) - v(x_0)$ is of order r, so Proposition 2.3 states that with high probability, there is a Δ-ball of radius r in which the increments $v(y) - v(x_0)$ are smaller than expected. This proposition is proved in Section 4.

In order to obtain results on polarity of points, we need an additional assumption.

ASSUMPTION 2.4. Let $I \subset \mathbb{R}^k$ be a closed box and $\varepsilon_0 > 0$ be as in Assumption 2.1.

(a) There is a constant $\tilde{c} > 0$ such that for all $x \in I^{(\varepsilon_0)}$, and $i = 1, \ldots, d$, we have $\|v_i(x)\|_{L^2} \geq \tilde{c}$.

(b) There is $\rho > 0$ with the following property. For $x \in I$, there are $x' \in I^{(\varepsilon_0)}$, $\delta_j \in]\alpha_j, 1]$, $j = 1, \ldots, k$, and $C > 0$ such that for all $i = 1, \ldots, d$, $y, \bar{y} \in I^{(\varepsilon_0)}$ with $\Delta(x, y) \leq 2\rho$ and $\Delta(x, \bar{y}) \leq 2\rho$,

$$\left| E\big[(v_i(y) - v_i(\bar{y}))v_i(x')\big] \right| \leq C \sum_{j=1}^{k} |y_j - \bar{y}_j|^{\delta_j}.$$

REMARK 2.5. (a) We do not require the lower bound $d(x, y) \geq c\Delta(x, y)$ on the canonical metric d. Without this lower bound, Theorem 2.6 below is valid, but Q defined in (2.5) might not be the critical dimension.

(b) Part (b) in Assumption 2.4 states that covariances are smoother than what one gets from the Cauchy–Schwarz inequality, Hölder continuity and Proposition 2.2:

$$\left| E\big[(v_i(y) - v_i(\bar{y}))v_i(x')\big] \right| \leq \|v_i(y) - v_i(\bar{y})\|_{L^2} \|v_i(x')\|_{L^2}$$

$$\leq \|v_i(x')\|_{L^2} \sum_{j=1}^{k} |y_j - \bar{y}_j|^{\alpha_j}.$$

This will be the case in the examples that we will consider.

The main result of this section is the following.

THEOREM 2.6. *Let Assumptions 2.1 and 2.4 hold for all sufficiently small boxes. Assume that $Q = d$. Then for any closed box J and for all $z \in \mathbb{R}^Q$,*

$$P\{\exists x \in J : v(x) = z\} = 0.$$

This theorem is proved in Section 5.

3. Preliminaries. Following [29], Section 2, we first set up some estimates that are needed.

Recall the number Q defined in (2.5). Let $I \subset \mathbb{R}^k$ be a closed box such that Assumption 2.1 is satisfied. For $x_0 \in I$, the number of balls in metric d of radius ε needed to cover the set

$$S_r(x_0) = \{x \in \mathbb{R}^k : \Delta(x, x_0) < r\}$$

is $\leq N_d(S_r, \varepsilon) = \bar{c} r^Q / \varepsilon^Q$ [indeed, $x \in S_r(x_0)$ implies that $|x_j - x_{0,j}| < r^{\alpha_j^{-1}}$, so the volume of $S_r(x_0)$ with respect to Lebesgue measure is $\leq r^Q$, and by Proposition 2.2, the volume of a d-ball of radius ε is $\geq \bar{c} \varepsilon^Q$].

LEMMA 3.1. *Let D be the diameter* (*in metric d*) *of a subset $S \subset \mathbb{R}^k$. There is a universal constant K_0 such that, for all $u > 0$, we have*

$$P\left\{\sup_{x,y \in S} |v(x) - v(y)| \geq K_0\left(u + \int_0^D \sqrt{\log N_d(S, \varepsilon)}\, d\varepsilon\right)\right\} \leq \exp\left(-\frac{u^2}{D^2}\right).$$

(*Note. There is a misprint in* [29], *Lemma* 2.1, *where D should be D^2.*)

PROOF. This is a consequence of inequality (11.4), page 302 in [23], which holds for Gaussian processes with $\psi(x) = e^{(x^2)} - 1$. □

LEMMA 3.2. *There is a constant $K > 0$* (*depending on c_0 in Assumption 2.1*) *such that, for all $u > 0$,*

$$P\left\{\sup_{x,y \in I} |v(x) - v(y)| \leq u\right\} \geq \exp\left(-\frac{1}{Ku^Q}\right).$$

PROOF. We use the small ball estimate for Gaussian processes (see [22], (7.13), page 257, or Lemma 2.2 of [29])

$$P\left\{\sup_{x,y \in I} |v(x) - v(y)| \leq u\right\} \geq \exp\left(-\frac{\psi(u)}{K}\right),$$

where $\psi(u) = u^{-Q}$. Indeed, a ball of radius ε (in the canonical metric d) has volume $\geq \tilde{c} \varepsilon^Q$, so the number of balls (in the canonical metric d) of radius ε needed to cover I is $\leq c_I \varepsilon^{-Q}$. □

LEMMA 3.3. *Consider $b > a > 1$, $\varepsilon_0 > r > 0$ and set*

$$A = \sum_{j=1}^k a^{\alpha_j^{-1}-1} r^{\alpha_j^{-1}} + b^{-1}.$$

There are constants A_0, $\tilde{K}$ and $\tilde{c}$ (depending on c_0 in Assumption 2.1) such that if $A \le A_0 r$ and

(3.1)
$$u \ge \tilde{K} A \log^{1/2}\left(\frac{r}{A}\right),$$

then

$$P\left\{ \sup_{x \in S_r(x_0)} |v(x) - v(x_0) - (v([a,b],x) - v([a,b],x_0))| \ge u \right\} \le \exp\left(-\frac{u^2}{\tilde{c}A^2}\right).$$

PROOF. Recall that $S_r = S_r(x_0) = \{x \in \mathbb{R}^k : \Delta(x, x_0) < r\}$, and set

$$\tilde{d}(x, y) = \|v(x) - v(y) - (v([a,b[,x) - v([a,b[,y))\|_{L^2}.$$

Then

$$\tilde{d}(x, y) \le \|v(x) - v(y)\|_{L^2} + \|v([a,b[,x) - v([a,b[,y)\|_{L^2}.$$

Since

$$v(x) - v(y) = (v([a,b[,x) - v([a,b[,y)) + (v(\mathbb{R}_+ \setminus [a,b[,x) - v(\mathbb{R}_+ \setminus [a,b[,y)),$$

and the two terms on the right-hand side are independent by Assumption 2.1(a), we see that

$$\|v([a,b[,x) - v([a,b[,y)\|_{L^2} \le \|v(x) - v(y)\|_{L^2}.$$

Finally,

$$\tilde{d}(x, y) \le 2\|v(x) - v(y)\|_{L^2} \le 8c_0 \Delta(x, y)$$

by Proposition 2.2. Therefore, for small $\varepsilon > 0$, the number of ε-balls (in metric $\tilde{d}$) needed to cover $S_r(x_0)$ is

$$N_{\tilde{d}}(S_r(x_0), \varepsilon) \le c \frac{r^Q}{\varepsilon^Q}.$$

For $x \in S_r(x_0)$, $|x_j - x_{0,j}| \le r^{\alpha_j^{-1}}$, so by Assumption 2.1(b), $\tilde{d}(x, x_0) \le c_0 A$ and, therefore, the diameter D of $S_r(x_0)$ satisfies $D \le 2c_0 A$. Assuming that we have chosen the constant A_0 and that $A \le A_0 r$, notice that for $D \le 2c_0 A \le 2c_0 A_0 r$, there is a constant $\tilde{K}'$ (depending on c and $c_0 A_0$) such that

$$\int_0^D \sqrt{\log N_{\tilde{d}}(S_r(x_0), \varepsilon)}\, d\varepsilon \le \tilde{K}' \int_0^D \sqrt{\log \frac{r}{\varepsilon}}\, d\varepsilon.$$

Recalling the elementary inequality $\int_x^{+\infty} u^2 e^{-u^2}\, du \le C x e^{-x^2}$ for x large, and using the change of variables $\varepsilon = r e^{-u^2}$ (r fixed), we see that there is a universal

constant K such that for all $D > 0$ and $r > 0$ with D/r sufficiently small (which is the case if A_0 is chosen sufficiently small),

$$\int_0^D \sqrt{\log \frac{r}{\varepsilon}} \, d\varepsilon \leq K D \sqrt{\log \frac{r}{D}},$$

so for D/r sufficiently small,

$$\int_0^D \sqrt{\log N_{\tilde{d}}(S_r(x_0), \varepsilon)} \, d\varepsilon \leq \tilde{K}' K D \sqrt{\log \frac{r}{D}}.$$

Let K_0 be the universal constant in Lemma 3.1. It follows that

$$u \geq K_0 \left(\frac{u}{2K_0} + \int_0^D \sqrt{\log N_{\tilde{d}}(S_r(x_0), \varepsilon)} \, d\varepsilon \right)$$

when

$$(3.2) \qquad\qquad u \geq 2 K_0 \tilde{K}' K D \sqrt{\log \frac{r}{D}},$$

so by Lemma 3.1 (applied to the random field $(v(x) - v([a, b[, x)))]$, when u satisfies (3.2),

$$P \left\{ \sup_{x \in S_r(x_0)} |v(x) - v(x_0) - (v([a, b], x) - v([a, b], x_0))| \geq u \right\}$$

$$\leq \exp \left(-\frac{(u/(2K_0))^2}{D^2} \right) \leq \exp \left(-\frac{u^2}{\tilde{c} A^2} \right).$$

In order to explain (3.1), notice that

$$D \sqrt{\log \frac{r}{D}} = [f(D)]^{1/2} \qquad \text{where } f(x) = \frac{x^2}{2} \log \frac{r^2}{x^2},$$

and

$$f'(x) = x \log \frac{r^2}{x^2} - \frac{x^2}{2} \frac{2}{x} = x \left(\log \frac{r^2}{x^2} - 1 \right),$$

so

$$f'(x) > 0 \qquad \text{if } \frac{r^2}{x^2} > e \qquad \text{that is, } x^2 < \frac{r^2}{e}.$$

Since $D \leq 2c_0 A$,

$$K D \sqrt{\log \frac{r}{D}} \leq \tilde{K}'' \left[\frac{A^2}{2} \log \left(\frac{r^2}{(2c_0 A)^2} \right) \right]^{1/2} \leq \tilde{K} A \log^{\frac{1}{2}} \left(\frac{r}{A} \right)$$

provided $(2c_0 A)^2 \leq r^2/e$, that is, $A \leq (2c_0 e)^{-1} r$, which is the case as long as A_0 is sufficiently small and $A \leq A_0 r$. In this case, (3.1) implies (3.2). $\quad\square$

LEMMA 3.4. *There is a constant K (depending on c_0 in Assumption 2.1) such that if $0 < u < r$, then for all $0 < a < b$,*

$$P\left\{ \sup_{x \in S_r(x_0)} |v([a, b[, x) - v([a, b[, x_0)| \le u \right\} \ge \exp\left(-K \frac{r^Q}{u^Q} \right).$$

PROOF. As in the proof of Lemma 3.2, we note that the number of balls of radius ε [in the canonical metric of $v(a, b, \cdot, \cdot)$] needed to cover $S_r(x_0)$ is $\le c\varepsilon^{-Q} r^Q$. Applying the same small ball estimate as in the proof of Lemma 3.2, we obtain the desired conclusion. $\square$

4. Proof of Proposition 2.3. Fix $U > 1$. Set $r_\ell = r_0 U^{-2\ell}$ and $a_\ell = U^{2\ell-1}/r_0$. Consider the largest integer ℓ_0 such that

$$(4.1) \qquad \ell_0 \le \frac{\log(1/r_0)}{2\log U}.$$

Then for $\ell \le \ell_0$, we have $r_\ell \ge r_0^2$.

It suffices to show that, for some large constant K_2,

$$P\left\{ \exists 1 \le \ell \le \ell_0 : \sup_{x \in S_{r_\ell}(x_0)} |v(x) - v(x_0)| \le K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/Q}} \right\}$$

$$\ge 1 - \exp\left(-\left(\log \frac{1}{r_0} \right)^{1/2} \right).$$

It follows from Lemma 3.4 that, for K_2 large enough so that $K/K_2^Q \le 1/4$,

$$P\left\{ \sup_{x \in S_{r_\ell}(x_0)} |v([a_\ell, a_{\ell+1}[, x) - v([a_\ell, a_{\ell+1}[, x_0)| \le K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/Q}} \right\}$$

$$(4.2) \qquad \ge \exp\left(-\frac{K}{K_2^Q} \frac{r_\ell^Q}{r_\ell^Q} \left(\log\log \frac{1}{r_\ell} \right) \right) \ge \exp\left(-\frac{1}{4} \left(\log\log \frac{1}{r_\ell} \right) \right)$$

$$= \left(\log \frac{1}{r_\ell} \right)^{-1/4}.$$

Thus, by independence of the $v([a_\ell, a_{\ell+1}[, \cdot), \ell = 1, \ldots, \ell_0$,

$$P\left\{ \exists \ell \le \ell_0 : \sup_{x \in S_{r_\ell}(x_0)} |v([a_\ell, a_{\ell+1}[, x) - v([a_\ell, a_{\ell+1}[, x_0)| \le K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/Q}} \right\}$$

$$(4.3) \quad = 1 - \prod_{1 \le \ell \le \ell_0} \left(1 - P\left\{ \sup_{x \in S_{r_\ell}(x_0)} |v([a_\ell, a_{\ell+1}[, x) - v([a_\ell, a_{\ell+1}[, x_0)| \right.\right.$$

$$\left.\left. \le K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/Q}} \right\} \right).$$

Apply (4.2) to see that this is greater than

$$1 - \prod_{\ell=1}^{\ell_0}\left[1 - \left[\log\frac{1}{r_\ell}\right]^{-\frac{1}{4}}\right] \geq 1 - \left[1 - \left[\log\frac{1}{r_0^2}\right]^{-\frac{1}{4}}\right]^{\ell_0}$$

(4.4)

$$\geq 1 - \exp\left[-\ell_0\left[\log\frac{1}{r_0^2}\right]^{-\frac{1}{4}}\right].$$

Set

$$A_\ell = \sum_{j=1}^{k} a_\ell^{\alpha_j^{-1}-1} r_\ell^{\alpha_j^{-1}} + a_{\ell+1}^{-1}.$$

Notice that $r_\ell a_\ell = U^{-1}$ and $r_\ell a_{\ell+1} = U$. Then

$$A_\ell r_\ell^{-1} = \sum_{j=1}^{k}(a_\ell r_\ell)^{\alpha_j^{-1}-1} + (a_{\ell+1}r_\ell)^{-1} = \sum_{j=1}^{k} U^{-(\alpha_j^{-1}-1)} + U^{-1} \leq (k+1)U^{-\beta},$$

with $\beta = \min(1, \min_{j=1,\ldots,k}(\alpha_j^{-1} - 1)) > 0$ since $\alpha_j < 1$, $j = 1,\ldots,k$. Therefore, for U large enough, $A_\ell \leq A_0 r_\ell$, and for $u \geq \tilde{K}r_\ell U^{-\beta}\sqrt{\log U}$, (3.1) is satisfied (with A there replaced by A_ℓ and r by r_ℓ), so by Lemma 3.3,

$$P\left\{\sup_{x\in S_{r_\ell}(x_0)}\left|v(x) - v(x_0) - v([a_\ell, a_{\ell+1}[, x) + v([a_\ell, a_{\ell+1}[, x_0)\right| \geq u\right\}$$

$$\leq \exp\left(-\frac{u^2}{\tilde{c}A_\ell^2}\right) \leq \exp\left(-\frac{u^2}{cr_\ell^2}U^{2\beta}\right).$$

Proceeding as in [29], (4.3), we take $u = K_2 r_\ell (\log\log\frac{1}{r_0})^{-1/Q}$, which is possible provided

$$K_2 r_\ell\left(\log\log\frac{1}{r_0}\right)^{-1/Q} \geq \tilde{K}r_\ell U^{-1}\sqrt{\log U},$$

that is, provided

(4.5) $$U(\log U)^{-1/2} \geq \frac{\tilde{K}}{K_2}\left(\log\log\frac{1}{r_0}\right)^{Q},$$

which holds if U is large enough, to get

$$P\left\{\sup_{x\in S_{r_\ell}(x_0)}\left|v(x) - v(x_0) - v([a_\ell, a_{\ell+1}[, x)\right.\right.$$

(4.6) $$\left.\left. + v([a_\ell, a_{\ell+1}[, x_0)\right| \geq K_2 r_\ell\left(\log\log\frac{1}{r_0}\right)^{-1/Q}\right\}$$

$$\leq \exp\left(-\frac{U^{2\beta}}{c(\log\log\frac{1}{r_0})^{2/Q}}\right).$$

Let

$$F_\ell = \left\{ \left| v([a_\ell, a_{\ell+1}[, x) - v([a_\ell, a_{\ell+1}[, x_0) \right| \leq \frac{K_2}{2} \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/\varrho}} \right\},$$

$$G_\ell = \left\{ \left| v(x) - v(x_0) - v([a_\ell, a_{\ell+1}[, x) \right.\right.$$

$$\left.\left. + v([a_\ell, a_{\ell+1}[, x_0) \right| \geq \frac{K_2}{2} \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/\varrho}} \right\}.$$

Then

$$P\left\{ \exists 1 \leq \ell \leq \ell_0 : \sup_{x \in S_{r_\ell}(x_0)} \left| v(x) - v(x_0) \right| \leq K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/\varrho}} \right\}$$

(4.7)
$$\geq P\left(\bigcup_{\ell=1}^{\ell_0} (F_\ell \cap G_\ell^c) \right) \geq P\left(\left(\bigcup_{\ell=1}^{\ell_0} F_\ell \right) \cap \left(\bigcap_{\ell=1}^{\ell_0} G_\ell^c \right) \right)$$

$$\geq P\left(\bigcup_{\ell=1}^{\ell_0} F_\ell \right) - P\left(\bigcup_{\ell=1}^{\ell_0} G_\ell \right).$$

By (4.4),

$$P\left(\bigcup_{\ell=1}^{\ell_0} F_\ell \right) \geq 1 - \exp\left(-\ell_0 \left(\log \frac{1}{r_0^2} \right)^{-1/4} \right),$$

and by (4.6),

$$P\left(\bigcup_{\ell=1}^{\ell_0} G_\ell \right) \leq \ell_0 \exp\left(-\frac{U^{2\beta}}{c(\log\log \frac{1}{r_0})^{2/\varrho}} \right).$$

Combining with (4.7), we get

$$P\left\{ \exists 1 \leq \ell \leq \ell_0 : \sup_{x \in S_{r_\ell}(x_0)} \left| v(x) - v(x_0) \right| \leq K_2 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/\varrho}} \right\}$$

$$\geq 1 - \exp\left(-\ell_0 \left(\log \frac{1}{r_0^2} \right)^{-1/4} \right) - \ell_0 \exp\left(-\frac{U^{2\beta}}{c(\log\log \frac{1}{r_0})^{2/\varrho}} \right).$$

Therefore, the proof of (2.6) will be complete provided

(4.8) $$\exp\left[-\ell_0 \left[\log \frac{1}{r_0^2} \right]^{-\frac{1}{4}} \right] + \ell_0 \exp\left[\frac{-U^{2\beta}}{c(\log\log \frac{1}{r_0})^{2/\varrho}} \right] \leq \exp\left[-\left[\log \frac{1}{r_0} \right]^{\frac{1}{2}} \right].$$

Recall the condition (4.5), and that ℓ_0 is defined in (4.1). Therefore, if we set

$$U = \left(\log \frac{1}{r_0} \right)^{1/(2\beta)},$$

then for r_0 small enough, by (4.1),

$$\ell_0 \geq \beta \left(\log \frac{1}{r_0} \right) \left(\log \log \frac{1}{r_0} \right)^{-1} \gg 1.$$

Therefore, the left-hand side of (4.8) is bounded above by

$$\exp\left[-\frac{(\log \frac{1}{r_0})^{3/4}}{c \log \log \frac{1}{r_0}} \right] + \left(1 + \log \frac{1}{r_0} \right) \exp\left[-\frac{\log \frac{1}{r_0}}{c(\log \log \frac{1}{r_0})^{2/Q}} \right]$$

$$\leq \exp\left[-\left(\log \frac{1}{r_0} \right)^{1/2} \right]$$

provided r_0 is small enough. This completes the proof of Proposition 2.3.

5. Proof of Theorem 2.6. The main effort in establishing Theorem 2.6 will be to prove the next proposition.

PROPOSITION 5.1. *Assume that $Q = d$. Let I be a sufficiently small box so that Assumptions 2.1 and 2.4 hold. Let $\varepsilon_0 > 0$ be as in Assumption 2.1 and let ρ be as in Assumption 2.4. Fix $x \in I$, and consider the following (random) subset of $\mathbb{R}^d$:*

$$M(\rho, x) = \{ v(y) : y \in \mathbb{R}^k \text{ and } \Delta(y, x) \leq \rho \}.$$

Then for any $z_0 \in \mathbb{R}^Q$, $P\{z_0 \in M(\rho, x)\} = 0$.

PROOF OF THEOREM 2.6 (ASSUMING PROPOSITION 5.1). Let J be a closed box and

$$M = \{ v(y) : y \in J \}.$$

Divide J into a finite union of small boxes I_ℓ for which Assumptions 2.1 and 2.4 hold. Let $\rho_\ell > 0$ be given by Assumption 2.4 for I_ℓ. Since $(S_{\rho_\ell}(x), x \in I_\ell)$ is an open cover of I_ℓ, there are $x_{\ell,1}, \ldots, x_{\ell,n_\ell} \in I_\ell$ such that $I_\ell \subset \bigcup_{i=1}^{n_\ell} S_{\rho_\ell}(x_{\ell,i})$. It follows that

$$M \subset \bigcup_\ell \bigcup_{i=1}^{n_\ell} M(\rho_\ell, x_{\ell,i}),$$

so for any $z_0 \in \mathbb{R}^Q$,

$$P\{z_0 \in M\} \leq \sum_\ell \sum_{i=1}^{n_\ell} P\{z_0 \in M(\rho_\ell, x_{\ell,i})\} = 0,$$

by Proposition 5.1. It follows that z_0 is polar for v. $\quad \square$

We now work toward proving Proposition 5.1. We proceed as in [30], Section 3. Set

$$B_\rho(x) = \{y \in \mathbb{R}^k : \Delta(y, x) \leq \rho\},$$

$$B'_\rho(x) = \{y \in \mathbb{R}^k : \Delta(y, x) \leq 2\rho\} = B_{2\rho}(x).$$

Let $x' \in \mathbb{R}^k$ be given by Assumption 2.4(b).

Define two $\mathbb{R}^d$-valued random fields:

$$v^2(y) = E(v(y) \mid v(x')), \qquad v^1(y) = v(y) - v^2(y).$$

REMARK 5.2. (a) Because they are Gaussian and orthogonal, the processes v^1 and v^2 are independent. Further, v^1 is independent of the random vector $v(x')$.

(b) If we only want to prove that *almost all points are polar for v* (i.e., the range of v has Lebesgue measure zero), then we would not need to introduce the process v^1. Here, we will prove that the range of v^1 has Lebesgue measure zero, and v^1 is quite a good approximation of v (so the range of v also has Lebesgue measure zero). Then we will use the independence of v^1 and $v(x')$ to deduce that *all* points are polar for v.

LEMMA 5.3. *The random field $v^2 = (v^2(y), y \in B'_\rho(x))$ has a continuous version, and there is a finite constant C such that, for $y \in B'_\rho(x)$ and $\bar{y} \in B'_\rho(x)$,*

$$|v^2(y) - v^2(\bar{y})| \leq C|v(x')| \sum_{j=1}^{k} |y_j - \bar{y}_j|^{\delta_j}.$$

PROOF. Let

(5.1) $$\alpha(y) = \frac{E(v_j(y)v_j(x'))}{E(v_j(x')^2)},$$

where the right-hand side does not depend on j. Since the components of $v(y)$ are independent, $v_j^2(y)$ is the orthogonal projection of $v_j(y)$ onto $v_j(x')$, therefore, for $j \in \{1, \ldots, d\}$,

(5.2) $$v_j^2(y) = \alpha(y)v_j(x'),$$

and $v^2(y) = (v_1^2(y), \ldots, v_d^2(y))$ is the continuous version of v^2. With this version, the conclusion follows from Assumption 2.4(a) and (b). □

LEMMA 5.4. *There is a number K (depending on d) such that, for $\varepsilon < 1/3$,*

$$P\left\{\forall y, \bar{y} \in \bar{I}, d(y, \bar{y}) \leq \varepsilon \Rightarrow |v(y) - v(\bar{y})| \leq K\varepsilon \log^{1/2} \frac{1}{\varepsilon}\right\} \geq 1 - \varepsilon.$$

PROOF.   The set $I^{(\varepsilon)}$ has finite diameter in the metric Δ, hence in the metric d by Proposition 2.2. According to [24], Theorem 6.3.3, page 258, there is a finite random variable Z such that, a.s., for all $y, \bar{y} \in \bar{I}$,

$$|v(y) - v(\bar{y})| \leq Z \int_0^{d(y,\bar{y})} \left[\left(\log \frac{1}{\lambda(B_d(y, u))} \right)^{1/2} + \left(\log \frac{1}{\lambda(B_d(\bar{y}, u))} \right)^{1/2} \right] du,$$

where λ denotes Lebesgue on $\bar{I}$ and $B_d(y, u)$ is the ball in metric d centered at y with radius u. Since $d(x, y) \leq 4c_0 \Delta(x, y)$ for small values of $\Delta(x, y)$,

$$|v(y) - v(\bar{y})| \leq Z \int_0^{d(y,\bar{y})} \left(\log \frac{1}{u^{\varrho}} \right)^{1/2} du.$$

Using the elementary inequality

$$\int_0^x \left(\log \frac{1}{u} \right)^{1/2} du \leq c_0 x \log^{1/2} \frac{1}{x},$$

which is valid for $0 < x < x_0$ with $x_0 > 0$, and the fact that $x \mapsto x \log^{1/2}(1/x)$ is increasing on $]0, 1/e[$, we see that $d(y, \bar{y}) \leq \varepsilon$ implies

$$\|v(y) - v(\bar{y})\| \leq c_0 Z \varepsilon \log^{1/2}(1/\varepsilon),$$

and this is $\leq K \varepsilon \log^{1/2}(1/\varepsilon)$ on the event $\{Z \leq K/c_0\}$. Since Z is finite a.s., this event has probability $\geq 1 - \varepsilon$ if K is large enough. $\square$

For $p \geq 1$, consider the random set

$$R_p = \Big\{ y \in B'_\rho(x) : \exists r \in [2^{-2p}, 2^{-p}[$$

$$\text{with} \sup_{\bar{y}: \Delta(\bar{y}, y) < r} |v(\bar{y}) - v(y)| \leq K_2 \frac{r}{(\log \log \frac{1}{r})^{\frac{1}{\varrho}}} \Big\},$$

and the event

$$\Omega_{p,1} = \Big\{ \lambda(R_p) \geq \lambda(B'_\rho(x)) \Big(1 - \exp\Big(-\frac{\sqrt{p}}{4} \Big) \Big) \Big\}$$

(here, λ denotes Lebesgue measure). Notice that $\Omega_{p,1}$ can be described as the event "a large portion of $B'_\rho(x)$ consists of points at which v is comparatively smooth." Then

$$(\Omega_{p,1})^c = \Big\{ \lambda(R_p) < \lambda(B'_\rho(x)) \Big(1 - \exp\Big(-\frac{\sqrt{p}}{4} \Big) \Big) \Big\}$$

$$= \Big\{ \lambda(B'_\rho(x) \setminus R_p) \geq \lambda(B'_\rho(x)) \exp\Big(-\frac{\sqrt{p}}{4} \Big) \Big\},$$

so by Markov's inequality

$$(5.3) \qquad P\big((\Omega_{p,1})^c\big) \leq \frac{E(\lambda(B'_\rho(x) \setminus R_p))}{\lambda(B'_\rho(x)) \exp(-\frac{\sqrt{p}}{4})}.$$

The numerator is equal to

$$E\bigg[\int_{B'_\rho(x)} 1_{B'_\rho(x) \setminus R_p}(y)\,dy\bigg] = \int_{B'_\rho(x)} P\{y \in B'_\rho(x) \setminus R_p\}\,dy.$$

By the definition of R_p and Proposition 2.3 (taking the log in base 2), for $y \in B'_\rho(x)$,

$$P\{y \notin R_p\} \leq \exp\bigg(-\Big(\log \frac{1}{2^{-p}}\Big)^{\frac{1}{2}}\bigg) = \exp(-\sqrt{p}),$$

therefore, by (5.3),

$$P\big((\Omega_{p,1})^c\big) \leq \exp\Big(-\frac{3}{4}\sqrt{p}\Big).$$

In particular,

$$(5.4) \qquad \sum_{p=1}^{\infty} P\big((\Omega_{p,1})^c\big) < +\infty.$$

Fix $\beta \in\,]0, \min(\min_{j=1,\ldots,k}(\delta_j\alpha_j^{-1} - 1), 1)[$ (which is possible since $\delta_j > \alpha_j$, $j = 1, \ldots, k$) and set

$$\Omega_{p,2} = \{|v(x')| \leq 2^{\beta p}\}.$$

Since $v(x')$ is a normal random vector, $\sum_{p \geq 1} P((\Omega_{p,2})^c) < +\infty$. In addition, on the event $\Omega_{p,2}$, the constant of Hölder continuity of v^2 is not too large. Indeed, by Lemma 5.3, for $y \in B'_\rho(x)$ and $\bar{y} \in B'_\rho(x)$, if $\Delta(y, x) \leq r$ and $\Delta(\bar{y}, x) \leq r$, then on $\Omega_{p,2}$,

$$|v^2(y) - v^2(\bar{y})| \leq C2^{\beta p} \sum_{j=1}^{k} |y_j - \bar{y}_j|^{\delta_j} \leq \tilde{C}2^{\beta p} \sum_{j=1}^{k} r^{\delta_j\alpha_j^{-1}}.$$

If $r \leq 2^{-p}$, then

$$r^{\delta_j\alpha_j^{-1}} 2^{\beta p} = r^{\delta_j\alpha_j^{-1}} (2^{-p})^{-\beta} \leq r^{\delta_j\alpha_j^{-1}} r^{-\beta} = r r^{\delta_j\alpha_j^{-1} - 1 - \beta},$$

and $\min_{j=1,\ldots,k}(\delta_j\alpha_j^{-1} - 1 - \beta) > 0$ by definition of β. Therefore, there is $K_3 > K_2$ such that on

$$\Omega_{p,3} \overset{\text{def}}{=} \Omega_{p,1} \cap \Omega_{p,2},$$

for each $y \in R_p$, there exists $r \in [2^{-2p}, 2^{-p}]$ such that

(5.5)
$$\sup_{\bar{y}: \Delta(\bar{y}, y) < r} |v^1(\bar{y}) - v^1(y)| \leq K_3 \frac{r}{(\log \log \frac{1}{r})^{\frac{1}{Q}}}.$$

Define an "anisotropic dyadic cube" of order ℓ as a box in $\mathbb{R}^k$ of the form

$$\prod_{j=1}^{k} [m_j 2^{-\ell \alpha_j^{-1}}, (m_j + 1) 2^{-\ell \alpha_j^{-1}}],$$

where $m_j \in \mathbb{N}$. For $y \in \mathbb{R}^k$, let $C_\ell(y)$ denote the anisotropic dyadic cube of order ℓ that contains y. This cube is called "good" if

(5.6)
$$\sup_{y, \bar{y} \in C_\ell(y) \cap B_\rho(x)} |v^1(y) - v^1(\bar{y})| \leq d_\ell,$$

where

$$d_\ell = \tilde{K}_3 \frac{2^{-\ell}}{(\log \log 2^\ell)^{\frac{1}{Q}}}$$

and $\tilde{K}_3 = k K_3$. By (5.5), when $\Omega_{p,3}$ occurs, we can find a family $\mathcal{H}_{1,p}$ of nonoverlapping good anisotropic dyadic cubes (they may have intersecting boundaries) of order $\ell \in [p, 2p]$ that covers R_p. This family only depends on the random field v^1.

Let $\mathcal{H}_{2,p}$ be the family of nonoverlapping dyadic cubes of order $2p$ that meet $B_\rho(x)$ but are not contained in any cube of $\mathcal{H}_{1,p}$. For p large enough, these cubes are contained in $B'_\rho(x)$, hence in $B'_\rho(x) \setminus R_p$. Therefore, when $\Omega_{p,3}$ occurs, their number is at most N_p, where

$$N_p 2^{-2pQ} \leq \lambda(B'_\rho(x)) \exp\left(-\frac{\sqrt{p}}{4}\right),$$

so

(5.7)
$$N_p \leq C 2^{2pQ} \exp\left(-\frac{\sqrt{p}}{4}\right),$$

where C does not depend on p.

Let $\Omega_{p,4}$ be the event "the inequality

(5.8)
$$\sup_{y, \bar{y} \in C} |v(y) - v(\bar{y})| \leq K_4 2^{-2p} \sqrt{p}$$

holds for each dyadic cube C of order $2p$ of $\mathbb{R}_+ \times \mathbb{R}$ that meets $B_\rho(x)$." We choose K_4 large enough so that $\sum_{p \geq 1} P((\Omega_{p,4})^c) < +\infty$: this is possible by Lemma 5.4.

Set $\mathcal{H}_p = \mathcal{H}_{1,p} \cup \mathcal{H}_{2,p}$. This family is well defined for all $p \geq 1$, and it is a nonoverlapping cover of $B_\rho(x)$ (because of how dyadic cubes fit together). Set

$$r_A = 4 d_\ell = 4 \tilde{K}_3 2^{-\ell} (\log \ell)^{-\frac{1}{Q}} \qquad \text{if } A \in \mathcal{H}_{1,p} \text{ and } A \text{ is of order } \ell \in [p, 2p],$$

$$r_A = K_4 2^{-2p} \sqrt{p} \qquad \text{if } A \in \mathcal{H}_{2,p}.$$

Define

$$\Omega_p = \Omega_{p,3} \cap \Omega_{p,4}.$$

LEMMA 5.5.   *Recall that $d = Q$. Let*

(5.9) $$f(x) = x^d \log\log \frac{1}{x}.$$

For p large enough, if $\Omega_{p,3}$ occurs, then

$$\sum_{A \in \mathcal{H}_p} f(r_A) \le K\lambda\big(B_\rho(x)\big).$$

PROOF.   For $A \in \mathcal{H}_{1,p}$,

$$f(r_A) \le K\left(\frac{2^{-\ell}}{(\log \ell)^{\frac{1}{Q}}}\right)^d \log\log 2^\ell \le K2^{-d\ell}\frac{\log \ell}{(\log \ell)^{d/Q}} = K2^{-Q\ell}$$

since $d = Q$, which is the volume of a anisotropic dyadic cube of order ℓ.

There is a constant K_5 such that, for p large enough and for all $A \in \mathcal{H}_{2,p}$,

$$f(r_A) \le K_5\big(2^{-2p}\sqrt{p}\big)^Q \log(2p).$$

If $\Omega_{p,3}$ occurs, then by (5.7), the total contribution of $\sum_{A \in \mathcal{H}_{2,p}} f(r_A)$ is bounded by

$$K2^{-2pQ}p^{Q/2}\log(2p)2^{2pQ}\exp\left(-\frac{\sqrt{p}}{4}\right) = p^{Q/2}\log(2p)\exp\left(-\frac{\sqrt{p}}{4}\right).$$

Therefore, since the cubes in $\mathcal{H}_{1,p}$ are nonoverlapping and intersect $B'_\rho(t,x)$, if $\Omega_{p,3}$ occurs, then

$$\sum_{A \in \mathcal{H}_p} f(r_A) \le K\lambda\big(B'_\rho(x)\big) + p^{Q/2}\log(2p)\exp\left(-\frac{\sqrt{p}}{4}\right).$$

Now $\lambda(B'_\rho(x)) \le 2^Q\lambda(B_\rho(x))$, and this quantity does not depend on p, so the lemma is proved. $\square$

For each $A \in \mathcal{H}_p$, we pick a distinguished point p_A in A (say the lower left corner). Let B_A be the Euclidean ball in $\mathbb{R}^d$ centered at $v(p_A)$ with radius r_A.

LEMMA 5.6.   *Let $\mathcal{F}_p$ be the family of balls $(B_A, A \in \mathcal{H}_p)$. For p large enough, on Ω_p, $\mathcal{F}_p$ covers $M(\rho, x)$.*

PROOF. Consider $z \in M(\rho, x)$. By definition, there is $y \in B_\rho(x)$ such that $v(y) = z$. Since $\mathcal{H}_p$ is a cover of $B_\rho(x)$, the point y belongs to a certain cube A of $\mathcal{H}_p$. We will show that $z \in B_A$.

Consider first the case $A \in \mathcal{H}_{1,p}$. Suppose that A is of order $\ell \in [p, 2p]$. By (5.6),

$$\left| v^1(p_A) - v^1(y) \right| \le d_\ell.$$

Thus, since $\ell \ge p$, on $\Omega_{p,3}$, by Lemma 5.3, letting $\underline{\gamma} = \min_{j=1,\dots,k}(\delta_j \alpha_j^{-1} - 1 - \beta) > 0$,

$$\left| v(p_A) - v(y) \right| \le d_\ell + \left| v^2(v_A) - v^2(y) \right| \le d_\ell + C2^{\beta p} \sum_{j=1}^{k} (2^{-\ell})^{\delta_j \alpha_j^{-1}}$$

$$\le d_\ell + Ck2^{\beta p} 2^{-\ell \underline{\gamma}} 2^{-\ell(1+\beta)} \le d_\ell + Ck2^{-\ell \underline{\gamma}} 2^{-\ell}$$

$$\le 2d_\ell$$

for p large enough, since $\underline{\gamma} > 0$. Since $v(y) = z$ and $r_A = 4d_\ell$, this implies that $z \in B_A$.

Now consider the case $A \in \mathcal{H}_2$. Then on $\Omega_{p,4}$, by (5.8),

$$\left| v(p_A) - z \right| = \left| v(p_A) - v(y) \right| \le K_4 2^{-2p} \sqrt{p} = r_A,$$

so $z \in B_A$. $\quad \square$

COROLLARY 5.7. *Almost surely, the set $M(\rho, x)$ has Lebesgue measure zero*: $\lambda(M(\rho, x)) = 0$ *a.s.*

PROOF. For p large enough so that Ω_p occurs, by the definition of f in (5.9) and Lemma 5.5,

$$\sum_{A \in \mathcal{H}_p} r_A^d \le \frac{1}{\log p} \sum_{A \in \mathcal{H}_p} f(r_A) \le \frac{K\lambda(B_\rho(x))}{\log p} \to 0$$

as $p \to +\infty$. Since the family of balls $(B_A, A \in \mathcal{H}_p)$ covers $M(\rho, x)$ by Lemma 5.6, we conclude that $\lambda(M(\rho, x)) = 0$ a.s. $\quad \square$

PROOF OF PROPOSITION 5.1. Fix $z_0 \in \mathbb{R}^Q$. Let $\alpha(y)$ be defined as in (5.1). Notice that for ρ small enough, $1/2 \le \alpha(y) \le 3/2$, and $y \mapsto \alpha(y)$ is Hölder continuous by Assumption 2.4(b). Define

$$v_3(y) = \frac{1}{\alpha(y)} (z_0 - v_1(y)).$$

Clearly, by (5.2),

(5.10) $v(y) = z_0 \quad \Longleftrightarrow \quad v_3(y) = v(x')$.

We are going to check that the range of v_3 has Lebesgue measure 0. Assuming this for the moment, let $f_{v(x')}$ be the probability density function of $v(x')$. Then

$$P\{z_0 \in M(\rho, x)\} = P\{\exists y \in B_\rho(x) : v_3(y) = v(x')\}$$
$$= \int_{\mathbb{R}^Q} dz f_{v(x')}(z) P\{\exists y \in B_\rho(x) : v_3(y) = z\},$$

where we have used the fact that v_1, hence v_3, is independent of $v(x')$ [see Remark 5.2(a)]. Since the range of v_3 has Lebesgue measure 0, the probability on the right-hand side vanishes for a.a. z, hence the integral is 0 and $P\{z_0 \in M(\rho, x)\} = 0$, as claimed in Proposition 5.1.

It remains to prove that the range of v_3 has Lebesgue measure 0. For $A \in \mathcal{H}_p$ and $y \in A$,

$$v_3(y) - v_3(p_A) = \frac{1}{\alpha(y)}(z_0 - v_1(y)) - \frac{1}{\alpha(p_A)}(z_0 - v_1(p_A)).$$

Recall that α is Hölder continuous and bounded above and below. If $A \in \mathcal{H}_{1,p}$ and A is of order ℓ, then for p sufficiently large, the right-hand side is

$$\leq c \sum_{j=1}^{k} |y_j - p_{A,j}|^{\delta_j} + 2d_\ell \leq c \sum_{j=1}^{k} 2^{-\ell \delta_j \alpha_j^{-1}} + 2d_\ell.$$

Since $\delta_j \alpha_j^{-1} > 1$, $j = 1, \ldots, k$, this is $\leq 3d_\ell \leq r_A$. If $A \in \mathcal{H}_{2,p}$, then for p sufficiently large, the right-hand side is

$$\leq c \left(\sum_{j=1}^{k} 2^{-2p\delta_j \alpha_j^{-1}} + 2K_4 2^{-2p} \sqrt{2p} \right) \leq \tilde{c} r_A.$$

This means that for some constant $\tilde{c}$, $(B(p_A, \tilde{c} r_A), A \in \mathcal{H}_p)$ covers the range of v_3. As in the proof of Corollary 5.7, we conclude that the Lebesgue measure of $\{v_3(y) : y \in B_1(x)\}$ is zero. $\square$

6. Warm-up: Polarity of points for fractional Brownian fields.

Recall that a (multiparameter) fractional Brownian motion (terminology of [29, 30]), which, following [7], we prefer to call a *fractional Brownian field with Hurst parameter* $H \in]0, 1[$, is a centered Gaussian field $v = (v(x), x \in \mathbb{R}^k)$ with values in $\mathbb{R}^d$, continuous sample paths and covariances given by

$$E(v_\ell(x) v_j(y)) = \delta_{\ell,j} \frac{1}{2}[|x|^{2H} + |y|^{2H} - |x - y|^{2H}],$$

where $\delta_{\ell,j}$ is the Kronecker symbol. This random field is such that

(6.1)
$$E(|v(x) - v(y)|^2) = d|x - y|^{2H}.$$

The papers [29, 30] of Talagrand do not explicitly discuss the issue of polarity of points for fractional Brownian fields in the critical dimension $d = k/H$ (though he was certainly aware of this result). In the case $k = 1$, this issue is handled in [26], Theorem 1. Here, we show how our Theorem 2.6 can be used to deduce polarity of points for the fractional Brownian field in the critical dimension. Obviously, since $v(0) = 0$, the parameter $0 \in \mathbb{R}^k$ (or the value $0 \in \mathbb{R}^d$) must be excluded.

THEOREM 6.1.   *Suppose that $d = k/H$. Then d is the critical dimension for hitting points and points are polar for v, that is, for all $z \in \mathbb{R}^{k/H}$,*

$$P\{\exists x \in \mathbb{R}^k \setminus \{0\} : v(x) = z\} = 0.$$

PROOF.   The fractional Brownian field has a moving average representation (see [7], Theorem 3.2.2 and (3.79)), but also a *harmonizable representation*

$$(6.2) \qquad v(x) = c \int_{\mathbb{R}^k} \frac{1 - e^{-ix \cdot \xi}}{|\xi|^{H+k/2}} W(d\xi),$$

where $W(d\xi)$ is a $\mathbb{C}^d$-valued white noise on $\mathbb{R}^k$, that is, $\mathrm{Re}(W)$ and $\mathrm{Im}(W)$ are independent $\mathbb{R}^d$-valued white noises on $\mathbb{R}^k$ based on Lebesgue measure (see [7], Definition 3.3.1). If one prefers to work with real numbers, then there is the equivalent representation (see [30], Section 2)

$$v(x) = c \int_{\mathbb{R}^k} \frac{1 - \cos(x \cdot \xi)}{|\xi|^{H+k/2}} W_1(d\xi) + c \int_{\mathbb{R}^k} \frac{\sin(x \cdot \xi)}{|\xi|^{H+k/2}} W_2(d\xi),$$

where W_1 and W_2 are independent $\mathbb{R}^d$-valued white noises on $\mathbb{R}^k$.

With this representation, one can define the random field $(v(A, x))$ by

$$v(A, x) = c \int_{|\xi|^H \in [a, b[} \frac{1 - \cos(x \cdot \xi)}{|\xi|^{H+k/2}} W_1(d\xi) + c \int_{|\xi|^H \in [a, b[} \frac{\sin(x \cdot \xi)}{|\xi|^{H+k/2}} W_2(d\xi).$$

Using fairly straightforward calculations that extend those in [29], Lemma 3.1, one checks that for all $0 \le a \le b \le +\infty$ and $x, y \in \mathbb{R}^k$,

$$\|v([a, b[, x) - v(x) - v([a, b[, y) + v(y)\|_{L^2} \le c_0 [a^{H^{-1}-1}|x - y| + b^{-1}],$$

that is, Assumption 2.1 is satisfied with $a_0 = 0$ and $\gamma_j = H^{-1} - 1 > 0$, $j = 1, \ldots, k$. This is also coherent with (6.1), that is, $\|v(x) - v(y)\|_{L^2} = \sqrt{d}|x - y|^H$.

Let $I \subset \mathbb{R}^k \setminus \{0\}$ be a closed box and let $\varepsilon_0 > 0$ be such that $I^{(\varepsilon_0)} \subset \mathbb{R}^k \setminus \{0\}$. Then Assumption 2.4(a) is satisfied since $\inf_{x \in I^{(\varepsilon_0)}} \|v_i(x)\|_{L^2} = \inf_{x \in I^{(\varepsilon_0)}} |x|^H > 0$. According to [30], Lemma 3.2, Assumption 2.4(b) is satisfied with $\rho = 2\varepsilon_0/3$ and $\delta_j = 1 \in \,]H, 1]$, $j = 1, \ldots, k$. It follows from Theorem 2.6 that if Q is defined by

$$Q = \sum_{j=1}^{k} (\gamma_j + 1) = \frac{k}{H},$$

since $d = Q$, then $P\{\exists x \in I : v(x) = z\} = 0$, for all $z \in \mathbb{R}^Q$. Since the closed box $I \subset \mathbb{R}^k \setminus \{0\}$ is arbitrary, points are polar for v. The fact that $d = Q$ is the critical dimension for hitting points follows from [2], Theorem 2.1. Theorem 6.1 is proved. $\qquad\square$

7. Polarity of points for systems of linear heat equations with constant coefficients.
Fix $k \geq 1$ and suppose $\beta \in]0, k \wedge 2[$ or $k = 1 = \beta$. Let $\mathcal{D}(\mathbb{R} \times \mathbb{R}^k)$ denote the space of C^∞-functions with compact support and $(\hat{W}(\varphi), \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^k))$ be a spatially homogeneous $\mathbb{R}^d$-valued Gaussian noise that is white in time, with spatial covariance given by the Riesz kernel $|x - y|^{-\beta}$, unless $k = 1 = \beta$, in which case $\hat{W}$ is space–time $\mathbb{R}^d$-valued Gaussian white noise based on Lebesgue measure. In both cases, $\hat{W}(\varphi) = (\hat{W}_1(\varphi), \ldots, \hat{W}_d(\varphi))$, and the components are independent.

Recall that in the spatially homogeneous case, the covariance of the noise is informally given by

$$E(\hat{W}_\ell(t, x)\hat{W}_j(s, y)) = \delta(t - s)|x - y|^{-\beta}\delta_{\ell, j},$$

where $\delta(\cdot)$ denotes the Dirac delta function and $\delta_{\ell, j}$ is the Kronecker symbol. More precisely, for any C^∞-test functions φ and ψ with compact support,

$$E(\hat{W}_\ell(\varphi)\hat{W}_j(\psi)) = \delta_{\ell, j} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz \varphi(r, y)|y - z|^{-\beta}\psi(r, z).$$

Using elementary properties of the Fourier transform (see (10) in [8]), this covariance can also be written

$$(7.1) \quad \begin{aligned} &E(\hat{W}_\ell(\varphi)\hat{W}_j(\psi)) \\ &= \delta_{\ell, j}c_{k, \beta} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^k} d\xi |\xi|^{\beta-k} \mathcal{F}_x\varphi(r, \cdot)(\xi)\overline{\mathcal{F}_x\psi(r, \cdot)(\xi)}, \end{aligned}$$

where $c_{k, \beta}$ is a constant and $\mathcal{F}_x\varphi(r, \cdot)(\xi)$ denotes the Fourier transform in the x-variable

$$\mathcal{F}_x\varphi(r, \cdot)(\xi) = \int_{\mathbb{R}^k} e^{-i\xi \cdot x}\varphi(r, x)\, dx.$$

This type of noise is discussed for instance in [11], Section 2. Space–time white noise in the case $k = 1$ corresponds formally to $\beta = 1$ in (7.1) or, equivalently,

$$E(\hat{W}_\ell(\varphi)\hat{W}_j(\psi)) = \delta_{\ell, j} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^k} dy \varphi(r, y)\psi(r, y).$$

Let $\hat{v} = (\hat{v}(t, x), t \in \mathbb{R}_+, x \in \mathbb{R})$ be the mild solution of a linear system of d uncoupled heat equations driven by this space–time white noise:

$$(7.2) \quad \begin{cases} \dfrac{\partial}{\partial t}\hat{v}_j(t, x) = \Delta\hat{v}_j(t, x) + \dot{W}_j(t, x) & j = 1, \ldots, d, \\ v(0, x) = 0 & x \in \mathbb{R}^k. \end{cases}$$

Here, $\hat{v}(t, x) = (\hat{v}_1(t, x), \ldots, \hat{v}_d(t, x))$ and Δ is the Laplacian in the spatial variables. The notion of *mild solution* is discussed in [11], Section 2 (see also [28], Chapter 6).

THEOREM 7.1.   *Suppose* $(4 + 2k)/(2 - \beta) = d$. *Then d is the critical dimension for hitting points and points are polar for* $\hat{v}$, *that is, for all* $z \in \mathbb{R}^{(4+2k)/(2-\beta)}$,

$$P\{\exists (t, x) \in]0, +\infty[\times \mathbb{R}^k : \hat{v}(t, x) = z\} = 0.$$

In particular, in the case where $k = 1 = \beta$, $\hat{W}$ *is space–time white noise and* $d = 6$, *then points are polar for* $\hat{v}$.

Let $W(d\tau, d\xi)$ be a $\mathbb{C}^d$-valued space–time white noise, that is, Re(W) and Im(W) are independent space–time white noises based on Lebesgue measure [Re(W) and Im(W) denote respectively the real and imaginary parts of W]. In particular,

$$E\big(W_\ell(A)\overline{W_j(B)}\big) = 2\lambda(A \cap B)\delta_{\ell, j}$$

[here, $W(A) = (W_1(A), \ldots, W_d(A))$].

We shall show in the next proposition that the process $(v(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ defined by

$$(7.3) \qquad v(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} e^{-i\xi \cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} |\xi|^{(\beta-k)/2} W(d\tau, d\xi),$$

is a solution of the stochastic heat equation. By analogy with the processes considered in [5], and with (6.2), we call formula (7.3) a *harmonizable representation* of the solution to (7.2). This type of representation also appears in [1], Section 4.

PROPOSITION 7.2.   *For* $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^k, \mathbb{C})$, *define*

$$\tilde{W}_j(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} W_j(d\tau, d\xi)|\xi|^{(\beta-k)/2} \mathcal{F}_{s,y}\varphi(\tau, \xi),$$

where $\mathcal{F}_{s,y}$ *denotes Fourier transform in the variables* (s, y).

(a) *For* $j = 1, \ldots, d$, *if* $k = 1 = \beta$, *then* $\tilde{W}_j$ *is a* $\mathbb{C}$-*valued space–time white noise; otherwise,* $\tilde{W}_j$ *is spatially homogeneous noise that is white in time with spatial covariance given by* $|x - y|^{-\beta}$.

(b) $(v(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ *defined in* (7.3) *is a* $\mathbb{C}$-*valued solution of*

$$(7.4) \qquad \begin{cases} \dfrac{\partial}{\partial t} v_j(t, x) = \Delta v_j(t, x) + \dot{\tilde{W}}_j(t, x) & j = 1, \ldots, d, \\ v(0, x) = 0 & x \in \mathbb{R}^k. \end{cases}$$

(c) $(\text{Re}(v(t, x)), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ *and* $(\hat{v}(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ *have the same law.*

PROOF.   (a) Consider first the case $k = 1 = \beta$. Observe that

$$E\big(\tilde{W}_j(\varphi)\overline{\tilde{W}_j(\psi)}\big) = \int_{\mathbb{R}}\int_{\mathbb{R}^k} d\tau\, d\xi \mathcal{F}_{s,y}\varphi(\tau,\xi)\overline{\mathcal{F}_{s,y}\psi(\tau,\xi)}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^k} ds\, dy \varphi(s,y)\overline{\psi(s,y)},$$

where we have used Plancherel's theorem, so $\tilde{W}_i$ is a space–time white noise.

Now consider the case $\beta \in {]}0, k \wedge 2{[}$. Then

$$E\big(\tilde{W}_j(\varphi)\overline{\tilde{W}_j(\psi)}\big) = \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}}\mathcal{F}_{s,y}\varphi(\tau,\xi)\overline{\mathcal{F}_{s,y}\psi(\tau,\xi)}$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz \varphi(s,y)\frac{1}{|y-z|^\beta}\overline{\psi(s,z)},$$

where we have used again formula (10) in [8], and property (a) is established.

(b) Let G be the fundamental solution of the heat equation. Notice that

$$\int_{\mathbb{R}\times\mathbb{R}^k} 1_{[0,t]}(s)G(t-s,x-y)\tilde{W}_j(ds,dy)$$

$$= \int_{\mathbb{R}\times\mathbb{R}^k} W_j(d\tau,d\xi)\mathcal{F}_{s,y}\big(1_{[0,t]}(\cdot)G(t-\cdot,x-\cdot)\big)(\tau,\xi)|\xi|^{(\beta-k)/2}.$$

Now, $\mathcal{F}_{s,y}(1_{[0,t]}(\cdot)G(t-\cdot,x-\cdot))(\tau,\xi)$ is equal to

$$\mathcal{F}_s\big(e^{-i\xi\cdot x}1_{[0,t]}(\cdot)\overline{\mathcal{F}_y G(t-\cdot,\cdot)(\xi)}\big)(\tau) = e^{-i\xi\cdot x}\mathcal{F}_s\big(e^{-(t-\cdot)|\xi|^2}1_{[0,t]}(\cdot)\big)(\tau)$$

$$= e^{-i\xi\cdot x - t|\xi|^2}\mathcal{F}_s\big(e^{s|\xi|^2}1_{[0,t]}(s)\big)(\tau).$$

The Fourier transform in the s-variable is easily calculated and one finds that

(7.5)
$$\int_{\mathbb{R}\times\mathbb{R}^k} 1_{[0,t]}(s)G(t-s,x-y)\tilde{W}_j(ds,dy)$$

$$= \int_{\mathbb{R}\times\mathbb{R}^k} W_j(d\tau,d\xi)e^{-i\xi\cdot x}\frac{e^{-i\tau t}-e^{-t|\xi|^2}}{|\xi|^2-i\tau}|\xi|^{(\beta-k)/2} = v_j(t,x).$$

By (7.3), $v_j(0,x) = 0$, so, following [28], Definition 6.1, we have checked that v is the (mild) solution of (7.4), and (b) is proved.

(c) Set $w = \mathrm{Re}(v)$. Then by (b), $w(0,x) = 0$, w satisfies $\frac{\partial w_j}{\partial t} - \Delta w_j = \mathrm{Re}(\tilde{W}_j(t,x))$. If $k = 1 = \beta$, then $\mathrm{Re}(\tilde{W}_j)$ is a real-valued space–time white noise such that $E[(\mathrm{Re}(\tilde{W}_j))^2] = \lambda(A)$, and otherwise, $\mathrm{Re}(\tilde{W}_j)$ is a spatially homogeneous noise with the appropriate covariance. This proves (c).   □

Let

(7.6)
$$\alpha_1 = \frac{2-\beta}{4}, \qquad \alpha_2 = \frac{2-\beta}{2} = 2\alpha_1.$$

[these are the Hölder exponents of $t \mapsto \hat{v}(t, x)$ and $x \mapsto \hat{v}(t, x)$, respectively, considered as functions with values in $L^2(\Omega, \mathcal{F}, P)$], and set

$$v(A, t, x) = \iint_{(\tau, \xi): \max(|\tau|^{\alpha_1}, |\xi|^{\alpha_2}) \in A} e^{-i\xi \cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} |\xi|^{(\beta-k)/2} W(d\tau, d\xi).$$

Clearly, the random field $(v(A, t, x), A \in \mathcal{B}(\mathbb{R}_+), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ satisfies Assumption 2.1(a) (with the generic variable $x \in \mathbb{R}^k$ replaced by $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^k$). In the next lemma, we check Assumption 2.1(b) (with $a_0 = 0$).

LEMMA 7.3. *Let*

$$\gamma_1 = \alpha_1^{-1} - 1 = \frac{2 + \beta}{2 - \beta}, \qquad \gamma_2 = \alpha_2^{-1} - 1 = \frac{\beta}{2 - \beta}.$$

There is a universal constant c_0 such that for all $0 \le a \le b$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^k$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^k$,

$$\|v([a, b[, t, x) - v(t, x) - v([a, b[, t_0, x_0) + v(t_0, x_0)\|_{L^2}$$

$$\le c_0 \left[a^{\gamma_1} |t - t_0| + a^{\gamma_2} \sum_{j=1}^{k} |x_j - x_{0,j}| + b^{-1} \right].$$

REMARK 7.4. Lemma 7.3 states in particular that for $b = \infty$, $(t, x) \mapsto v(t, x) - v([a, \infty[, t, x)$ is Lipschitz continuous in $L^2(\Omega, \mathcal{F}, P)$. However, the Lipschitz constants in t and x are of different orders of magnitude, which reflects the (α_1, α_2)-Hölder exponents of $(t, x) \mapsto v(t, x)$.

PROOF. Let

$$v_1(a, t, x) = \iint_{\max(|\tau|^{\alpha_1}, |\xi|^{\alpha_2}) < a} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{\xi^2 - i\tau} |\xi|^{(\beta-k)/2} W(d\tau, d\xi),$$

$$v_2(b, t, x) = \iint_{\max(|\tau|^{\alpha_1}, |\xi|^{\alpha_2}) > b} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{\xi^2 - i\tau} |\xi|^{(\beta-k)/2} W(d\tau, d\xi).$$

Then

(7.7)
$$v([a, b[, t, x) - v(t, x) - v([a, b[, t_0, x_0) + v(t_0, x_0)$$
$$= v_1(a, t_0, x_0) - v_1(a, t, x) + v_2(b, t_0, x_0) - v_2(b, t, x).$$

Set

$$f_1(a, t, x, t_0, x_0) = E[|v_1(a, t, x) - v_1(a, t_0, x_0)|^2],$$

$$f_2(b, t, x, t_0, x_0) = E[|v_2(b, t, x) - v_2(b, t_0, x_0)|^2].$$

We shall estimate these two quantities separately. First, set

$$D_1(a) = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^k : \max(|\tau|^{\alpha_1}, |\xi|^{\alpha_2}) < a\}.$$

Then

$$f_1(a, t, x, t_0, x_0)$$

$$= d \iint_{D_1(a)} \left| e^{-i\xi \cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} - e^{-i\xi \cdot x_0} \frac{e^{-i\tau t_0} - e^{-t_0|\xi|^2}}{|\xi|^2 - i\tau} \right|^2 |\xi|^{\beta-k} \, d\tau \, d\xi$$

$$(7.8) \quad = d \iint_{D_1(a)} \left| \frac{e^{-i\tau t} - e^{-t|\xi|^2} - e^{-i\xi \cdot (x_0-x) - i\tau t_0} + e^{-t_0|\xi|^2} e^{-i\xi \cdot (x_0-x)}}{|\xi|^2 - i\tau} \right|^2$$

$$\times |\xi|^{\beta-k} \, d\tau \, d\xi$$

$$= d \iint_{D_1(a)} \frac{\varphi_1(t, x, \tau, \xi)^2 + \varphi_2(t, x, \tau, \xi)^2}{|\xi|^4 + \tau^2} |\xi|^{\beta-k} \, d\tau \, d\xi,$$

where

$$\varphi_1(t, x, \tau, \xi) = \cos(\tau t) - e^{-t|\xi|^2} - \cos(\xi \cdot (x_0 - x) + \tau t_0)$$
$$+ e^{-t_0|\xi|^2} \cos(\xi \cdot (x_0 - x)),$$

$$\varphi_2(t, x, \tau, \xi) = -\sin(\tau t) + \sin(\xi \cdot (x_0 - x) + \tau t_0) - e^{-t_0\xi^2} \sin(\xi \cdot (x_0 - x)).$$

Observe that $\varphi_1(t_0, x_0, \tau, \xi) = 0 = \varphi_2(t_0, x_0, \tau, \xi)$, and

$$\frac{\partial \varphi_1}{\partial t} = -\tau \sin(\tau t) + |\xi|^2 e^{-t|\xi|^2},$$

$$\frac{\partial \varphi_1}{\partial x_j} = -\xi_j \sin(\xi \cdot (x_0 - x) + \tau t_0) + \xi_j e^{-t_0\xi^2} \sin(\xi \cdot (x_0 - x)),$$

$$\frac{\partial \varphi_2}{\partial t} = -\tau \cos(\tau t),$$

$$\frac{\partial \varphi_2}{\partial x_j} = -\xi_j \cos(\xi \cdot (x_0 - x) + \tau t_0) + \xi_j e^{-t_0|\xi|^2} \cos(\xi \cdot (x_0 - x)).$$

Therefore, for $\ell = 1, 2$,

$$\left| \frac{\partial \varphi_\ell}{\partial t} \right| \le |\tau| + |\xi|^2, \qquad \left| \frac{\partial \varphi_\ell}{\partial x_j} \right| \le 2|\xi|,$$

and the mean value theorem implies that

$$|\varphi_\ell(t, x, \tau, \xi)| \le (|\tau| + |\xi|^2)|t - t_0| + 2|\xi||x - x_0|,$$

so

$$f_1(a, t, x, t_0, x_0) \leq d \iint_{D_1(a)} [4(\tau^2 + |\xi|^4)(t - t_0)^2$$

$$+ 8|\xi|^2 |x - x_0|^2] \frac{|\xi|^{\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi$$

(7.9)

$$\leq 4d \cdot (t - t_0)^2 \iint_{D_1(a)} |\xi|^{\beta-k} \, d\tau \, d\xi$$

$$+ 8d \cdot |x - x_0|^2 \iint_{D_1(a)} \frac{|\xi|^{2+\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi.$$

For the first integral, pass to polar coordinates $r = |\xi|$ and use the fact that $\alpha_2 = 2\alpha_1$ to get

$$c_k \iint_{\max(|\tau|, r^2) < a^{\alpha_1^{-1}}} r^{\beta-k+k-1} \, d\tau \, dr = c_k(A_1 + A_2),$$

where

$$A_1 = \iint_{r^2 < |\tau| < a^{\alpha_1^{-1}}} r^{\beta-1} \, d\tau \, dr, \qquad A_2 = \iint_{|\tau| < r^2 < a^{\alpha_1^{-1}}} r^{\beta-1} \, d\tau \, dr.$$

Clearly,

$$A_1 = \int_0^{a^{\alpha_1^{-1}}} d\tau \int_0^{\sqrt{\tau}} dr \, r^{\beta-1} = ca^{2(2+\beta)/(2-\beta)} = ca^{2\gamma_1},$$

and

$$A_2 = \int_0^{a^{\alpha_2^{-1}}} dr \, r^{\beta-1} \int_0^{r^2} d\tau = ca^{2(2+\beta)/(2-\beta)} = ca^{2\gamma_1}.$$

We conclude that

(7.10)
$$\iint_{D_1(a)} |\xi|^{\beta-k} \, d\tau \, d\xi \leq \tilde{c} a^{2\gamma_1}.$$

For the second integral, pass to polar coordinates $r = |\xi|$:

$$\iint_{D_1(a)} \frac{|\xi|^{2+\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi = c_k \int_{\max(|\tau|^{\alpha_1}, r^{\alpha_2}) \leq a} \frac{r^{2+\beta-k}}{r^4 + \tau^2} r^{k-1} \, d\tau \, dr,$$

then set $w = r^2$ to get

$$\int_{\max(|\tau|^{\alpha_1}, w^{\alpha_1}) \leq a} \frac{w^{(\beta+1)/2}}{w^2 + \tau^2} \frac{d\tau \, dw}{2\sqrt{w}} = \int_{\max(|\tau|, w) \leq a^{\alpha_1^{-1}}} \frac{w^{\beta/2}}{w^2 + \tau^2} \, d\tau \, dw$$

$$\leq \int_{\max(|\tau|, w) \leq a^{\alpha_1^{-1}}} |(w, \tau)|^{\frac{\beta}{2}-2} \, d\tau \, dw.$$

Pass to polar coordinates $\rho = |(w, \tau)|$ to see that

$$(7.11) \qquad \iint_{D_1(a)} \frac{|\xi|^{2+\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi \leq c \int_0^{a^{\alpha_1^{-1}}} d\rho \rho^{\frac{\beta}{2}-1} = ca^{2\beta/(2-\beta)} = ca^{2\gamma_2}.$$

We conclude that

$$f_1(a, t, x, t_0, x_0) \leq c[a^{2\gamma_1}(t - t_0)^2 + a^{2\gamma_2}|x - x_0|^2].$$

We now examine f_2. Set

$$D_2(b) = \{(\tau, \xi) : \max(|\tau|^{\alpha_1}, |\xi|^{\alpha_2}) > b\}.$$

Notice that, as in (7.8),

$$(7.12) \qquad \begin{aligned} &f_2(b, t, x, t_0, x_0) \\ &= d \iint_{D_2(b)} \frac{\varphi_1(t, x, \tau, \xi)^2 + \varphi_2(t, x, \tau, \xi)^2}{|\xi|^4 + \tau^2} |\xi|^{\beta-k} \, d\tau \, d\xi. \end{aligned}$$

Observing that $|\varphi_1| \leq 4$ and $|\varphi_2| \leq 3$, we see that

$$f_2(b, t, x, t_0, x_0) \leq 25d \iint_{D_2(b)} \frac{|\xi|^{\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi.$$

Let

$$A_1 = \{(\tau, \xi) : |\tau|^{\alpha_1} \geq |\xi|^{\alpha_2} \text{ and } |\tau|^{\alpha_1} > b\},$$
$$A_2 = \{(\tau, \xi) : |\tau|^{\alpha_1} < |\xi|^{\alpha_2} \text{ and } |\xi|^{\alpha_2} > b\},$$

so that $A_1 \cup A_2 = D_2(b)$. Passing to polar coordinates $\rho = |\xi|$, notice that

$$(7.13) \qquad \begin{aligned} \iint_{A_1} \frac{|\xi|^{\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi &\leq 4c_k \int_{b^{\alpha_1^{-1}}}^{\infty} d\tau \int_0^{\sqrt{\tau}} d\rho \frac{\rho^{k-1+\beta-k}}{\tau^2} \\ &= cb^{2\frac{\beta-2}{2-\beta}} = cb^{-2} \end{aligned}$$

and

$$(7.14) \qquad \iint_{A_2} \frac{|\xi|^{\beta-k}}{|\xi|^4 + \tau^2} \, d\tau \, d\xi \leq c_k \int_{b^{\alpha_2^{-1}}}^{\infty} d\rho \rho^{k-1} \int_0^{\rho^2} d\tau \frac{\rho^{\beta-k}}{\rho^4} = \tilde{c}b^{2\frac{\beta-2}{2-\beta}} = \tilde{c}b^{-2},$$

therefore, $f_2(b, t, x, t_0, x_0) \leq cb^{-2}$. This proves Lemma 7.3. $\square$

We now check Assumption 2.4 for the process $\hat{v}$. In the context of this section, in agreement with Lemma 7.3 and Assumption 2.1(b),

$$(7.15) \qquad \Delta((t, x), (s, y)) = |t - s|^{\frac{2-\beta}{4}} + |x - y|^{\frac{2-\beta}{2}}.$$

It is well known (see [9], Lemma 4.2) that for any compact box $I \subset]0, \infty[\times \mathbb{R}$, there is $c > 0$ such that for all $(t, x) \in I$ and $(s, y) \in I$,

$$(7.16) \qquad \|\hat{v}(t, x) - \hat{v}(s, y)\|_{L^2} \geq c\Delta((t, x), (s, y)),$$

and $Q = (4 + 2k)/(2 - \beta)$ is the critical dimension for hitting points. Further,

$$
\begin{aligned}
\|\hat{v}(t, x)\|_{L^2}^2 &= d \int_0^t ds \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz G(s, y) \frac{1}{|y - z|^\beta} G(s, z) \\
&\geq c^2 \int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} |\mathcal{F}_y G(s, \cdot)(\xi)|^2 \\
(7.17) \qquad &= c^2 \int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} e^{-s|\xi|^2} \\
&= c_0 t^{\frac{2-\beta}{2}},
\end{aligned}
$$

so Assumption 2.4(a) is satisfied for the box I. In the next lemma, we check Assumption 2.4(b).

LEMMA 7.5. *Let $I \subset]0, \infty[\times \mathbb{R}^k$ be a compact box. Fix $(t, x) \in I$. Let $t' = t - 2(2\rho)^{\alpha_1^{-1}}$ and $x' = x$ (where ρ is small enough so that $t' > 0$). There is a number C_1 (depending possibly on ρ, β, k and d) such that for all $(s_1, y_1), (s_2, y_2) \in B_\rho'(t, x)$ [the open Δ-ball in $\mathbb{R}_+ \times \mathbb{R}^k$ of radius 2ρ centered at (t, x)], and $j \in \{1, \ldots, d\}$,*

$$E[(\hat{v}_j(s_1, y_1) - \hat{v}_j(s_2, y_2))\hat{v}_j(t', x')] \leq C_1(|s_1 - s_2| + |y_1 - y_2|).$$

PROOF. For $(s, y) \in B_\rho'(t, x)$, define

$$f(s, y) = E(\hat{v}_j(s, y)\hat{v}_j(t', x')).$$

Case 1: $k = 1 = \beta$. In this case,

$$f(s, y) = C \int_0^{t'} dr \int_{\mathbb{R}} d\bar{y} G(s - r, y - \bar{y}) G(t' - r, x' - \bar{y})$$

(notice that the right-hand side does not depend on j). Then

$$\frac{\partial f}{\partial y}(s, y) = \int_0^{t'} dr \int_{\mathbb{R}} d\bar{y} \frac{\partial G}{\partial y}(s - r, y - \bar{y}) G(t' - r, x' - \bar{y}).$$

Notice that

$$\frac{\partial G}{\partial y}(s - r, y - \bar{y}) = \frac{y - \bar{y}}{s - r} G(s - r, y - \bar{y}).$$

Since $(s, y) \in B_\rho'(t, x)$, $s \geq t - (2\rho)^{\alpha_1^{-1}}$, and since $t - t' = 2(2\rho)^{\alpha_1^{-1}}$, it follows that for $r \leq t'$, $s - r \geq (2\rho)^{\alpha_1^{-1}}$. Therefore, $|\frac{\partial f}{\partial y}|$ is bounded over $B_\rho'(t, x)$ [with a bound that depends on ρ but does not depend on $(t, x) \in I$].

Similarly, since

$$\frac{\partial G}{\partial s}(s - r, y - \bar{y}) = -\frac{1}{2}\left(\frac{1}{s - r} + \frac{(y - \bar{y})^2}{(s - r)^2}\right)G(s - r, y - \bar{y}),$$

we see that $|\frac{\partial f}{\partial s}|$ is also bounded over $B'_\rho(t, x)$. By the mean value theorem, we conclude that

$$\left|f(s_1, y_1) - f(s_2, y_2)\right| \leq C\left(|s_1 - s_2| + |y_1 - y_2|\right),$$

and this proves the lemma in this case.

Case 2: $\beta \in]0, 2 \wedge k[$. In this case,

$$f(s, y) = \int_0^{t'} dr \int_{\mathbb{R}^k} d\bar{y} \int_{\mathbb{R}^k} d\bar{z} G(s - r, y - \bar{y})\frac{1}{|\bar{y} - \bar{z}|^\beta}G(t' - r, x' - \bar{z}),$$

so

$$\frac{\partial f}{\partial y_j}(s, y) = \int_0^{t'} dr \int_{\mathbb{R}^k} d\bar{y} \int_{\mathbb{R}^k} d\bar{z} \frac{\partial G}{\partial y_j}(s - r, y - \bar{y})\frac{1}{|\bar{y} - \bar{z}|^\beta}G(t' - r, x' - \bar{z})$$

$$= C \int_0^{t'} dr \int_{\mathbb{R}^k} d\xi \, |\xi|^{\beta - k} e^{-i\xi \cdot (x' - y)} i\xi_j e^{-(s-r)|\xi|^2} e^{-(t'-r)|\xi|^2}.$$

Use, as above, the fact that for $r \leq t'$, $s - r \geq (2\rho)^{\alpha_1^{-1}} =: c$, so that $e^{-(s-r)|\xi|^2} \leq e^{-c|\xi|^2}$, then bound the integrand by its modulus, permute the integrals and compute the dr-integral, to conclude that $\frac{\partial f}{\partial y_j}(s, y)$ is bounded [with a bound that depends on ρ but does not depend on $(t, x) \in I$], as is $\frac{\partial f}{\partial s}$, so the conclusion follows as in Case 1. □

PROOF OF THEOREM 7.1. We have already mentioned (before Lemma 7.5) that $d = (4 + 2k)/(2 - \beta)$ is the critical dimension. By Lemma 7.3 and the sentences that precede this lemma, for any compact box $I \subset]0, \infty[\times \mathbb{R}^k$, Assumption 2.1 is satisfied for Re(v), with exponents $\gamma_1 = \frac{2+\beta}{2-\beta}$ and $\gamma_j = \frac{\beta}{2-\beta}$, $j = 2, \ldots, k + 1$, so that $\alpha_1 = \frac{2-\beta}{4}$ and $\alpha_j = \frac{2-\beta}{2}$, $j = 2, \ldots, k + 1$. By Lemma 7.5 and the comments that precede this lemma, Assumption 2.4 is satisfied by $\hat{v}$ (with $\delta_j \equiv 1$), hence by Re(v) by Proposition 7.2(c). Since $Q = \alpha_1^{-1} + k\alpha_2^{-1} = (4 + 2k)/(2 - \beta) = d$, it follows from Theorem 2.6 that for all $z \in \mathbb{R}^Q$,

$$P\{\exists (t, x) \in I : \hat{v}(t, x) = z\} = P\{\exists (t, x) \in I : \text{Re}(v(t, x)) = z\} = 0.$$

Since this holds for all compact rectangles $I \subset]0, \infty[\times \mathbb{R}^k$, Theorem 7.1 is proved.
□

8. Polarity of points for systems of linear heat equations with nonconstant coefficients. For $j = 1, \ldots, k$, let $\sigma_j : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be continuous functions such that, for all $T \in \mathbb{R}_+$, there are $0 < c_T < C_T < \infty$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$(8.1) \qquad\qquad c_T \leq \sigma_j(t, x) \leq C_T.$$

Let $\hat{W}$ be as in Section 7 and let $\hat{v} = (\hat{v}(t, x), t \in \mathbb{R}_+, x \in \mathbb{R}^k)$ be the solution of a linear system of d independent heat equations with deterministic coefficients:

$$(8.2) \qquad \begin{cases} \dfrac{\partial}{\partial t} \hat{v}_j(t, x) = \Delta \hat{v}_j(t, x) + \sigma_j(t, x) \dot{\hat{W}}_j(t, x) & j = 1, \ldots, d, \\ v(0, x) = 0 & x \in \mathbb{R}^k. \end{cases}$$

Set

$$\tilde{G}_{t,x}(s, y) = 1_{[0,t]}(s) G(t - s, x - y).$$

As a consequence of (8.1), in either of the cases $\beta \in]0, k \wedge 2[$ or $k = 1 = \beta$,

$$\int_{\mathbb{R}} d\tau \int_{\mathbb{R}^k} d\xi\, |\xi|^{\beta-k} \big| \mathcal{F}_{s,y}(\tilde{G}_{t,x}\sigma_j)(\tau, \xi) \big|^2 < \infty.$$

Indeed, in the case $\beta \in]0, k \wedge 2[$, for instance, the integral is equal to

$$\int_{\mathbb{R}} ds \int_{\mathbb{R}^k} dy_1 \int_{\mathbb{R}^k} dy_2 \tilde{G}_{t,x}(s, y_1)\sigma_j(s, y_1) \frac{1}{|y_1 - y_2|^\beta} G_{t,x}(s, y_2)\sigma_j(s, y_2),$$

and then (8.1) can be used.

We also make the following technical assumption on σ_j. This assumption can be checked for specific choices of β, k and σ_j, as in Proposition 8.6 below, for instance.

ASSUMPTION 8.1. (a) $\mathcal{F}_{s,y}\sigma_j$ is a measure μ_j with finite total variation.
(b) Similar to (7.11), for large a,

$$\iint_{\mathbb{R}\times\mathbb{R}^k} |\mu_j|(dr, dz) \iint_{D_1(a)} d\tau\, d\xi |\xi|^{\beta-k} \frac{|\xi - z|^2}{|\xi - z|^4 + |\tau - r|^2} \leq ca^{2\gamma_2}.$$

(c) Similar to (7.13) and (7.14), for large b,

$$\iint_{\mathbb{R}\times\mathbb{R}^k} |\mu_j|(dr, dz) \iint_{D_2(b)} d\tau\, d\xi \frac{|\xi|^{\beta-k}}{|\xi - z|^4 + |\tau - r|^2} \leq cb^{-2}.$$

THEOREM 8.2. *Suppose that* $d = (4 + 2k)/(2 - \beta)$ *and Assumption* 8.1 *is satisfied. Then* d *is the critical dimension for hitting points and points are polar for* $\hat{v}$.

Recall from the calculations that led to (7.5) that

$$\mathcal{F}_{s,y}\tilde{G}_{t,x}(\tau,\xi) = e^{-i\xi\cdot x}\frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau}.$$

Define $\tilde{W}$ as in Proposition 7.2, and set

$$v_j(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}^k} W_j(d\tau, d\xi)|\xi|^{(\beta-k)/2}(\mathcal{F}_{s,y}\tilde{G}_{t,x} * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi).$$

PROPOSITION 8.3. *The random field $v = (v(t,x) = (v_1(t,x),\ldots,v_d(t,x)))$ is the solution of the SPDE (8.2) with $\dot{W}$ replaced by $\dot{\tilde{W}}$.*

PROOF. Observe that by definition of $\tilde{W}_j$,

$$\iint 1_{[0,t]}(s)G(t-s,x-y)\sigma(s,y)\tilde{W}_j(ds,dy)$$
$$= \iint W_j(d\tau,d\xi)\mathcal{F}_{s,y}(\tilde{G}_{t,x}\sigma_j)(\tau,\xi)|\xi|^{(\beta-k)/2} = v_j(t,x),$$

and $v_j(0,x) = 0$. Therefore, v is the mild solution of (8.2) (with $\dot{W}$ replaced by $\dot{\tilde{W}}$). This completes the proof. $\square$

Define α_1 and α_2 as in (7.6) and let

$$v_j(A,t,x) = \iint_{\max(|\tau|^{\alpha_1},|\xi|^{\alpha_2})\in A} W_j(d\tau,d\xi)|\xi|^{(\beta-k)/2}(\mathcal{F}_{s,y}\tilde{G}_{t,x} * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi)$$

and $(v(A,t,x) = (v_1(A,t,x),\ldots,v_d(A,t,x)))$.

PROPOSITION 8.4. *Under Assumption 8.1(a)–(c), the random field $(v(A,t,x))$ satisfies Assumption 2.1 for any compact box $I \subset]0,\infty[\times \mathbb{R}^k$.*

PROOF. Assumption 2.1(a) is clearly satisfied, so we check Assumption 2.1(b). Set

$$v_{1,j}(a,t,x) = \iint_{D_1(a)} W_j(d\tau,d\xi)|\xi|^{(\beta-k)/2}(\mathcal{F}_{s,y}\tilde{G}_{t,x} * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi).$$

Define

$$f_{1,j}(a,t,x,t_0,x_0)$$
$$:= E\big((v_{1,j}(a,t,x) - v_{1,j}(a,t_0,x_0))^2\big)$$
$$= \iint_{D_1(a)} d\tau\,d\xi\,|\xi|^{\beta-k}\big|((\mathcal{F}_{s,y}\tilde{G}_{t,x} - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}) * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi)\big|^2,$$

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and notice that, by the Cauchy–Schwarz inequality,

$$
\big|((\mathcal{F}_{s,y}\tilde{G}_{t,x} - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}) * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi)\big|^2
$$

$$
(8.3) \qquad = \left| \iint_{\mathbb{R}\times\mathbb{R}^k} (\mathcal{F}_{s,y}\tilde{G}_{t,x}(\tau - r, \xi - z) - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}(\tau - r, \xi - z))\mu_j(dr, dz) \right|^2
$$

$$
\leq |\mu_j|(\mathbb{R}\times\mathbb{R}^k) \iint_{\mathbb{R}\times\mathbb{R}^k} |\mathcal{F}_{s,y}\tilde{G}_{t,x}(\tau - r, \xi - z)
$$

$$
- \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}(\tau - r, \xi - z)|^2 |\mu_j|(dr, dz),
$$

so

$$
f_{1,j}(a,t,x,t_0,x_0) \leq C \iint_{\mathbb{R}\times\mathbb{R}^k} |\mu_j|(dr,dz) \iint_{D_1(a)} d\tau\, d\xi\, |\xi|^{\beta-k}
$$

$$
\times \big|\mathcal{F}_{s,y}\tilde{G}_{t,x}(\tau - r, \xi - z) - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}(\tau - r, \xi - z)\big|^2.
$$

By (7.8) and (7.9), the inner integral is equal to

$$
d \iint_{D_1(a)} d\tau\, d\xi\, |\xi|^{\beta-k} \frac{\varphi_1(t,x,\tau-r,\xi-z)^2 + \varphi_2(t,x,\tau-r,\xi-z)^2}{|\xi-z|^4 + |\tau-r|^2}
$$

$$
\leq d \iint_{D_1(a)} d\tau\, d\xi\, |\xi|^{\beta-k} \left[4(t-t_0)^2 + 8 \frac{|x-x_0|^2 |\xi-z|^2}{|\xi-z|^4 + |\tau-r|^2} \right].
$$

By (7.10), this is

$$
\leq c_1 (t-t_0)^2 a^{2\gamma_1} + c_2 |x-x_0|^2 \iint_{D_1(a)} d\tau\, d\xi\, |\xi|^{\beta-k} \frac{|\xi-z|^2}{|\xi-z|^4 + |\tau-r|^2}.
$$

This establishes in particular (2.2) for any $a_0 \geq 0$.

By Assumption 8.1(a) and (b), we conclude that, for large a,

$$
(8.4) \qquad
\begin{aligned}
f_1(a,t,x,t_0,x_0) &:= E\big(|v_1(a,t,x) - v_1(a,t_0,x_0)|^2\big) \\
&\leq c_1 a^{2\gamma_1}(t-t_0)^2 + c_2 a^{2\gamma_2}|x-x_0|^2.
\end{aligned}
$$

Set

$$
v_{2,j}(a,t,x) = \iint_{D_2(b)} W_j(d\tau, d\xi) |\xi|^{(\beta-k)/2} (\mathcal{F}_{s,y}\tilde{G}_{t,x} * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi).
$$

Then

$$
\begin{aligned}
f_{2,j}(b,t,x,t_0,x_0) \\
&:= E\big((v_{2,j}(b,t,x) - v_{2,j}(b,t_0,x_0))^2\big) \\
&= \iint_{D_2(b)} d\tau\, d\xi\, |\xi|^{\beta-k} \big|((\mathcal{F}_{s,y}\tilde{G}_{t,x} - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}) * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi)\big|^2.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality as in (8.3), we find that

$$f_{2,j}(b, t, x, t_0, x_0) \leq C \iint |\mu_j|(dr, dz) \iint_{D_2(b)} d\tau \, d\xi \, |\xi|^{\beta-k}$$
$$\times |\mathcal{F}_{s,y}\tilde{G}_{t,x}(\tau - r, \xi - z) - \mathcal{F}_{s,y}\tilde{G}_{t_0,x_0}(\tau - r, \xi - z)|^2$$

and by (7.12), the inner integral is equal to

$$\iint_{D_2(b)} d\tau \, d\xi \, |\xi|^{\beta-k} \frac{\varphi_1(t, x, \tau - r, \xi - z)^2 + \varphi_2(t, x, \tau - r, \xi - z)^2}{|\xi - z|^4 + |\tau - r|^2}$$
$$\leq 25 \iint_{D_2(b)} d\tau \, d\xi \, \frac{|\xi|^{\beta-k}}{|\xi - z|^4 + |\tau - r|^2}.$$

By Assumption 8.1(c), for large b,

$$f_2(b, t, x, t_0, x_0) := E\big(|v_2(b, t, x) - v_2(b, t_0, x_0)|^2\big) \leq cb^{-2}.$$

Putting this together with (8.4), we conclude that Assumption 2.1(b) is satisfied.
□

In the context of this section, $\Delta((t, x), (s, y))$ is defined as in (7.15), and because of the lower bound in (8.1), the inequality (7.16) remains satisfied. Therefore, by [2], Theorem 2.1, $d = (4 + 2k)/(2 - \beta)$ is the critical dimension for hitting points.

LEMMA 8.5. $(\hat{v}(t, x))$ *defined in* (8.2) *satisfies Assumption* 2.4 *for any compact box* $I \subset]0, \infty[\times \mathbb{R}^k$.

PROOF. Observe that

$$\|\hat{v}(t, x)\|_{L^2}^2 = d \int_0^t ds \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz G(s, y)\sigma(s, y)\frac{1}{|y - z|^\beta}G(s, z)\sigma(s, z)$$
$$\geq c_T^2 \int_0^t ds \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz G(s, y)\frac{1}{|y - z|^\beta}G(s, z) \geq c_0 t^{\frac{2-\beta}{2}}$$

by the same calculation as in (7.17), so Assumption 2.4(a) is satisfied.

Since σ is bounded above by (8.1), the proof of Assumption 2.4(b) follows the proof of Lemma 7.5. □

PROOF OF THEOREM 8.2. We have observed before Lemma 8.5 that $d = (4 + 2k)/(2 - \beta)$ is the critical dimension for hitting points. By Proposition 8.4, Assumption 2.1 is satisfied for Re(v), with exponents $\alpha_1 = \frac{2-\beta}{4}$ and $\alpha_2 = \frac{2-\beta}{2}$, for any compact box $I \subset]0, \infty[\times \mathbb{R}^k$. By Lemma 8.5, Assumption 2.4 is satisfied for $\hat{v}$. Since Re(v) and $\hat{v}$ have the same law by Proposition 8.3, the conclusion follows from Theorem 2.6. □

Sufficient conditions for Assumption 8.1(b) *and* (c)

PROPOSITION 8.6. *Suppose that $k = 1 = \beta$, and $\mu_j = \mathcal{F}_{s,y}\sigma_j$ is a measure with compact support and finite total variation. Then Assumption 8.1 is satisfied.*

PROOF. It is clear that Assumption 8.1(a) holds. We check Assumption 8.1(b). Note that $k = 1 = \beta$, so $\alpha_1 = \frac{1}{4}$ and $\alpha_2 = \frac{1}{2}$, and observe that

$$
\iint_{D_1(a)} d\tau \, d\xi \, \frac{|\xi - z|^2}{|\xi - z|^4 + |\tau - r|^2}
$$
$$
\leq \iint_{\max(|\tau - r|^{\frac{1}{4}}, |\xi - z|^{\frac{1}{2}}) \leq a + r^{\frac{1}{4}} + z^{\frac{1}{2}}} d\tau \, d\xi \, \frac{|\xi - z|^2}{|\xi - z|^4 + |\tau - r|^2}
$$
$$
= \iint_{D_1(a + r^{\frac{1}{4}} + z^{\frac{1}{2}})} d\tau \, d\xi \, \frac{|\xi|^2}{|\xi|^4 + |\tau|^2}.
$$

By (7.11) in the case $k = 1 = \beta$ (so $\gamma_2 = 1$), we conclude that this integral is $\leq c(a + r^{\frac{1}{4}} + z^{\frac{1}{2}})^2$ and, therefore,

$$
\iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz) \iint_{D_1(a)} d\tau \, d\xi \, \frac{|\xi - z|^2}{|\xi - z|^4 + |\tau - r|^2}
$$
(8.5)
$$
\leq c \iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz)(a + r^{\frac{1}{4}} + z^{\frac{1}{2}})^2
$$
$$
= ca^2 \iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz)\left(1 + \frac{r^{\frac{1}{4}}}{a} + \frac{z^{\frac{1}{2}}}{a}\right)^2
$$
$$
\leq ca^2 \iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz)\left(1 + r^{\frac{1}{4}} + z^{\frac{1}{2}}\right)^2
$$

provided $a \geq 1$, and the integral is finite under the assumptions of this proposition. This establishes Assumption 8.1(b).

We now check Assumption 8.1(c) in the case $k = 1 = \beta$. Use the change of variables $s = \tau - r$, $y = \xi - z$ to see that

$$
\iint_{D_2(b)} d\tau \, d\xi \, \frac{1}{|\xi - z|^4 + |\tau - r|^2} \leq \iint_{\max(|s+r|^{\frac{1}{4}}, |y+z|^{\frac{1}{2}}) > b} ds \, dy \, \frac{1}{|y|^4 + |s|^2}
$$
$$
\leq \iint_{D_2(\psi(b,r,z))} d\tau \, d\xi \, \frac{1}{|\xi|^4 + |\tau|^2},
$$

where $\psi(b, r, z) = \min(|b^4 - |r||^{1/4}, |b^2 - |z||^{1/2})$.

By (7.13) and (7.14), we conclude that

$$
\iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz) \iint_{D_2(b)} d\tau \, d\xi \, \frac{1}{|\xi - z|^4 + |\tau - r|^2}
$$
(8.6)
$$
\leq c \iint_{\mathbb{R} \times \mathbb{R}} |\mu_j|(dr, dz)(\psi(b, r, z))^{-2},
$$

and clearly,

$$\psi(b, r, z) = b \min\left(\left|1 - \frac{|r|}{b^4}\right|^{1/4}, \left|1 - \frac{|z|}{b^2}\right|^{1/2}\right).$$

If b is large enough so that the inequalities $|z| \le b^2/2$ and $|r| \le b^4/2$ are satisfied for all (r, z) in the support of μ_j, then $\psi(b, r, z) \ge b/2$, and so the right-hand side of (8.6) is $\le 4cb^{-2}$. This establishes Assumption 8.1(c). $\square$

COROLLARY 8.7. *Suppose that $d = 6$, $k = 1$, $\hat{W}$ is space–time white noise and $\mathcal{F}_{s,y}\sigma_j$ is a measure with compact support and finite total variation. Then points are polar for the solution $(\hat{v}(t, x))$ of the stochastic heat equation (8.2) with nonconstant deterministic coefficients σ_i.*

PROOF. This is an immediate consequence of Theorem 8.2 (with $\beta = 1$) and Proposition 8.6. $\square$

9. Polarity of points for systems of linear wave equations with constant coefficients.
Fix $k \ge 1$ and $\beta \in]0, k \wedge 2[$ or $k = 1 = \beta$, and let $\hat{W}$ be spatially homogeneous $\mathbb{R}^d$-valued Gaussian noise as in the beginning of Section 7. We assume that

$$(9.1) \qquad\qquad\qquad \beta \ge 1.$$

Let $\hat{v}$ be the solution of the stochastic wave equation in spatial dimension k driven by $\hat{W}$:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\hat{v}_j(t, x) = \Delta\hat{v}_j(t, x) + \dot{\hat{W}}_j(t, x) & j = 1, \ldots, d, \\ \hat{v}(0, x) = 0, \quad \dfrac{\partial}{\partial t}\hat{v}(0, x) = 0 & x \in \mathbb{R}^k. \end{cases}$$

THEOREM 9.1. *Suppose $k = 1 = \beta$ or $1 < \beta < k \wedge 2$, and $d = \frac{2(k+1)}{2-\beta}$. Then d is the critical dimension for hitting points and points are polar for $\hat{v}$, that is, for all $z \in \mathbb{R}^d$,*

$$P\{\exists(t, x) \in]0, +\infty[\times \mathbb{R}^k : \hat{v}(t, x) = z\} = 0.$$

In particular, in the case where $k = 1 = \beta$, $\hat{W}$ is space–time white noise and $d = 4$, then points are polar for $\hat{v}$.

Define

$$F(t, x, \tau, \xi) = \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|}\left[\frac{1 - e^{it(\tau + |\xi|)}}{\tau + |\xi|} - \frac{1 - e^{it(\tau - |\xi|)}}{\tau - |\xi|}\right].$$

The next proposition gives the harmonizable representation of $\hat{v}$. This representation also appears in [1], Section 6.

PROPOSITION 9.2. *Set*

$$v(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} W(d\tau, d\xi) |\xi|^{(\beta-k)/2} F(t, x, \tau, \xi),$$

and let $\tilde{W}_j(\varphi)$ *be defined as in Proposition* 7.2. *Then* $(v(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ *is a* $\mathbb{C}$-*valued solution of*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} v_j(t, x) = \Delta v_j(t, x) + \dot{\tilde{W}}_j(t, x) & j = 1, \ldots, d, \\ v(0, x) = 0, \qquad \dfrac{\partial}{\partial t} v(0, x) = 0 & x \in \mathbb{R}^k. \end{cases}$$

In particular, $\mathrm{Re}(v)$ *and* $\hat{v}$ *have the same law.*

PROOF. Let $S(s, y)$ be the fundamental solution of the wave equation. Since $\beta \in]0, k \wedge 2[$ or $k = 1 = \beta$, the stochastic integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}^k} \tilde{W}_j(ds, dy) 1_{[0,t]}(s) S(t - s, x - y)$$

is well defined in all spatial dimensions $k \geq 1$ (see [8], Example 6), and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^k} \tilde{W}_j(ds, dy) 1_{[0,t]}(s) S(t - s, x - y)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^k} W_j(d\tau, d\xi) |\xi|^{(\beta-k)/2} \mathcal{F}_{s,y}(1_{[0,t]}(\cdot) S(t - \cdot, x - \cdot))(\tau, \xi).$$

Now for $s \in [0, t]$, according to [8], Example 6,

$$\mathcal{F}_y S(t - s, x - \cdot)(\xi) = e^{-i\xi \cdot x} \mathcal{F}_y S(t - s, \cdot)(-\xi) = e^{-i\xi \cdot x} \frac{\sin((t - s)|\xi|)}{|\xi|},$$

and $\mathcal{F}_{s,y}(1_{[0,t]}(\cdot) S(t - \cdot, x - \cdot))(\tau, \xi)$ is equal to

$$\frac{e^{-i\xi \cdot x}}{|\xi|} \int_0^t e^{-i\tau s} \sin((t - s)|\xi|) \, ds = \frac{e^{-i\xi \cdot x}}{|\xi|} \int_0^t e^{-i\tau(t-r)} \sin(r|\xi|) \, dr$$
$$= \frac{e^{-i\xi \cdot x - i\tau t}}{|\xi|} \int_0^t e^{i\tau r} \frac{e^{ir|\xi|} - e^{-ir|\xi|}}{2i} \, dr$$
$$= \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left[\frac{1 - e^{it(\tau+|\xi|)}}{\tau + |\xi|} + \frac{e^{it(\tau-|\xi|)} - 1}{\tau - |\xi|} \right].$$

Therefore,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{W}_j(ds, dy) 1_{[0,t]}(s) S(t - s, x - y) = v_j(t, x), \qquad j = 1, \ldots, d.$$

This proves the proposition. $\square$

Let

$$\alpha = \frac{2-\beta}{2},$$

and set

$$v(A, t, x) = \iint_{\max(|\tau|^\alpha, |\xi|^\alpha) \in A} |\xi|^{(\beta-k)/2} F(t, x, \tau, \xi) W(d\tau, d\xi).$$

Clearly, the random field $(v(A, t, x), A \in \mathcal{B}(\mathbb{R}_+), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$ satisfies Assumption 2.1(a) (with the generic variable $x \in \mathbb{R}^k$ replaced by $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^k$). In the next lemma, we check Assumption 2.1(b).

LEMMA 9.3.   *Let* $I \subset]0, T] \times \mathbb{R}$ *be a compact box. Assume that* (9.1) *holds. Then the random field* $(v(A, t, x), A \in \mathcal{B}(\mathbb{R}_+), (t, x) \in I)$ *satisfies the conditions of Assumption* 2.1, *with exponents*

$$\gamma_1 = \gamma_2 = \alpha^{-1} - 1 = \frac{\beta}{2-\beta} =: \gamma.$$

In particular, there is a universal constant c_0 *and* $a_0 \in \mathbb{R}_+$ *such that for all* $a_0 \le a \le b$, $(t_0, x_0) \in I$, $(t, x) \in I$,

(9.2)
$$\left\| v([a, b[, t, x) - v(t, x) - v([a, b[, t_0, x_0) + v(t_0, x_0) \right\|_{L^2}$$
$$\le c_0 \left[a^\gamma |t - t_0| + a^\gamma \sum_{j=1}^k |x_j - x_{0,j}| + b^{-1} \right]$$

and

(9.3)

$$\left\| v([0, a_0], t, x) - v([0, a_0], t_0, x_0) \right\|_{L^2} \le c_0 \left[|t - t_0| + \sum_{j=1}^k |x_j - y_j| \right].$$

PROOF.   Assumption 2.1(a) is clearly satisfied, so we check Assumption 2.1(b). Let

$$D_1(a) = \left\{ (\tau, \xi) : \max(|\tau|^\alpha, |\xi|^\alpha) < a \right\},$$
$$D_2(b) = \left\{ (\tau, \xi) : \max(|\tau|^\alpha, |\xi|^\alpha) > b \right\},$$

and for $\ell = 1, 2$,

$$v_\ell(a, t, x) = \iint_{D_\ell(a)} |\xi|^{(\beta-k)/2} F(t, x, \tau, \xi) W(d\tau, d\xi).$$

As in (7.7),

(9.4)
$$v([a, b], t, x) - v(t, x) - v([a, b], t_0, x_0) + v(t_0, x_0)$$
$$= v_1(a, t_0, x_0) - v_1(a, t, x) + v_2(b, t_0, x_0) - v_2(b, t, x).$$

So for $\ell = 1, 2$, we let

$$f_\ell(a, t, x, t_0, x_0) = E\big[|v_\ell(a, t_0, x_0) - v_\ell(a, t, x)|^2\big].$$

Clearly,

$$f_1(a, t, x, t_0, x_0) = d \iint_{D_1(a)} |F(t_0, x_0, \tau, \xi) - F(t, x, \tau, \xi)|^2 |\xi|^{\beta-k} \, d\tau \, d\xi.$$

Using Lemma 9.4(a) below, we see that

$$f_1(a, t, x, t_0, x_0) \le c\left(|t - t_0|^2 + \sum_{j=1}^{k} |x_j - x_{0,j}|^2\right)$$

$$\times \iint_{D_1(a)} \left[\frac{1}{1 + \frac{1}{4}(\tau + |\xi|)^2} + \frac{1}{1 + \frac{1}{4}(\tau - |\xi|)^2}\right] |\xi|^{\beta-k} \, d\tau \, d\xi.$$

Change to polar coordinates $r = |\xi|$ to see that the double integral is equal to

$$C \iint_{0 \le \max(|\tau|, r) < a^{1/\alpha}, r > 0} \left[\frac{1}{1 + \frac{1}{4}(\tau + r)^2} + \frac{1}{1 + \frac{1}{4}(\tau - r)^2}\right] r^{\beta-1} \, d\tau \, dr.$$

Use the change of variables $u = (\tau + r)/2$, $v = (\tau - r)/2$ to see that the double integral is equal to

$$2 \iint_{\max(|u+v|, u-v) \le 2a^{1/\alpha}, u-v > 0} \left[\frac{1}{1 + u^2} + \frac{1}{1 + v^2}\right] (u - v)^{\beta-1} \, du \, dv$$

$$\le \iint_{\max(|u|, |v|) \le 2a^{1/\alpha}} \left[\frac{1}{1 + u^2} + \frac{1}{1 + v^2}\right] |u - v|^{\beta-1} \, du \, dv.$$

By Lemma 9.5(a) below, this is $\le Ca^{\beta/\alpha} = Ca^{2\gamma}$. We conclude that

$$(9.5) \qquad \|v_1(a, t_0, x_0) - v_1(a, t, x)\|_{L^2} \le Ca^{\gamma}\left[|t - t_0| + \sum_{j=1}^{k} |x_j - x_{0,j}|\right].$$

This establishes in particular (9.3), for any $a_0 \ge 0$.

We now turn to the second term:

$$f_2(b, t, x, t_0, x_0)$$

$$(9.6) \qquad = d \iint_{D_2(b)} |F(t_0, x_0, \tau, \xi) - F(t, x, \tau, \xi)|^2 |\xi|^{\beta-k} \, d\tau \, d\xi$$

$$\le 2d \iint_{D_2(b)} \big[(F(t_0, x_0, \tau, \xi))^2 + (F(t, x, \tau, \xi))^2\big] |\xi|^{\beta-k} \, d\tau \, d\xi.$$

By Lemma 9.4(b) below, the double integral is bounded above by

$$C_T \iint_{D_2(b)} \left[\frac{1}{1 + \frac{1}{4}|\tau + |\xi||^2} + \frac{1}{1 + \frac{1}{4}|\tau - |\xi||^2}\right] \frac{|\xi|^{\beta-k}}{1 + |\xi|^2} \, d\tau \, d\xi.$$

Change again to polar coordinates $r = |\xi|$ to see that this is bounded by

$$(9.7) \qquad \iint_{\max(|\tau|,r)>b^{1/\alpha},r>0} \left[\frac{1}{1 + \frac{1}{4}(\tau + r)^2} + \frac{1}{1 + \frac{1}{4}(\tau - r)^2} \right] \frac{r^{\beta-1}}{1 + r^2} \, d\tau \, dr.$$

By Lemma 9.5(b) below, this is $\leq cb^{-2}$.

We conclude from (9.6) and the above estimate (9.7) that, for large b,

$$(9.8) \qquad f_2(b, t, x, t_0, x_0) \leq Cb^{-2}.$$

Putting together (9.4), (9.5) and (9.8), we conclude that for a_0 large enough and $a_0 \leq a \leq b$, the conclusion of Lemma 9.3 holds. $\quad\square$

The following two lemmas were used in the proof of Lemma 9.3.

LEMMA 9.4. *Fix $T > 0$. There is a constant C_T such that for all (t, x), $(t_0, x_0) \in [0, T] \times \mathbb{R}^k$, and all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^k$, the following inequalities hold*:
(a)

$$\big| F(t_0, x_0, \tau, \xi) - F(t, x, \tau, \xi) \big|$$

$$\leq C_T \left(|t - t_0| + \sum_{j=1}^k |x_j - x_{0,j}| \right) \left[\frac{1}{1 + \frac{1}{2}|\tau + |\xi||} + \frac{1}{1 + \frac{1}{2}|\tau - |\xi||} \right];$$

(b)

$$\big| F(t, x, \tau, \xi) \big| \leq C_T \left[\frac{1}{1 + \frac{1}{2}|\tau + |\xi||} + \frac{1}{1 + \frac{1}{2}|\tau - |\xi||} \right] \frac{1}{1 + |\xi|}.$$

PROOF. (a) Notice that

$$\frac{\partial F}{\partial x_j}(t, x, \tau, \xi) = -i\xi_j F(t, x, \tau, \xi)$$

$$= \frac{-i\xi_j}{2|\xi|} e^{-i\xi \cdot x - i\tau t} \left[\frac{1 - e^{it(\tau + |\xi|)}}{\tau + |\xi|} - \frac{1 - e^{it(\tau - |\xi|)}}{\tau - |\xi|} \right].$$

Observe that there is $c > 0$ such that for all $u \in \mathbb{R}$ and $t \in [0, T]$,

$$\left| \frac{1 - e^{itu}}{u} \right| \leq \frac{c}{1 + \frac{1}{2}|u|},$$

so

$$(9.9) \qquad \left| \frac{\partial F}{\partial x_j}(t, x, \tau, \xi) \right| \leq \frac{c}{2} \left[\frac{1}{1 + \frac{1}{2}|\tau + |\xi||} + \frac{1}{1 + \frac{1}{2}|\tau - |\xi||} \right].$$

Similarly,

$$\frac{\partial F}{\partial t}(t, x, \tau, \xi)$$

$$= -i\tau F(t, x, \tau, \xi) + \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left[-i e^{it(\tau + |\xi|)} + i e^{it(\tau - |\xi|)} \right]$$

$$= \frac{-i}{2|\xi|} e^{-i\xi \cdot x} \left[\frac{\tau e^{-i\tau t} - \tau e^{it|\xi|}}{\tau + |\xi|} - \frac{\tau e^{-i\tau t} - \tau e^{-it|\xi|}}{\tau - |\xi|} + e^{it|\xi|} - e^{-it|\xi|} \right].$$

We notice that the term in brackets vanishes when $|\xi| = 0$, and remains bounded when $\tau \pm |\xi| \to 0$, so $\frac{\partial F}{\partial t}$ is locally bounded. In fact, reducing to a common denominator, rearranging terms and simplifying, one finds that

$$\frac{\partial F}{\partial t}(t, x, \tau, \xi) = \frac{i}{2} e^{-i\xi \cdot x - i\tau t} \left[\frac{1 - e^{it(\tau - |\xi|)}}{\tau - |\xi|} + \frac{1 - e^{it(\tau + |\xi|)}}{\tau + |\xi|} \right],$$

therefore, as in (9.9),

$$(9.10) \qquad \left| \frac{\partial F}{\partial t}(t, x, \tau, \xi) \right| \le \frac{c}{2} \left[\frac{1}{1 + \frac{1}{2}|\tau + |\xi||} + \frac{1}{1 + \frac{1}{2}|\tau - |\xi||} \right].$$

Using (9.9), (9.10) and the mean value theorem, we see that (a) holds.

(b) Let $u = (\tau + |\xi|)/2$, $v = (\tau - |\xi|)/2$ and notice that

$$(9.11) \qquad \left| F(t, x, \tau, \xi) \right| = \frac{|\varphi_t(2u) - \varphi_t(2v)|}{2|u - v|},$$

where

$$\varphi_t(u) = \frac{1 - e^{itu}}{u}, \qquad u \ne 0.$$

Setting $\varphi_t(0) = -it$, then $\varphi_t \in C^1(\mathbb{R}, \mathbb{C})$, and

$$\varphi_t'(u) = \frac{-1 + e^{itu} - itu e^{itu}}{u^2} \qquad \text{if } u \ne 0,$$

and $\varphi_t'(0) = t^2/2$. It follows that for all $(t, u) \in [0, T] \times \mathbb{R}$,

$$(9.12) \qquad \max(|\varphi_t(u)|, |\varphi_t'(u)|) \le \frac{C_T}{1 + |u|}.$$

In particular, we claim that for all $(t, u) \in [0, T] \times \mathbb{R}$ with $|u - v| \le 1/2$,

$$(9.13) \qquad \frac{|\varphi_t(2u) - \varphi_t(2v)|}{|u - v|} \le C_T \left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|} \right].$$

Indeed, by the mean value theorem,

$$\left| \varphi_t(2u) - \varphi_t(2v) \right| \le 2|u - v| |\varphi_t'(\xi)|,$$

for some ξ between u and v. If both u and v have the same sign, say if $0 < u < v$, then by (9.12),

$$|\varphi_t'(\xi)| \leq \frac{C}{1 + |\xi|} \leq \frac{C}{1 + |u|} \leq C\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right].$$

The case where u and v are both negative is handled similarly. Finally, if $u < 0 < v$, then since $|u - v| \leq 1/2$, we have $|u| \leq 1/2$ and $|v| \leq 1/2$, so

$$|\varphi_t'(\xi)| \leq \frac{C}{1 + |\xi|} \leq C = C\left[\frac{3/4}{1 + \frac{1}{2}} + \frac{3/4}{1 + \frac{1}{2}}\right] \leq \tilde{C}\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right].$$

This proves (9.13).

We now claim that there is a constant $C_T < \infty$ such that for all $(t, u, v) \in \mathbb{R}_+ \times \mathbb{R}^2$,

(9.14) $$\frac{|\varphi_t(2u) - \varphi_t(2v)|}{2|u - v|} \leq C_T\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right]\frac{1}{1 + |u - v|}.$$

Indeed, assume first that $|u - v| \leq 1/2$. Then by (9.13), the left-hand side is

$$\leq C_T\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right] \leq C_T\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right]\frac{3/2}{1 + |u - v|}.$$

Now assume that $|u - v| \geq 1/2$. Then the left-hand side of (9.14) is

$$\leq \frac{3/2}{1 + |u - v|}(|\varphi_t(2u)| + |\varphi_t(2v)|) \leq \frac{\tilde{C}_T}{1 + |u - v|}\left[\frac{1}{1 + |u|} + \frac{1}{1 + |v|}\right],$$

where we have used (9.12). This completes the proof of (b). $\square$

LEMMA 9.5. (a) *For* $\beta \in]0, 2[$,

$$\iint_{\max(|u|,|v|) \leq 2a^{1/\alpha}}\left[\frac{1}{1 + u^2} + \frac{1}{1 + v^2}\right]|u - v|^{\beta-1}\,du\,dv \leq Ca^{\beta/\alpha}.$$

(b) *If* $\beta \geq 1$, *then for large* b,

$$\iint_{\max(|\tau|,r) > b^{1/\alpha}, r > 0}\left[\frac{1}{1 + \frac{1}{4}(\tau + r)^2} + \frac{1}{1 + \frac{1}{4}(\tau - r)^2}\right]\frac{r^{\beta-1}}{1 + r^2}\,d\tau\,dr \leq Cb^{-2}.$$

PROOF. (a) It suffices to consider the two integrals

$$A_1 = \iint_{\max(u,v) \leq 2a^{1/\alpha}, u > 0, v > 0}\left[\frac{1}{1 + u^2} + \frac{1}{1 + v^2}\right]|u - v|^{\beta-1}\,du\,dv,$$

$$A_2 = \iint_{\max(u,v) \leq 2a^{1/\alpha}, u > 0, v > 0}\left[\frac{1}{1 + u^2} + \frac{1}{1 + v^2}\right]|u + v|^{\beta-1}\,du\,dv.$$

By symmetry, $A_1 = 2A_{1,1}$, where

$$A_{1,1} = \int_0^{2a^{1/\alpha}} du \int_0^u dv \left[\frac{1}{1+u^2} + \frac{1}{1+v^2} \right] (u-v)^{\beta-1} \, du \, dv$$

$$\leq 2 \int_0^{2a^{1/\alpha}} du \int_0^u dv \frac{1}{1+v^2} (u-v)^{\beta-1}.$$

By Fubini's theorem, this is equal to

$$2 \int_0^{2a^{1/\alpha}} \frac{dv}{1+v^2} \int_v^{2a^{1/\alpha}} du (u-v)^{\beta-1}$$

$$= 2 \int_0^{2a^{1/\alpha}} \frac{dv}{1+v^2} (2a^{1/\alpha} - v)^\beta \leq C \int_0^{2a^{1/\alpha}} \frac{dv}{1+v^2} (2a^{1/\alpha})^\beta$$

$$\leq C a^{\beta/\alpha} \int_0^\infty \frac{dv}{1+v^2}.$$

Turning to A_2, by symmetry,

$$A_2 = 2 \int_0^{2a^{1/\alpha}} dv \int_0^v du \left[\frac{1}{1+u^2} + \frac{1}{1+v^2} \right] |u+v|^{\beta-1}$$

$$\leq C \int_0^{2a^{1/\alpha}} dv \int_0^v du \frac{1}{1+u^2} (u+v)^{\beta-1}.$$

By Fubini's theorem, this is equal to

$$C \int_0^{2a^{1/\alpha}} \frac{du}{1+u^2} \int_u^{2a^{1/\alpha}} dv (u+v)^{\beta-1}$$

$$= C \int_0^{2a^{1/\alpha}} \frac{du}{1+u^2} [(2a^{1/\alpha} + u)^\beta - (2u)^\beta]$$

$$\leq C \int_0^{2a^{1/\alpha}} \frac{du}{1+u^2} (3a^{1/\alpha})^\beta \leq \tilde{C} a^{\beta/\alpha}.$$

This proves (a).

(b) We need to integrate over two regions:

$$r > b^{1/\alpha}, |\tau| < r, \quad \text{and} \quad |\tau| > b^{1/\alpha}, 0 < r < |\tau|.$$

Concerning the first region, we have to consider

$$\int_{b^{1/\alpha}}^\infty dr \int_{-r}^r d\tau \frac{1}{1 + \frac{1}{4}(\tau \pm r)^2} \frac{r^{\beta-1}}{1+r^2},$$

and, by symmetry, it suffices to consider

$$\int_{b^{1/\alpha}}^\infty dr \frac{r^{\beta-1}}{1+r^2} \int_0^r d\tau \frac{1}{1 + \frac{1}{4}(\tau \pm r)^2} \leq \frac{\pi}{2} \int_{b^{1/\alpha}}^\infty dr r^{\beta-3} = b^{(\beta-2)/\alpha} = b^{-2}.$$

For the second region, we consider

$$\int_{b^{1/\alpha}}^{\infty} d\tau \int_0^{\tau} dr \frac{1}{1 + \frac{1}{4}(\tau \pm r)^2} \frac{r^{\beta-1}}{1 + r^2} = \int_0^{b^{1/\alpha}} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}}^{\infty} d\tau \frac{1}{1 + \frac{1}{4}(\tau \pm r)^2}$$

$$+ \int_{b^{1/\alpha}}^{\infty} dr \frac{r^{\beta-1}}{1 + r^2} \int_r^{\infty} d\tau \frac{1}{1 + \frac{1}{4}(\tau \pm r)^2}.$$

The second integral is

$$\leq \int_{b^{1/\alpha}}^{\infty} dr\, r^{\beta-3} \frac{\pi}{2} \leq Cb^{(\beta-2)/\alpha} = Cb^{-2}.$$

Concerning the first integral, in the case of a "+" sign, it is

$$\leq \int_0^{b^{1/\alpha}} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}}^{\infty} d\tau \frac{1}{1 + \frac{1}{4}\tau^2} = \int_0^{b^{1/\alpha}} dr \frac{r^{\beta-1}}{1 + r^2} \left[\frac{\pi}{2} - \arctan\left(\frac{b^{1/\alpha}}{2} \right) \right].$$

Using the property

$$\lim_{x \to \infty} x \left[\frac{\pi}{2} - \arctan(x) \right] = 1,$$

we see that for all $b \geq 1$, this is

$$\leq \tilde{c} b^{-1/\alpha} \int_0^{\infty} dr \frac{r^{\beta-1}}{1 + r^2} \leq Cb^{-2},$$

since $\frac{1}{\alpha} = \frac{2}{2-\beta} \geq 2$ because $\beta \geq 1$.

In the case of a "−" sign, we write the first integral as $I_1(b) + I_2(b)$, where

$$I_1(b) = \int_0^{b^{1/\alpha}/2} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}}^{\infty} d\tau \frac{1}{1 + \frac{1}{4}(\tau - r)^2},$$

$$I_2(b) = \int_{b^{1/\alpha}/2}^{b^{1/\alpha}} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}}^{\infty} d\tau \frac{1}{1 + \frac{1}{4}(\tau - r)^2}.$$

Then

$$I_1(b) = \int_0^{b^{1/\alpha}/2} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}-r}^{\infty} du \frac{1}{1 + \frac{1}{4}u^2}$$

$$\leq \int_0^{b^{1/\alpha}/2} dr \frac{r^{\beta-1}}{1 + r^2} \int_{b^{1/\alpha}/2}^{\infty} du \frac{1}{1 + \frac{1}{4}u^2} \leq Cb^{-1/\alpha} \int_0^{b^{1/\alpha}/2} dr \frac{r^{\beta-1}}{1 + r^2}$$

$$\leq cb^{-2},$$

since $\beta \geq 1$, and

$$I_2(b) \leq \int_{b^{1/\alpha}/2}^{b^{1/\alpha}} dr \frac{r^{\beta-1}}{1+r^2} \int_{-\infty}^{\infty} d\tau \frac{1}{1+\frac{1}{4}(\tau-r)^2} \leq C \int_{b^{1/\alpha}/2}^{b^{1/\alpha}} dr r^{\beta-3}$$

$$\leq C b^{(\beta-2)/\alpha} = C b^{-2}.$$

This completes the proof of (b). □

We now turn to Assumption 2.4. In the context of this section, in agreement with Lemma 9.3 and Assumption 2.1(b),

$$\Delta\big((t,x),(s,y)\big) = |t-s|^{\frac{2-\beta}{2}} + |x-y|^{\frac{2-\beta}{2}}.$$

It is well known (see [15], Proposition 1.4) that for any compact box $I \subset \,]0,\infty[\,\times\mathbb{R}^k$, there is $c > 0$ such that

$$\|\hat{v}(t,x) - \hat{v}(s,y)\|_{L^2}^2 \geq c\Delta\big((t,x),(s,y)\big),$$

and by [15], Theorems 4.4 and 4.5, $d = 2(k+1)/(2-\beta)$ is the critical dimension for hitting points.

Further, using the change of variables $r = t-s$, $\eta = (t-s)\xi$, we see that

$$\|\hat{v}(t,x)\|_{L^2}^2 = d \int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} \frac{\sin^2((t-s)|\xi|)}{|\xi|^2}$$

$$= d \int_0^t dr r^{2-\beta} \int_{\mathbb{R}^k} \frac{d\eta}{|\eta|^{k+2-\beta}} \sin^2(|\eta|)$$

$$= c t^{3-\beta},$$

so Assumption 2.4(a) is satisfied for the box I. In the next lemma, we check Assumption 2.4(b).

LEMMA 9.6. *Let $I \subset \,]0,\infty[\,\times\mathbb{R}^k$ be a compact box. Fix $(t,x) \in I$. Let $t' = t - 2(2\rho)^{\alpha^{-1}}$ and $x' = x$ (where ρ is small enough so that $t' > 0$). Assume that $k = 1 = \beta$ or $1 < \beta < k \wedge 2$. There is a number C_1 (depending on ρ, β, k and d) such that for all $(s_1,y_1),(s_2,y_2) \in B'_\rho(t,x)$ (the open Δ-ball of radius 2ρ centered at (t,x)) and $j \in \{1,\ldots,d\}$,*

$$(9.15) \quad \big|E\big[(\hat{v}_j(s_1,y_1) - \hat{v}_j(s_2,y_2))\hat{v}_j(t',x')\big]\big| \leq C_1\big(|s_1-s_2|^\delta + |y_1-y_2|^\delta\big),$$

where $\delta = 2-\beta$.

REMARK 9.7. The conclusion of this lemma would not be possible for $\beta \in \,]0,1[$, since it would mean that $(s,y) \mapsto E[\hat{v}_j(s,y)\hat{v}_j(t',x')]$ would be Hölder-continuous with exponent $2-\beta > 1$.

PROOF. Consider first the case $k = 1 = \beta$ (space–time white noise in spatial dimension $k = 1$). Then

$$(9.16) \qquad E[\hat{v}_j(s, y)\hat{v}_j(t', x')] = \int_0^{t'} dr \int_{\mathbb{R}} dz\, S(s - r, y - z) S(t' - r, x' - z),$$

where $S(r, z) = \frac{1}{2} 1_{\{|z| < r\}}$ is the fundamental solution of the wave equation. For $(s, y) \in B'_\rho(t, x)$,

$$|s - t| \le (2\rho)^{\alpha^{-1}}, \qquad |y - x| \le (2\rho)^{\alpha^{-1}},$$

and since $t' = t - 2(2\rho)^{\alpha^{-1}}$ and $x' = x$, one checks immediately that if, in addition, $|x' - z| < t' - r$, then

$$|y - z| \le |y - x| + |x - z| < (2\rho)^{\alpha^{-1}} + t' - r = t - (2\rho)^{\alpha^{-1}} - r \le s - r,$$

so the right-hand side of (9.16) is equal to

$$\int_0^{t'} dr \int_{\mathbb{R}} dz \frac{1}{4} 1_{\{|x' - z| < t' - r\}}$$

and, therefore, $(s, y) \mapsto E[\hat{v}_j(s, y)\hat{v}_j(t', x')]$ is constant over $B'_\rho(t, x)$ and (9.15) is trivially satisfied.

Now consider the case where $1 < \beta < k \wedge 2$. Then for $s \ge t'$,

$$E[(\hat{v}_j(s, y)\hat{v}_j(t', x')]$$
$$= \int_0^{t'} dr \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k - \beta}} \mathcal{F} S(s - r, y - \cdot)(\xi) \overline{\mathcal{F} S(t' - r, x' - \cdot)(\xi)}$$
$$= \int_0^{t'} dr \int_{\mathbb{R}^k} d\xi |\xi|^{\beta - 2 - k} e^{-i\xi \cdot (y - x')} \sin((s - r)|\xi|) \sin((t' - r)|\xi|)$$
$$= \int_0^{t'} dr \int_{\mathbb{R}^k} d\xi |\xi|^{\beta - 2 - k} e^{-i\xi \cdot (y - x')} \sin(r|\xi|) \sin((h + r)|\xi|),$$

where we have set $h = s - t'$. We now permute the two integrals and calculate the dr-integral explicitly. As in the proof of Lemma A.12 in [16], this gives

$$(9.17) \qquad E[\hat{v}_j(s, y)\hat{v}_j(t', x')] = (t')^{3 - \beta} \int_{\mathbb{R}^k} d\eta |\eta|^{\beta - 2 - k} e^{-i\eta \cdot u} g_0(\lambda, |\eta|),$$

where $u = (y - x')/t'$, $\lambda = (s - t')/t'$, and

$$g_0(\lambda, r) = \cos(\lambda r) - \frac{\sin(r)}{r} \cos((\lambda + 1)r).$$

Case 1 (time increments): $s_1 \ne s_2$, $y_1 = y_2 = y$. Set $\lambda_1 = (s_1 - t')/t'$, $\lambda_2 = (s_2 - t')/t'$, and $u = (y - x')/t'$. Then by (9.17),

$$(9.18) \qquad \begin{aligned} &E[(\hat{v}_j(s_1, y) - \hat{v}_j(s_2, y))\hat{v}_j(t', x')] \\ &\quad = (t')^{3 - \beta} \int_{\mathbb{R}^k} d\eta |\eta|^{\beta - 2 - k} e^{-i\eta \cdot u} (g_0(\lambda_1, |\eta|) - g_0(\lambda_2, |\eta|)). \end{aligned}$$

Because $\cos(\cdot)$ is Lipschitz, we see that

$$\left|g_0(\lambda_1, r) - g_0(\lambda_2, r)\right| \leq 4\big((|\lambda_1 - \lambda_2|r) \wedge 1\big),$$

therefore, the right-hand side of (9.18) is bounded above by $4(t')^{3-\beta}(I_1 + I_2)$, where

$$I_1 = |\lambda_1 - \lambda_2| \int_0^{|\lambda_1 - \lambda_2|^{-1}} r^{\beta-2}\, dr, \qquad I_2 = \int_{|\lambda_1 - \lambda_2|^{-1}}^{\infty} r^{\beta-3}\, dr.$$

Clearly, since $\beta > 1$,

$$I_1 = |\lambda_1 - \lambda_2| \frac{|\lambda_1 - \lambda_2|^{1-\beta}}{\beta - 1} = c|s_1 - s_2|^{2-\beta}$$

and

$$I_2 = \frac{|\lambda_1 - \lambda_2|^{2-\beta}}{2 - \beta} = c|s_1 - s_2|^{2-\beta}.$$

We conclude that

$$(9.19) \qquad E\big[(\hat{v}_j(s_1, y) - \hat{v}_j(s_2, y))\hat{v}_j(t', x')\big] \leq c|s_1 - s_2|^{2-\beta}.$$

Case 2 (spatial increments): $s_1 = s_2 = s$, $y_1 \neq y_2$. Set $\lambda = (s - t')/t'$, $u_1 = (y_1 - x')/t'$, $u_2 = (y_2 - x')/t'$. By (9.17),

$$(9.20) \quad \begin{aligned} &E\big[(\hat{v}_j(s, y_1) - \hat{v}_j(s, y_2))\hat{v}_j(t', x')\big] \\ &= (t')^{3-\beta} \int_{\mathbb{R}^k} d\eta |\eta|^{\beta-2-k}\big(e^{-i\eta \cdot u_1} - e^{-i\eta \cdot u_2}\big) g_0(\lambda, |\eta|). \end{aligned}$$

Notice that

$$\left|e^{-i\eta \cdot u_1} - e^{-i\eta \cdot u_2}\right| \leq 2\big((|u_1 - u_2||\eta|) \wedge 1\big)$$

and $g_0(\lambda, r) \leq 2$, so the right-hand side of (9.20) is bounded above by $4(t')^{3-\beta}(J_1 + J_2)$, where

$$J_1 = |u_1 - u_2| \int_0^{|u_1 - u_2|^{-1}} r^{\beta-2}\, dr, \qquad J_2 = \int_{|u_1 - u_2|^{-1}}^{\infty} r^{\beta-3}\, dr.$$

Clearly the same calculations as for I_1 and I_2 show that

$$J_1 + J_2 \leq \tilde{c}|u_1 - u_2|^{2-\beta} = c|y_1 - y_2|^{2-\beta}.$$

We conclude that

$$(9.21) \qquad E\big[(\hat{v}_j(s, y_1) - \hat{v}_j(s, y_2))\hat{v}_j(t', x')\big] \leq c|y_1 - y_2|^{2-\beta}.$$

Putting together (9.19) and (9.21) establishes (9.15). This proves Lemma 9.6. $\qquad \square$

PROOF OF THEOREM 9.1.   We have already observed (before Lemma 9.6) that $d = 2(k+1)/(2-\beta)$ is the critical dimension for hitting points. By Lemma 9.3 and the sentences that precede this lemma, for any compact box $I \subset]0, \infty[\times \mathbb{R}^k$, Assumption 2.1 is satisfied for $\mathrm{Re}(v)$, with exponents $\gamma_1 = \frac{\beta}{2-\beta} = \gamma_j$, $j = 2, \ldots, k+1$, so that $\alpha_1 = \frac{2-\beta}{2} = \alpha_j$, $j = 2, \ldots, k+1$. By Lemma 9.6 and the comments that precede this lemma, Assumption 2.4 is satisfied by $\hat{v}$ (with $\delta_j = 2 - \beta > \alpha_j$), hence by $\mathrm{Re}(v)$ by Proposition 9.2. Since $Q = \alpha_1^{-1} + k\alpha_2^{-1} = (2+2k)/(2-\beta) = d$, it follows from Theorem 2.6 that for all $z \in \mathbb{R}^Q$,

$$P\{\exists (t,x) \in I : \hat{v}(t,x) = z\} = P\{\exists (t,x) \in I : \mathrm{Re}(v(t,x)) = z\} = 0.$$

Since this holds for all compact boxes $I \subset]0, \infty[\times \mathbb{R}^k$, Theorem 9.1 is proved.
□

REFERENCES

[1] BALAN, R. M. (2012). Linear SPDEs driven by stationary random distributions. *J. Fourier Anal. Appl.* **18** 1113–1145. MR3000977

[2] BIERMÉ, H., LACAUX, C. and XIAO, Y. (2009). Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. Lond. Math. Soc.* **41** 253–273. MR2496502

[3] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory. Pure and Applied Mathematics* **29**. Academic Press, New York. MR0264757

[4] CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183. MR0420845

[5] CAMBANIS, S. and LIU, B. (1970). On harmonizable stochastic processes. *Inf. Control* **17** 183–202. MR0282401

[6] CLARKE DE LA CERDA, J. and TUDOR, C. A. (2014). Hitting times for the stochastic wave equation with fractional colored noise. *Rev. Mat. Iberoam.* **30** 685–709. MR3231213

[7] COHEN, S. and ISTAS, J. (2013). *Fractional Fields and Applications. Mathématiques & Applications* (*Berlin*) **73**. Springer, Heidelberg. MR3088856

[8] DALANG, R. C. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4** no. 6, 29 pp. MR1684157

[9] DALANG, R. C., KHOSHNEVISAN, D. and NUALART, E. (2007). Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 231–271.

[10] DALANG, R. C., KHOSHNEVISAN, D. and NUALART, E. (2009). Hitting probabilities for systems for non-linear stochastic heat equations with multiplicative noise. *Probab. Theory Related Fields* **144** 371–427. MR2496438

[11] DALANG, R. C., KHOSHNEVISAN, D. and NUALART, E. (2013). Hitting probabilities for systems of non-linear stochastic heat equations in spatial dimension $k \geq 1$. *Stoch. Partial Differ. Equ., Anal. Computat.* **1** 94–151. MR3327503

[12] DALANG, R. C., KHOSHNEVISAN, D., NUALART, E., WU, D. and XIAO, Y. (2012). Critical Brownian sheet does not have double points. *Ann. Probab.* **40** 1829–1859. MR2978539

[13] DALANG, R. C. and MUELLER, C. (2015). Multiple points of the Brownian sheet in critical dimensions. *Ann. Probab.* **43** 1577–1593. MR3353809

[14] DALANG, R. C. and NUALART, E. (2004). Potential theory for hyperbolic SPDEs. *Ann. Probab.* **32** 2099–2148. MR2073187

[15] DALANG, R. C. and SANZ-SOLÉ, M. (2010). Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli* **16** 1343–1368. MR2759182

[16] DALANG, R. C. and SANZ-SOLÉ, M. (2015). Hitting probabilities for nonlinear systems of stochastic waves. *Mem. Amer. Math. Soc.* **237** v+75. MR3401290

[17] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York. MR0058896

[18] ITÔ, K. (1954). Stationary random distributions. *Mem. Coll. Sci.*, *Univ. Kyoto, Ser.* 1: *Math* **28** 291–326.

[19] KHOSHNEVISAN, D. (2002). *Multiparameter Processes. An Introduction to Random Fields*. Springer, New York.

[20] KHOSHNEVISAN, D. (2003). Intersections of Brownian motions. *Expo. Math.* **21** 97–114. MR1978059

[21] KHOSHNEVISAN, D. and SHI, Z. (1999). Brownian sheet and capacity. *Ann. Probab.* **27** 1135–1159. MR1733143

[22] LEDOUX, M. (1996). Isoperimetry and Gaussian analysis. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1994). *Lecture Notes in Math.* **1648** 165–294. Springer, Berlin. MR1600888

[23] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces. Ergebnisse der Mathematik und Ihrer Grenzgebiete* (3) **23**. Springer, Berlin. MR1102015

[24] MARCUS, M. B. and ROSEN, J. (2006). *Markov Processes, Gaussian Processes, and Local Times. Cambridge Studies in Advanced Mathematics* **100**. Cambridge Univ. Press, Cambridge.

[25] MUELLER, C. and TRIBE, R. (2002). Hitting properties of a random string. *Electron. J. Probab.* **7**. 29 pp.

[26] MUELLER, C. and WU, Z. (2009). A connection between the stochastic heat equation and fractional Brownian motion, and a simple proof of a result of talagrand. *Electron. Commun. Probab.* **14** 55–65. MR2481666

[27] NUALART, E. and VIENS, F. (2009). The fractional stochastic heat equation on the circle: Time regularity and potential theory. *Stochastic Process. Appl.* **119** 1505–1540. MR2513117

[28] SANZ-SOLÉ, M. (2005). *Malliavin Calculus. With Applications to Stochastic Partial Differential Equations*. EPFL Press, Lausanne.

[29] TALAGRAND, M. (1995). Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23** 767–775. MR1334170

[30] TALAGRAND, M. (1998). Multiple points of trajectories of multiparameter fractional Brownian motion. *Probab. Theory Related Fields* **112** 545–563. MR1664704

[31] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *Ecole D'Eté de Probabilités de Saint-Flour XIV*. *Lect. Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085

[32] WU, D. (2011). On the solution process for a stochastic fractional partial differential equation driven by space–time white noise. *Statist. Probab. Lett.* **81** 1161–1172. MR2803759

[33] XIAO, Y. (1997). Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Theory Related Fields* **109** 129–157. MR1469923

[34] XIAO, Y. (2009). Sample path properties of anisotropic Gaussian random fields. In *A Minicourse on Stochastic Partial Differential Equations* (D. Khoshnevisan and F. Rassoul-Agha, eds.). *Lecture Notes in Math.* **1962** 145–212. Springer, Berlin. MR2508776

[35] YAGLOM, A. M. (1957). Some classes of random fields in n-dimensional space, related to stationary random processes. *Theory Probab. Appl.* **2** 273–320.

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