
Sample Path Properties of Anisotropic Gaussian Random Fields

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Summary. Anisotropic Gaussian random fields arise in probability theory and in various applications. Typical examples are fractional Brownian sheets, operator-scaling Gaussian fields with stationary increments, and the solution to the stochastic heat equation.

This paper is concerned with sample path properties of anisotropic Gaussian random fields in general. Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a Gaussian random field with values in \mathbf{R}^d and with parameters H_1, \dots, H_N . Our goal is to characterize the anisotropic nature of X in terms of its parameters explicitly.

Under some general conditions, we establish results on the modulus of continuity, small ball probabilities, fractal dimensions, hitting probabilities and local times of anisotropic Gaussian random fields. An important tool for our study is the various forms of strong local nondeterminism.

1 Introduction

Gaussian random fields have been extensively studied in probability theory and applied in a wide range of scientific areas including physics, engineering, hydrology, biology, economics and finance. Two of the most important Gaussian random fields are respectively the Brownian sheet and fractional Brownian motion.

The Brownian sheet $W = \{W(t), t \in \mathbf{R}_+^N\}$, which was first introduced by a statistician J. Kitagawa in 1951, is a centered Gaussian random field with values in \mathbf{R}^d and covariance function given by

$$\mathbb{E}[W_i(s)W_j(t)] = \delta_{ij} \prod_{k=1}^N (s_k \wedge t_k), \quad \forall s, t \in \mathbf{R}_+^N, \quad (1)$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. When $N = 1$, W is the ordinary Brownian motion in \mathbf{R}^d . For $N \geq 2$, W has independent increments over disjoint intervals in \mathbf{R}_+^N and such increments are stationary. We refer to [1; 49] for systematic accounts on properties of the Brownian sheet and to [94] and the articles in this volume for its important roles in stochastic partial differential equations.

For a fixed constant $0 < \alpha < 1$, an (N, d) -fractional Brownian motion with index α is a centered Gaussian random field $X^\alpha = \{X^\alpha(t), t \in \mathbf{R}^N\}$ with values in \mathbf{R}^d and covariance function given by

$$\mathbb{E}[X_i^\alpha(s)X_j^\alpha(t)] = \frac{1}{2}\delta_{ij}\left(|s|^{2\alpha} + |t|^{2\alpha} - |s - t|^{2\alpha}\right), \quad \forall s, t \in \mathbf{R}^N, \quad (2)$$

where $|\cdot|$ denotes the Euclidean norm in \mathbf{R}^N . The existence of X^α follows from the positive semi-definiteness of the kernel on the right hand side of (2); see [82] for a proof. When $N = 1$ and $\alpha = 1/2$, X^α is again the Brownian motion in \mathbf{R}^d ; when $N > 1$, $\alpha = 1/2$ and $d = 1$, it is the multiparameter Brownian motion introduced by P. Lévy; see [46; 82] for more historical information, probabilistic and statistical properties of fractional Brownian motion.

By using (2) one can verify that X^α is self-similar with exponent α , i.e. for every constant $c > 0$,

$$\{X^\alpha(ct), t \in \mathbf{R}^N\} \stackrel{\text{d}}{=} \{c^\alpha X^\alpha(t), t \in \mathbf{R}^N\}, \quad (3)$$

where $\stackrel{\text{d}}{=}$ means equality in finite dimensional distributions. Moreover, X^α has stationary increments in the *strong* sense; see Section 8.1 of [82]. In particular, X is isotropic in the sense that the distribution of $X(s) - X(t)$ depends only on the Euclidean distance $|s - t|$. Fractional Brownian motion is naturally related to long range dependence which makes it important for modelling phenomena with self-similarity and/or long memory properties. In the last decade the literature on statistical analysis and applications of fractional Brownian motion has grown rapidly [30].

On the other hand, many data sets from various areas such as image processing, hydrology, geostatistics and spatial statistics have anisotropic nature in the sense that they have different geometric and probabilistic characteristics along different directions, hence fractional Brownian motion is not adequate for modelling such phenomena. Many people have proposed to apply anisotropic Gaussian random fields as more realistic models [11; 18; 29].

Several classes of anisotropic Gaussian random fields have been introduced for theoretical and application purposes. For example, Kamont [47] introduced fractional Brownian sheets [see the definition in Section 2.1] and studied some of their regularity properties. Benassi et al. [10] and Bonami and Estrade [18] considered some anisotropic Gaussian random fields with stationary increments. Biermé et al. [17] constructed a large class of operator self-similar Gaussian or stable random fields with stationary increments. Anisotropic Gaussian random fields also arise naturally in stochastic partial

differential equations [20; 73; 74; 75]; and in studying the most visited sites of symmetric Markov processes [35]. Hence it is of importance in both theory and applications to investigate the probabilistic and statistical properties of anisotropic random fields.

This paper is concerned with sample path properties of anisotropic Gaussian random fields in general. From the recent works on fractional Brownian sheets [see Section 2.1 for a list of references] it is known that the behavior of anisotropic Gaussian random fields may differ significantly from those of the Brownian sheet and fractional Brownian motion. Our objective is to gather and develop some general methods for studying the analytic and geometric properties of anisotropic Gaussian fields. In particular our results are applicable to the solutions of stochastic partial differential equations including the stochastic heat and wave equations. In a similar vein, Pitt and Robeva [78], Robeva and Pitt [79], Balan and Kim [8] have proposed to study the Markov properties of (anisotropic) Gaussian random fields and the solutions to the stochastic heat equations.

The rest of this paper is organized as follows. Section 2 contains definitions and basic properties of several classes of anisotropic Gaussian random fields including fractional Brownian sheets, Gaussian random fields with stationary increments and solutions to stochastic partial differential equations. We also provide the general conditions [i.e., Conditions (C1), (C2), (C3) and (C3')] for the Gaussian random fields that will be studied in this paper.

An important technical tool in this paper is the properties of strong local nondeterminism for anisotropic Gaussian random fields, extending the concept of *local nondeterminism* first introduced by Berman [14] for Gaussian processes. In Section 3, we recall the recent result of Wu and Xiao [97] on the property of sectorial local nondeterminism for fractional Brownian sheets; and we prove a sufficient condition for an anisotropic Gaussian field with stationary increments to be strongly locally nondeterministic (with respect to an appropriate metric).

Section 4 is concerned with analytic and asymptotic properties of the sample functions of anisotropic Gaussian fields. We summarize three methods for deriving a sharp modulus of continuity for any anisotropic Gaussian random field satisfying Condition (C1). The first method is to use an extension, due to Arnold and Imkeller [2], Funaki, Kikuchi and Potthoff [39], Dalang, Khoshnevisan and Nualart [22], of the powerful Garsia-Rodemich-Rumsey continuity lemma; the second is the “minorizing metric” method of Kwapień and Rysiński [60]; and the third is based on the Gaussian isoperimetric inequality. While the first two methods have wider applicability, the third method produces more precise results for Gaussian random fields.

Section 5 provides an application of strong local nondeterminism in studying small ball probabilities of anisotropic Gaussian fields.

In Section 6, we consider the Hausdorff and packing dimensions of the range $X([0, 1]^N) = \{X(t) : t \in [0, 1]^N\}$ and graph $\text{Gr}X([0, 1]^N) = \{(t, X(t)) : t \in [0, 1]^N\}$ of X . Due to anisotropy, these results are different from the

corresponding results for fractional Brownian motion and the Brownian sheet. We also establish an explicit formula for the Hausdorff dimension of the image $X(E)$ in terms of the generalized Hausdorff dimension of E (with respect to an appropriate metric) and the Hurst index H . Moreover, when $H = (\alpha, \dots, \alpha) \in (0, 1)^N$, we prove the following uniform Hausdorff dimension result for the images of X : If $N \leq \alpha d$, then with probability one,

$$\dim_{\mathcal{H}} X(E) = \frac{1}{\alpha} \dim_{\mathcal{H}} E \quad \text{for all Borel sets } E \subseteq (0, \infty)^N. \quad (4)$$

This extends the previous results of [51; 70; 72] for fractional Brownian motion and the Brownian sheet, respectively, and is another application of the strong local nondeterminism.

In Section 7, we determine the Hausdorff and packing dimensions of the level sets, and establish estimates on the hitting probabilities of Gaussian random fields X satisfying Conditions (C1) and (C2).

In Section 8, we study the existence and joint continuity of local times of anisotropic Gaussian random fields under Conditions (C3) and (C3'). Moreover, we discuss local and uniform Hölder conditions of the local times in the set variable and show their applications in evaluating the exact Hausdorff measure of the level sets of X .

We end the Introduction with some notation. Throughout this paper, the underlying parameter space is \mathbf{R}^N or $\mathbf{R}_+^N = [0, \infty)^N$. We use $|\cdot|$ to denote the Euclidean norm in \mathbf{R}^N . The inner product and Lebesgue measure in \mathbf{R}^N are denoted by $\langle \cdot, \cdot \rangle$ and λ_N , respectively. A typical parameter, $t \in \mathbf{R}^N$ is written as $t = (t_1, \dots, t_N)$, or as $\langle c \rangle$ if $t_1 = \dots = t_N = c$. For any $s, t \in \mathbf{R}^N$ such that $s_j < t_j$ ($j = 1, \dots, N$), $[s, t] = \prod_{j=1}^N [s_j, t_j]$ is called a closed interval (or a rectangle). We will let \mathcal{A} denote the class of all closed intervals in \mathbf{R}^N . For two functions f and g , the notation $f(t) \asymp g(t)$ for $t \in T$ means that the function $f(t)/g(t)$ is bounded from below and above by positive constants that do not depend on $t \in T$.

We will use c to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \dots$.

2 Examples and General Assumptions

In this section, we give some important examples of anisotropic Gaussian random fields, among them, fractional Brownian sheets are the most studied. We will show that the methods for studying fractional Brownian sheets can be modified to investigate sample path properties of anisotropic Gaussian random fields in general. In §2.4, we provide the general conditions for the Gaussian random fields that will be studied in this paper.

Even though anisotropic random fields generally do not satisfy the ordinary self-similarity (3), they may have certain *operator-scaling* properties. Following the terminology of Biermé et al. [17], we say that a random field $X = \{X(t), t \in \mathbf{R}^N\}$ is *operator-self-similar* [or *operator-scaling*] in the *time variable* if there exist a linear operator A on \mathbf{R}^N with positive real parts of the eigenvalues and some constant $\beta > 0$ such that

$$\{X(c^A t), t \in \mathbf{R}^N\} \stackrel{d}{=} \{c^\beta X(t), t \in \mathbf{R}^N\} \quad \forall c > 0. \tag{5}$$

In the above, c^A is the linear operator defined by $c^A = \sum_{n=0}^\infty (\ln c)^n A^n / n!$. The linear operator A is called a self-similarity exponent [which may not be unique].

There is also a notion of operator-self-similarity in the *space variable* [65; 98]. We will not discuss this topic in this paper.

2.1 Fractional Brownian Sheets

Fractional Brownian sheets were first introduced by Kamont [47], who also studied some of their regularity properties. For $H = (H_1, \dots, H_N) \in (0, 1)^N$, an $(N, 1)$ -fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbf{R}^N\}$ with Hurst index H is a real-valued, centered Gaussian random field with covariance function given by

$$E[B_0^H(s)B_0^H(t)] = \prod_{j=1}^N \frac{1}{2} \left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j} \right), \quad s, t \in \mathbf{R}^N. \tag{6}$$

It follows from (6) that $B_0^H(t) = 0$ a.s. for every $t \in \mathbf{R}^N$ with at least one zero coordinate.

Note that if $N = 1$, then B_0^H is a fractional Brownian motion in \mathbf{R} with Hurst index $H_1 \in (0, 1)$; if $N > 1$ and $H = \langle 1/2 \rangle$, then B^H is the Brownian sheet in \mathbf{R} . Hence B_0^H can be regarded as a natural generalization of one parameter fractional Brownian motion as well as a generalization of the Brownian sheet.

It follows from (6) that B_0^H has the following operator-scaling property: For all constants $c > 0$,

$$\{B_0^H(c^A t), t \in \mathbf{R}^N\} \stackrel{d}{=} \{c^N B_0^H(t), t \in \mathbf{R}^N\}, \tag{7}$$

where $A = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/H_i$ for all $1 \leq i \leq N$ and $a_{ij} = 0$ if $i \neq j$. Thus, B_0^H is operator-self-similar with exponent A and $\beta = N$.

The covariance structure of B_0^H is more complicated than those of fractional Brownian motion and the Brownian sheet. The following stochastic integral representations are useful. They were established by Ayache et al. [4]

and Herbin [43], respectively, and can be verified by checking the covariance functions.

• **Moving average representation:**

$$B_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} g(t, s) W(ds), \quad (8)$$

where $W = \{W(s), s \in \mathbf{R}^N\}$ is a standard real-valued Brownian sheet and

$$g(t, s) = \prod_{j=1}^N \left[((t_j - s_j)_+)^{H_j-1/2} - ((-s_j)_+)^{H_j-1/2} \right]$$

with $s_+ = \max\{s, 0\}$, and where $\kappa_H > 0$ is a normalization constant.

To give a harmonizable representation for B_0^H , let us recall briefly the definition of a complex-valued Gaussian measure. Let (E, \mathcal{E}, Δ) be a measure space and let $\mathcal{A} = \{A \in \mathcal{E} : \Delta(A) < \infty\}$. We say that $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian measure on (E, \mathcal{E}, Δ) if $\{\widetilde{\mathcal{M}}(A), A \in \mathcal{A}\}$ is a centered complex-valued Gaussian process satisfying

$$\mathbb{E} \left(\widetilde{\mathcal{M}}(A) \overline{\widetilde{\mathcal{M}}(B)} \right) = \Delta(A \cap B) \quad \text{and} \quad \widetilde{\mathcal{M}}(-A) = \overline{\widetilde{\mathcal{M}}(A)}, \quad (9)$$

for all $A, B \in \mathcal{A}$. The measure Δ is called the *control measure* of $\widetilde{\mathcal{M}}$. For any complex valued function $\widetilde{f} \in L^2(E, \mathcal{E}, \Delta)$, the stochastic integral $\int_E \widetilde{f}(\xi) \widetilde{\mathcal{M}}(d\xi)$ can be defined; see, e.g., Section 7.2.2 of [82]. With this notion, we give the following:

• **Harmonizable representation:**

$$B_0^H(t) = K_H^{-1} \int_{\mathbf{R}^N} \psi_t(\lambda) \widetilde{W}(d\lambda), \quad (10)$$

where \widetilde{W} is a centered complex-valued Gaussian random measure in \mathbf{R}^N with Lebesgue control measure and

$$\psi_t(\lambda) = \prod_{j=1}^N \frac{e^{it_j \lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}}, \quad (11)$$

where $K_H > 0$ is a constant. Recently, Wang [95] gives another stochastic integral representation for B_0^H .

Let B_1^H, \dots, B_d^H be d independent copies of B_0^H . Then the (N, d) -fractional Brownian sheet with Hurst index $H = (H_1, \dots, H_N)$ is the Gaussian random field $B^H = \{B^H(t) : t \in \mathbf{R}^N\}$ with values in \mathbf{R}^d defined by

$$B^H(t) = (B_1^H(t), \dots, B_d^H(t)), \quad t \in \mathbf{R}^N. \quad (12)$$

Several authors have studied various properties of fractional Brownian sheets. For example, Ayache et al. [4] provided the moving average representation (8) for B_0^H and studied its sample path continuity as well as its continuity in H . Dunker [32], Mason and Shi [66], Belinski and Linde [9], Kühn and Linde [59] studied the small ball probabilities of B_0^H . Mason and Shi [66] also computed the Hausdorff dimension of some exceptional sets related to the oscillation of the sample paths of B_0^H . Ayache and Taqqu [5] derived an optimal wavelet series expansion for fractional Brownian sheet B_0^H ; see also [33; 59] for other optimal series expansions for B_0^H . Øksendal and Zhang [75], and Hu, Øksendal and Zhang [45] studied stochastic partial differential equations driven by fractional Brownian sheets.

For fractal properties, Kamont [47] and Ayache [3] studied the box and Hausdorff dimensions of the graph set of an $(N, 1)$ -fractional Brownian sheet. Ayache and Xiao [7] investigated the uniform and local asymptotic properties of B^H by using wavelet methods, and determined the Hausdorff dimensions of the image $B^H([0, 1]^N)$, the graph $\text{Gr}B^H([0, 1]^N)$ and the level set $L_x = \{t \in (0, \infty)^N : B^H(t) = x\}$, where $x \in \mathbf{R}^d$. Further results on the geometric and Fourier analytic properties of the images of B^H can be found in Wu and Xio [97].

Xiao and Zhang [110] studied the existence of local times of an (N, d) -fractional Brownian sheet B^H and proved a sufficient condition for the joint continuity of the local times. Ayache, Wu and Xiao [6] established the joint continuity of the local times under the optimal condition and studied the local and uniform Hölder conditions for the maximum local times. Related to the above results, we mention that Tudor and Xiao [93] have obtained results on Hausdorff dimensions of the sample paths, local times and their chaos expansion for (N, d) -bifractional Brownian sheets.

2.2 Anisotropic Gaussian Random Fields with Stationary Increments

Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a real-valued, centered Gaussian random field with $X(0) = 0$. We assume that X has stationary increments and continuous covariance function $R(s, t) = \mathbb{E}[X(s)X(t)]$. According to Yaglom [111], $R(s, t)$ can be represented as

$$R(s, t) = \int_{\mathbf{R}^N} (e^{i\langle s, \xi \rangle} - 1)(e^{-i\langle t, \xi \rangle} - 1)\Delta(d\xi) + \langle s, \Theta t \rangle, \tag{13}$$

where $\langle x, y \rangle$ is the ordinary inner product in \mathbf{R}^N , Θ is an $N \times N$ non-negative definite matrix and $\Delta(d\xi)$ is a nonnegative symmetric measure on $\mathbf{R}^N \setminus \{0\}$ that satisfies

$$\int_{\mathbf{R}^N} \frac{|\xi|^2}{1 + |\xi|^2} \Delta(d\xi) < \infty. \tag{14}$$

The measure Δ and its density (if it exists) $f(\xi)$ are called the *spectral measure* and *spectral density* of X , respectively.

It follows from (13) that X has the following stochastic integral representation:

$$X(t) = \int_{\mathbf{R}^N} (e^{i\langle t, \xi \rangle} - 1) \widetilde{\mathcal{M}}(d\xi) + \langle \mathbf{Y}, t \rangle, \quad (15)$$

where \mathbf{Y} is an N -dimensional Gaussian random vector with mean 0 and covariance matrix Θ , and where $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian random measure in \mathbf{R}^N with control measure Δ , which is independent of \mathbf{Y} . Since the linear term $\langle \mathbf{Y}, t \rangle$ in (15) will not have any effect on the problems considered in this paper, we will from now on assume $\mathbf{Y} = 0$. This is equivalent to assuming $\Theta = 0$ in (13). Consequently, we have

$$\sigma^2(h) = \mathbb{E}[(X(t+h) - X(t))^2] = 2 \int_{\mathbf{R}^N} (1 - \cos \langle h, \xi \rangle) \Delta(d\xi). \quad (16)$$

It is important to note that $\sigma^2(h)$ is a negative definite function [12] and, by the Lévy-Khintchine formula, can be viewed as the characteristic exponent of a symmetric infinitely divisible distribution.

If the function $\sigma^2(h)$ depends only on $|h|$, then X is called an *isotropic* random field. We say that a Gaussian random field X is *approximately isotropic* if $\sigma^2(h) \asymp \phi(|h|)$ in a neighborhood of $h = 0$ for some nonnegative function ϕ . Sample path properties of such Gaussian random fields have been studied widely. See [83; 102; 108] and the references therein for more information. The results in [7; 97] on fractional Brownian sheets indicate that the properties of anisotropic Gaussian random fields can be very different and often more difficult to be established.

Many Gaussian random fields can be constructed by choosing the spectral measures appropriately. For example, if we consider the spectral density

$$f(\xi) = \frac{1}{\left(\sum_{j=1}^N |\xi_j|^{H_j}\right)^{2+Q}} \quad \forall \xi \in \mathbf{R}^N \setminus \{0\}, \quad (17)$$

where the constants $H_j \in (0, 1)$ for $j = 1, \dots, N$ and $Q = \sum_{j=1}^N H_j^{-1}$, then the corresponding Gaussian random field X has stationary increments and is operator-self-similar with exponent $A = (a_{ij})$, where $a_{ii} = H_i^{-1}$ and $a_{ij} = 0$ if $i \neq j$ and $\beta = 1$. This Gaussian random field is similar to that in Example 3 of [18].

The following class of Gaussian random fields constructed by Biermé, Meerschaert and Scheffler [17, Section 4] is more general.

Theorem 2.1. *Let A be a real $N \times N$ matrix with the real parts of the eigenvalues $1 < a_1 \leq a_2 \leq \dots \leq a_N$ and let $Q = \text{trace}(A)$. If $\psi : \mathbf{R}^N \rightarrow [0, \infty)$ is a continuous, A' -homogeneous function [i.e., $\psi(c^{A'} \xi) = c\psi(\xi)$ for all $c > 0$ and $\xi \in \mathbf{R}^N$. Here A' denotes the transpose of A] such that $\psi(\xi) > 0$ for $\xi \neq 0$. Then the Gaussian random field*

$$X_\psi(t) = \text{Re} \int_{\mathbf{R}^N} (e^{i\langle t, \xi \rangle} - 1) \frac{\widetilde{\mathcal{W}}(d\xi)}{\psi(\xi)^{1+Q/2}}, \quad x \in \mathbf{R}^N, \quad (18)$$

where $\widetilde{\mathcal{W}}$ is a centered complex-valued Gaussian random measure in \mathbf{R}^N with Lebesgue control measure, has stationary increments and is operator-self-similar in the sense of (5) with exponent A and $\beta = 1$.

Compared with (15), we see that the spectral measure of X_ψ is $\Delta(d\xi) = \psi(\xi)^{-(2+Q)} d\xi$. As the results of this paper will suggest, the sample functions of X_ψ share many properties with fractional Brownian sheets and many of them can be described in terms of the real parts of the eigenvalues of A . See [15] for more details.

2.3 Solutions to Stochastic Partial Differential Equations

Gaussian random fields arise naturally as solutions to stochastic partial differential equations. In the following we list as examples the solutions to the stochastic heat equation and stochastic wave equation, and discuss possible ways to study their sample path properties using general methods for Gaussian random fields. We refer to [20; 21; 22; 23; 24; 27; 28; 73; 94], and the articles in this volume for more information.

2.3.1 The Stochastic Heat Equation

Funaki’s model for random string in \mathbf{R} is specified by the following stochastic heat equation:

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + \dot{W}, \tag{19}$$

where $\dot{W}(x, t)$ is an \mathbf{R} -valued space-time white noise, which is assumed to be adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, where \mathcal{F} is complete and the filtration $\{\mathcal{F}_t, t \geq 0\}$ is right continuous [38; 73].

Recall from [73] that a solution of (19) is defined as an \mathcal{F}_t -adapted, continuous random field $\{u(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ with values in \mathbf{R} satisfying the following properties:

- (i) $u(0, \cdot) \in \mathcal{E}_{\text{exp}}$ almost surely and is adapted to \mathcal{F}_0 , where $\mathcal{E}_{\text{exp}} = \cup_{\lambda > 0} \mathcal{E}_\lambda$ and

$$\mathcal{E}_\lambda = \left\{ f \in C(\mathbf{R}) : |f(x)| e^{-\lambda|x|} \rightarrow 0 \text{ as } |x| \rightarrow \infty \right\};$$
- (ii) For every $t > 0$, there exists $\lambda > 0$ such that $u(s, \cdot) \in \mathcal{E}_\lambda$ for all $s \leq t$, almost surely;
- (iii) For every $t > 0$ and $x \in \mathbf{R}$, the following Green’s function representation holds

$$u(t, x) = \int_{\mathbf{R}} G(t, x - y) u(0, y) dy + \int_0^t \int_{\mathbf{R}} G(t - r, x - y) W(dy dr), \tag{20}$$

where $G(t, x) = (4\pi t)^{-1/2} \exp\{-x^2/(4t)\}$ is the fundamental solution of the heat equation.

We call each solution $\{u(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ of (19) a random string process with values in \mathbf{R} , or simply a random string as in [73]. Note that, in general, a random string may not be Gaussian, a powerful step in the proofs of [73] is to reduce the problems about a general random string process to those of the stationary pinned string $U_0 = \{U_0(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$, obtained by taking the initial function $u(0, \cdot)$ in (20) to be defined by

$$u(0, x) = \int_0^\infty \int_{\mathbf{R}} (G(r, x - z) - G(r, z)) \widehat{W}(dz dr), \quad (21)$$

where \widehat{W} is a space-time white noise independent of the white noise \dot{W} . Consequently, the stationary pinned string is a continuous version of the following Gaussian field

$$\begin{aligned} U_0(t, x) &= \int_0^\infty \int_{\mathbf{R}} (G(t + r, x - z) - G(t + r, z)) \widehat{W}(dz dr) \\ &\quad + \int_0^t \int_{\mathbf{R}} G(r, x - z) W(dz dr), \end{aligned} \quad (22)$$

Mueller and Tribe [73] proved that the Gaussian field $U_0 = \{U_0(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ has stationary increments and satisfies the Conditions (C1) and (C2) in Section 2.4. Let U_1, \dots, U_d be d independent copies of U_0 , and consider the Gaussian random field $U = \{U(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ with values in \mathbf{R}^d defined by $U(t, x) = (U_1(t, x), \dots, U_d(t, x))$. Mueller and Tribe [73] found necessary and sufficient conditions [in terms of the dimension d] for U to hit points or to have double points of various types. They also studied the question of recurrence and transience for $\{U(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$. Continuing the work of Mueller and Tribe [73], Wu and Xiao [96] studied the fractal properties of various random sets generated by the random string processes. Further results on hitting probabilities of non-linear stochastic heat equations can be found in [22; 23].

On the other hand, Robeva and Pitt [79, Proposition 3] showed that the Gaussian random field

$$u_0(t, x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{e^{i(\xi_1 t + \xi_2 x)} - 1}{i\xi_1 + \xi_2^2} \widetilde{\mathcal{W}}(d\xi_1 d\xi_2), \quad \forall t \in \mathbf{R}_+, x \in \mathbf{R} \quad (23)$$

is another solution to (19) satisfying $u_0(0, 0) = 0$. Here, $\widetilde{\mathcal{W}}$ is a centered complex Gaussian random measure in \mathbf{R}^2 with Lebesgue control measure. This Gaussian random field has stationary increments with spectral density

$$f(\xi) = \frac{1}{\xi_1^2 + \xi_2^4}. \quad (24)$$

This density function is comparable to (17) with $H_1 = 1/4$, $H_2 = 1/2$ and $Q = 6$. Hence, it follows from Theorem 3.2 that the Gaussian field $\{u_0(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ satisfies the Conditions (C1) and (C3') in §2.4.

If we define a $(2, d)$ -Gaussian random field $\{u(t, x), t \in \mathbf{R}_+, x \in \mathbf{R}\}$ by $u(t, x) = (u_1(t, x), \dots, u_d(t, x))$, where u_1, \dots, u_d are independent copies of u_0 , then many of its sample path properties follow from the results in later sections of this paper.

If $x \in \mathbf{R}^N$ and $N \geq 2$, the stochastic heat equation (19) has no process solution [the solution is a random Schwartz distribution]. It might be helpful to remark that our random field notation is different from that in the references on s.p.d.e.'s: now the parameter $(t, x) \in \mathbf{R}^{N+1}$ and \mathbf{R}^d is reserved for the state space of random fields.

The approach of Dalang [20] is to replace \dot{W} by a Gaussian noise \dot{F} which is white in time and has spatial covariance induced by a kernel f [not to be confused with the spectral density above], which is the Fourier transform of a tempered measure μ in \mathbf{R}^N . The covariance of F is of the form

$$E(F(dt dx)F(ds dy)) = \delta(t - s)f(x - y), \tag{25}$$

where $\delta(\cdot)$ is the Dirac delta function. The case $f(r) = \delta(r)$ would correspond to the case of space-time white noise. More precisely, let $\mathcal{D}(\mathbf{R}^{N+1})$ be the topological space of functions $\phi \in C_0^\infty(\mathbf{R}^{N+1})$ with the topology that corresponds to the following notion of convergence: $\phi_n \rightarrow \phi$ if and only if the following two conditions hold:

- (i) There exists a compact set $K \subseteq \mathbf{R}^{N+1}$ such that $\text{supp}(\phi_n - \phi) \subseteq K$ for all $n \geq 1$, and
- (ii) $\lim_{n \rightarrow \infty} D^\alpha \phi_n = D^\alpha \phi$ uniformly on K for every multi-index α .

Let $F = \{F(\phi), \phi \in \mathcal{D}(\mathbf{R}^{N+1})\}$ be an $L^2(\Omega, \mathcal{F}, P)$ -valued, centered Gaussian process with covariance of the form $(\phi, \psi) \mapsto E(F(\phi)F(\psi)) = J(\phi, \psi)$, where

$$J(\phi, \psi) = \int_{\mathbf{R}_+} dt \int_{\mathbf{R}^N} dx \int_{\mathbf{R}^N} \phi(t, x)f(x - y)\psi(t, y) dy. \tag{26}$$

As shown by Dalang [20], $\phi \mapsto F(\phi)$ can be extended to a worthy martingale measure $(t, A) \mapsto M_t(A)$ in the sense of Walsh [94, pp. 289–290], with covariance measure

$$\begin{aligned} Q([0, t] \times A \times B) &= \langle M(A); M(B) \rangle_t \\ &= t \int_{\mathbf{R}^N} dx \int_{\mathbf{R}^N} \mathbf{1}_A(x)f(x - y)\mathbf{1}_B(y) dy, \end{aligned} \tag{27}$$

and dominating measure $K \equiv Q$ such that

$$F(\phi) = \int_{\mathbf{R}_+} \int_{\mathbf{R}^N} \phi(t, x) M(dt dx), \quad \forall \phi \in \mathcal{D}(\mathbf{R}^{N+1}). \tag{28}$$

Moreover, Dalang [20] constructed generalized stochastic integrals with respect to the martingale measure M .

Now we consider the stochastic heat equation with vanishing initial conditions, written formally as

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + \dot{F}, \quad \forall (t, x) \in (0, T) \times \mathbf{R}^N \quad (29)$$

and $u(0, \cdot) \equiv 0$. Here $T > 0$ is any fixed constant and \dot{F} is the Gaussian noise defined above.

Dalang [20] proved that (29) has a process solution if and only if

$$\int_{\mathbf{R}^N} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty. \quad (30)$$

Under this condition, the mean zero Gaussian field $u = \{u(t, x); t \in [0, T], x \in \mathbf{R}^N\}$ defined by

$$u(t, x) = \int_0^T \int_{\mathbf{R}^N} G(t-s, x-y) M(ds dy) \quad (31)$$

is the *process solution* of the stochastic heat equation (29) with vanishing initial condition. In the above, $G(r, x) = (4\pi r)^{-N/2} \exp(-|x|^2/(4r))$ ($r > 0, x \in \mathbf{R}^N$) is the fundamental solution of the heat equation.

Many interesting examples can be constructed by choosing $\mu(d\xi)$ suitably [8; 20]. As we mentioned in the Introduction, [8; 79] studied the Markov property of the solution of stochastic heat equation (29). In view of the results in this paper, it would be interesting to see when the solutions of (29) satisfy Conditions (C3) or (C3') in §2.4.

2.3.2 The Stochastic Wave Equation

The stochastic wave equation in one spatial dimension [i.e., $N = 1$]

$$\frac{\partial^2 u(t, x)}{\partial t^2} - \frac{\partial^2 u(t, x)}{\partial x^2} = \dot{W}(t, x), \quad t > 0, x \in \mathbf{R}, \quad (32)$$

driven by the white noise was considered by Walsh [94] and many other authors [21; 24]. In spatial dimension two or higher, however, the stochastic wave equation driven by the white noise has no solution in the space of real valued measurable processes [94].

For $N = 2$, Dalang and Frangos [21] considered the stochastic wave equation driven by the Gaussian noise \dot{F} with covariance (25):

$$\begin{cases} \frac{\partial^2 u(t, x)}{\partial t^2} = \Delta u(t, x) + \dot{F}, \\ u(0, x) = 0, \\ \frac{\partial u}{\partial t}(0, x) = 0, \end{cases} \quad \forall (t, x) \in (0, \infty) \times \mathbf{R}^2. \quad (33)$$

They proved that (33) has a process solution $u = \{u(t, x) : t \geq 0, x \in \mathbf{R}^2\}$ if and only if

$$\int_{0+} f(r) r \log\left(\frac{1}{r}\right) dr < \infty, \tag{34}$$

where f is the kernel in (25). Under the latter condition, $u = \{u(t, x) : t \geq 0, x \in \mathbf{R}^2\}$ can be represented as

$$u(t, x) = \int_0^t \int_{\mathbf{R}^2} S(t-s, x-y) M(ds dy), \tag{35}$$

where $S(t, x) = (2\pi)^{-1}(t^2 - |x|^2)^{-1/2} \mathbf{1}_{\{|x| < t\}}$. Sample path regularity of the solution $\{u(t, x) : t \geq 0, x \in \mathbf{R}^2\}$ has been investigated by Dalang and Frangos [21] and Dalang and Sanz-Solé [27].

For the stochastic wave equation with spatial dimension three, we refer to [24; 28] for information on the existence of a process solution and its sample path regularities. It seems that, in all the cases considered so far, the questions on fractal properties, existence and regularity of the local times of the solutions remain to be investigated.

2.4 General Assumptions

Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a Gaussian random field in \mathbf{R}^d defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ by

$$X(t) = (X_1(t), \dots, X_d(t)), \quad t \in \mathbf{R}^N, \tag{36}$$

where X_1, \dots, X_d are independent copies of X_0 . We assume that X_0 is a mean zero Gaussian random field with $X_0(0) = 0$ a.s.

Let $(H_1, \dots, H_N) \in (0, 1)^N$ be a fixed vector. In order to study anisotropic Gaussian fields, we have found the following metric ρ on \mathbf{R}^N is often more convenient than the Euclidean metric:

$$\rho(s, t) = \sum_{j=1}^N |s_j - t_j|^{H_j}, \quad \forall s, t \in \mathbf{R}^N. \tag{37}$$

For any $r > 0$ and $t \in \mathbf{R}^N$, we denote by $B_\rho(t, r) = \{s \in \mathbf{R}^N : \rho(s, t) \leq r\}$ the closed (or open) ball in the metric ρ .

Let $I \in \mathcal{A}$ be a fixed closed interval, and we will consider various sample path properties of $X(t)$ when $t \in I$. For simplicity we will mostly assume $I = [\varepsilon, 1]^N$, where $\varepsilon \in (0, 1)$ is fixed. Typically, the assumption for I to be away from the axis is only needed for Gaussian fields similar to fractional Brownian sheets. Even in these later cases, many results such as those on Hausdorff and packing dimensions remain to be true for $I = [0, 1]^N$.

Many sample path properties of X can be determined by the following function:

$$\sigma^2(s, t) = \mathbf{E} \left(|X_0(s) - X_0(t)|^2 \right), \quad \forall s, t \in \mathbf{R}^N. \tag{38}$$

In this paper, we will make use of the following general conditions on X_0 :

(C1) There exist positive constants $c_{2,1}, \dots, c_{2,4}$ such that

$$c_{2,1} \leq \sigma^2(t) := \sigma^2(0, T) \leq c_{2,2} \quad \forall t \in I \quad (39)$$

and

$$c_{2,3} \sum_{j=1}^N |s_j - t_j|^{2H_j} \leq \sigma^2(s, t) \leq c_{2,4} \sum_{j=1}^N |s_j - t_j|^{2H_j}, \quad (40)$$

for all $s, t \in I$. It may be helpful to note that (40) is in terms of $\rho(s, t)^2$.

(C2) There exists a constant $c_{2,5} > 0$ such that for all $s, t \in I$,

$$\text{Var}(X_0(t) \mid X_0(s)) \geq c_{2,5} \rho(s, t)^2.$$

Here and in the sequel, $\text{Var}(Y \mid Z)$ denotes the conditional variance of Y given Z .

(C3) There exists a constant $c_{2,6} > 0$ such that for all integers $n \geq 1$ and all $u, t^1, \dots, t^n \in I$,

$$\text{Var}(X_0(u) \mid X_0(t^1), \dots, X_0(t^n)) \geq c_{2,6} \sum_{j=1}^N \min_{0 \leq k \leq n} |u_j - t_j^k|^{2H_j},$$

where $t_j^0 = 0$ for every $j = 1, \dots, N$.

(C3') There exists a constant $c_{2,7} > 0$ such that for all integers $n \geq 1$ and all $u, t^1, \dots, t^n \in I$,

$$\text{Var}(X_0(u) \mid X_0(t^1), \dots, X_0(t^n)) \geq c_{2,7} \min_{0 \leq k \leq n} \rho(u, t^k)^2,$$

where $t^0 = 0$.

Remark 2.2. The following are some remarks about the above conditions.

- Conditions (C1)–(C3) can be significantly weakened and/or modified in various parts of the paper to obtain more general results. The present formulation of these conditions has the advantage that it is more convenient and produces cleaner results.
- Condition (39) assumes that X is non-degenerate on I . If (40) holds for $s = 0$ as well, then (39) is true automatically.
- Under condition (C1), X has a version which has continuous sample functions on I almost surely. Henceforth we will assume without loss of generality that the Gaussian random field X has continuous sample paths.
- Conditions (C1) and (C2) are related. It is easy to see that (C1) implies that $\text{Var}(X_0(t) \mid X_0(s)) \leq c_{2,4} \sum_{j=1}^N |s_j - t_j|^{2H_j}$ for all $s, t \in I$ and, on the other hand, (C2) implies $\sigma^2(s, t) \geq c_{2,5} \sum_{j=1}^N |s_j - t_j|^{2H_j}$. Moreover, if the function $\sigma(0, t)$ satisfies certain smoothness condition, say, it has continuous first order derivatives on I , then one can show that (C1) implies

(C2) by using the following fact [which can be easily verified]: If (U, V) is a Gaussian vector, then

$$\text{Var}(U | V) = \frac{(\rho_{U,V}^2 - (\sigma_U - \sigma_V)^2)((\sigma_U + \sigma_V)^2 - \rho_{U,V}^2)}{4\sigma_V^2}, \quad (41)$$

where $\rho_{U,V}^2 = E[(U - V)^2]$, $\sigma_U^2 = E(U^2)$ and $\sigma_V^2 = E(V^2)$.

- Pitt [77] proved that fractional Brownian motion X^α satisfies Condition (C3') for all $I \in \mathcal{A}$ with $H = \langle \alpha \rangle$; Khoshnevisan and Xiao [55] proved that the Brownian sheet satisfies the property (C3) with $H = \langle 1/2 \rangle$ for all $I \in \mathcal{A}$ which are away from the boundary of \mathbf{R}_+^N . It has been proved in [7; 97] that, for every $\varepsilon \in (0, 1)$, fractional Brownian sheets satisfy Conditions (C1), (C2) and (C3) for all $I \subseteq [\varepsilon, \infty)^N$.
- Let X be a Gaussian random field with stationary increments and spectral density comparable to (17). Then one can verify that X satisfies Condition (C1). In the next section, we will prove that X satisfies Condition (C3') [thus it also satisfies (C2)]. Therefore, all the results in this paper are applicable to such Gaussian random fields.
- Note that Condition (C3') implies (C3). It can be verified that the converse does not even hold for the Brownian sheet [this is an exercise]. Roughly speaking, when $H = \langle \alpha \rangle$, the behavior of a Gaussian random field X satisfying conditions (C1) and (C3') is comparable to that of a fractional Brownian motion of index α ; while the behavior of a Gaussian random field X satisfying conditions (C1) and (C3) [but not (C3')] is comparable to that of a fractional Brownian sheet. Hence, in analogy to the terminology respectively for fractional Brownian motion and the Brownian sheet, Condition (C3') will be called *the strong local nondeterminism* [in metric ρ] and Condition (C3) will be called *the sectorial local nondeterminism*.
- It is well-known that there is a close relation between Gaussian processes and operators in Hilbert spaces [62]. Recently Linde [64] has extended the notion of strong local nondeterminism to a linear operator $u : \mathcal{H} \rightarrow C(T)$, where \mathcal{H} is a real Hilbert space and $C(T)$ is the Banach space of continuous functions on the compact metric space T , and applied this property to derive a lower bound for the entropy number of u . As examples, Linde [64] showed that the integral operators related to fractional Brownian motion and fractional Brownian sheets are strongly locally nondeterministic in his sense. Following this line, it would be interesting to further study the properties of strong local nondeterminism analogous to (C3) and (C3') for linear operators related to anisotropic Gaussian random fields such as the solutions to the stochastic heat and wave equations.

3 Properties of Strong Local Nondeterminism

One of the main difficulties in studying sample path properties of anisotropic Gaussian random fields such as fractional Brownian sheets is the complexity of their dependence structure. For example, unlike fractional Brownian motion which is locally nondeterministic [77] or the Brownian sheet which has independent increments, a fractional Brownian sheet has neither of these properties. The same is true for anisotropic Gaussian random fields in general. The main technical tool which we will apply to study anisotropic Gaussian random fields is the properties of strong local nondeterminism [SLND] and sectorial local nondeterminism.

Recall that the concept of local nondeterminism was first introduced by Berman [14] to unify and extend his methods for studying local times of real-valued Gaussian processes, and then extended by Pitt [77] to Gaussian random fields. The notion of strong local nondeterminism was later developed to investigate the regularity of local times, small ball probabilities and other sample path properties of Gaussian processes and Gaussian random fields. We refer to [107; 108] for more information on the history and applications of the properties of local nondeterminism.

For Gaussian random fields, the aforementioned properties of local nondeterminism can only be satisfied by those with approximate isotropy. It is well-known that the Brownian sheet does not satisfy the properties of local nondeterminism in the senses of Berman or Pitt. Because of this, many problems for fractional Brownian motion and the Brownian sheet have to be investigated using different methods.

Khoshnevisan and Xiao [55] have recently proved that the Brownian sheet satisfies the sectorial local nondeterminism [i.e., (C3) with $H = \langle 1/2 \rangle$] and applied this property to study various analytic and geometric properties of the Brownian sheet; see also [51].

Wu and Xiao [97] extended the result of [55] and proved that fractional Brownian sheet B_0^H satisfies Condition (C3).

Theorem 3.1. *Let $B_0^H = \{B_0^H(t), t \in \mathbf{R}^N\}$ be an $(N, 1)$ -fractional Brownian sheet with index $H = (H_1, \dots, H_N) \in (0, 1)^N$. For any fixed number $\varepsilon \in (0, 1)$, there exists a positive constant $c_{3,1}$, depending on ε, H and N only, such that for all positive integers $n \geq 1$, and all $u, t^1, \dots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\text{Var} (B_0^H(u) \mid B_0^H(t^1), \dots, B_0^H(t^n)) \geq c_{3,1} \sum_{j=1}^N \min_{0 \leq k \leq n} |u_j - t_j^k|^{2H_j}, \quad (42)$$

where $t_j^0 = 0$ for $j = 1, \dots, N$.

Proof. While the argument of [55] relies on the property of independent increments of the Brownian sheet and its connection to Brownian motion, the proof

for B_0^H is based on a Fourier analytic argument in [46, Chapter 18] and the harmonizable representation (10) of B_0^H . We refer to [97] for details. \square

Now we prove a sufficient condition for an anisotropic Gaussian random field with stationary increments to satisfy Condition (C3').

Theorem 3.2. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a centered Gaussian random field in \mathbf{R} with stationary increments and spectral density $f(\lambda)$. Assume that there is a vector $H = (H_1, \dots, H_N) \in (0, 1)^N$ such that*

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{2+Q}} \quad \forall \lambda \in \mathbf{R}^N \setminus \{0\}, \tag{43}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$. Then there exists a constant $c_{3.2} > 0$ such that for all $n \geq 1$, and all $u, t^1, \dots, t^n \in \mathbf{R}^N$,

$$\text{Var}(X(u) \mid X(t^1), \dots, X(t^n)) \geq c_{3.2} \min_{0 \leq k \leq n} \rho(u, t^k)^2, \tag{44}$$

where $t^0 = 0$.

Remark 3.3. The following are some comments about Theorem 3.2.

- (i) When $H_1 = \dots = H_N$, (44) is of the same form as the SLND of fractional Brownian motion [77]. As shown by Xiao [102; 108] and Shieh and Xiao [83], many sample path properties of such Gaussian random fields are similar to those of fractional Brownian motion.
- (ii) Condition (43) can be significantly weakened. In particular, one can prove that similar results hold for Gaussian random fields with stationary increments and discrete spectrum measures; see [109] for details.
- (iii) It would be interesting to study under which conditions the solutions to the stochastic heat and wave equations (29) and (33) are strongly local nondeterministic.

Proof of Theorem 3.2. Denote $r \equiv \min_{0 \leq k \leq n} \rho(u, t^k)$. Since the conditional variance in (44) is the square of the $L^2(\mathbf{P})$ -distance of $X(u)$ from the subspace generated by $\{X(t^1), \dots, X(t^n)\}$, it is sufficient to prove that for all $a_k \in \mathbf{R}$ ($1 \leq k \leq n$),

$$\mathbf{E} \left(\left| X(u) - \sum_{k=1}^n a_k X(t^k) \right|^2 \right) \geq c_{3.2} r^2, \tag{45}$$

and $c_{3.2} > 0$ is a constant which may only depend on H and N .

By the stochastic integral representation (15) of X , the left hand side of (45) can be written as

$$\begin{aligned} & \mathbb{E} \left(\left| X(u) - \sum_{k=1}^n a_k X(t^k) \right|^2 \right) \\ &= \int_{\mathbf{R}^N} \left| e^{i\langle u, \lambda \rangle} - 1 - \sum_{k=1}^n a_k \left(e^{i\langle t^k, \lambda \rangle} - 1 \right) \right|^2 f(\lambda) d\lambda. \end{aligned} \quad (46)$$

Hence, we need to only show that

$$\int_{\mathbf{R}^N} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^n a_k e^{i\langle t^k, \lambda \rangle} \right|^2 f(\lambda) d\lambda \geq c_{3,2} r^2, \quad (47)$$

where $t^0 = 0$ and $a_0 = -1 + \sum_{k=1}^n a_k$.

Let $\delta(\cdot) : \mathbf{R}^N \rightarrow [0, 1]$ be a function in $C^\infty(\mathbf{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0, 1)$ in the metric ρ . Denote by $\widehat{\delta}$ the Fourier transform of δ . Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbf{R}^N)$ as well and $\widehat{\delta}(\lambda)$ decays rapidly as $|\lambda| \rightarrow \infty$.

Let E be the diagonal matrix with $H_1^{-1}, \dots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q} \delta(r^{-E}t)$. Then the inverse Fourier transform and a change of variables yield

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{-i\langle t, \lambda \rangle} \widehat{\delta}(r^E \lambda) d\lambda. \quad (48)$$

Since $\min\{\rho(u, t^k) : 0 \leq k \leq n\} \geq r$, we have $\delta_r(u - t^k) = 0$ for $k = 0, 1, \dots, n$. This and (48) together imply that

$$\begin{aligned} J &:= \int_{\mathbf{R}^N} \left(e^{i\langle u, \lambda \rangle} - \sum_{k=0}^n a_k e^{i\langle t^k, \lambda \rangle} \right) e^{-i\langle u, \lambda \rangle} \widehat{\delta}(r^E \lambda) d\lambda \\ &= (2\pi)^N \left(\delta_r(0) - \sum_{k=0}^n a_k \delta_r(u - t^k) \right) \\ &= (2\pi)^N r^{-Q}. \end{aligned} \quad (49)$$

On the other hand, by the Cauchy–Schwarz inequality and (46), we have

$$\begin{aligned} J^2 &\leq \int_{\mathbf{R}^N} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^n a_k e^{i\langle t^k, \lambda \rangle} \right|^2 f(\lambda) d\lambda \cdot \int_{\mathbf{R}^N} \frac{1}{f(\lambda)} \left| \widehat{\delta}(r^E \lambda) \right|^2 d\lambda \\ &\leq \mathbb{E} \left(\left| X(u) - \sum_{k=1}^n a_k X(t^k) \right|^2 \right) \cdot r^{-Q} \int_{\mathbf{R}^N} \frac{1}{f(r^{-E} \lambda)} \left| \widehat{\delta}(\lambda) \right|^2 d\lambda \\ &\leq c \mathbb{E} \left(\left| X(u) - \sum_{k=1}^n a_k X(t^k) \right|^2 \right) \cdot r^{-2Q-2}, \end{aligned} \quad (50)$$

where $c > 0$ is a constant which may only depend on H and N .

We square both sides of (49) and use (50) to obtain

$$(2\pi)^{2N} r^{-2Q} \leq c r^{-2Q-2} \mathbb{E} \left(\left| X(u) - \sum_{k=1}^n a_k X(t^k) \right|^2 \right). \tag{51}$$

Hence (47) holds. This finishes the proof of the theorem.

Given a Gaussian vector (Z_1, \dots, Z_n) , we denote the determinant of its covariance matrix by $\det \text{Cov}(Z_1, \dots, Z_n)$. If $\det \text{Cov}(Z_1, \dots, Z_n) > 0$, then we have the identity

$$\frac{(2\pi)^{n/2}}{\det \text{Cov}(Z_1, \dots, Z_n)} = \int_{\mathbf{R}^n} \mathbb{E} \exp \left(-i \sum_{k=1}^n u_k Z_k \right) du_1 \cdots du_n. \tag{52}$$

By using the fact that, for every k , the conditional distribution of Z_k given Z_1, \dots, Z_{k-1} is still Gaussian with mean $\mathbb{E}(Z_k | Z_1, \dots, Z_{k-1})$ and variance $\text{Var}(Z_k | Z_1, \dots, Z_{k-1})$, one can evaluate the integral in the right-hand side of (52) and thus verify the following formula:

$$\det \text{Cov}(Z_1, \dots, Z_n) = \text{Var}(Z_1) \prod_{k=2}^n \text{Var}(Z_k | Z_1, \dots, Z_{k-1}). \tag{53}$$

A little thought reveals that (53) still holds when $\det \text{Cov}(Z_1, \dots, Z_n) = 0$. Note that the left-hand side of (53) is permutation invariant for Z_1, \dots, Z_n , one can represent $\det \text{Cov}(Z_1, \dots, Z_n)$ in terms of the conditional variances in $n!$ different ways.

Combined with (42) or (44), the identity (53) can be applied to estimate the joint distribution of the Gaussian random variables $X(t^1), \dots, X(t^n)$, where $t^1, \dots, t^n \in \mathbf{R}^N$. This is why the properties of strong local nondeterminism are not only essential in this paper, but will also be useful in studying self-intersection local times, exact Hausdorff measure of the sample paths and other related problems [68; 69].

The following simple result will be needed in Section 8.

Lemma 3.4. *Let X be a Gaussian random field satisfying Condition (C3') [resp., (C3)]. Then for all integers $n \geq 1$ and for all distinct points $t^1, \dots, t^n \in [\varepsilon, 1]^N$ [resp., all points $t^1, \dots, t^n \in [\varepsilon, 1]^N$ with distinct coordinates, i.e., $t_i^k \neq t_j^l$ when $(i, k) \neq (j, l)$], the Gaussian random variables $X(t^1), \dots, X(t^n)$ are linearly independent.*

Proof. We assume Condition (C3') holds and let $t^1, \dots, t^n \in [\varepsilon, 1]^N$ be n distinct points. Then it follows from (53) that $\det \text{Cov}(X(t^1), \dots, X(t^n)) > 0$. This proves the lemma. Similar conclusion holds when Condition (C3) is satisfied. □

4 Modulus of Continuity

It is sufficient to consider real-valued Gaussian random fields. Ayache and Xiao [7] established a sharp modulus of continuity (i.e., including the logarithmic correction) for fractional Brownian sheets as a consequence of their wavelet expansion for B_0^H . Since the wavelet method depends on the stochastic integral representation (8), it can not be easily applied to Gaussian random fields in general. In this section, we describe several ways to establish sharp modulus of continuity for all anisotropic Gaussian random fields satisfying Condition (C1). The first two methods, i.e., the extended Garsia-Rodemich-Rumsey continuity lemma and the minorization metric method of Kwapien and Rosinski [60], can be applied to random fields which are not necessarily Gaussian. Hence they can be more convenient when applied to solutions of stochastic partial differential equations. The third method, which is based on Dudley's entropy theorem and the Gaussian isoperimetric inequality, provides a stronger result in the sense that the upper bound is a constant instead of a random variable [cf. (69)].

Theorem 4.1 is an extension of the well-known Garsia-Rodemich-Rumsey continuity lemma [40]. It follows from Theorem 2.1 of [39], which is slightly more general [because of its freedom in choosing the function p] than an analogous result of [2]. A similar result can also be found in [22].

For our purpose, we have formulated Theorem 4.1 in terms of the metric ρ defined in (37). Let $T \subseteq \mathbf{R}^N$ be a fixed closed interval. For any $r > 0$ and $s \in T$, recall that $B_\rho(s, r) = \{t \in T : \rho(t, s) \leq r\}$ denotes the closed ball (in T) with center s and radius r in the metric ρ .

Theorem 4.1. *Suppose that $Y : T \rightarrow \mathbf{R}$ is a continuous mapping. If there exist two strictly increasing functions Ψ and p on \mathbf{R}_+ with $\Psi(0) = p(0) = 0$ and $\lim_{u \rightarrow \infty} \Psi(u) = \infty$ such that*

$$K := \int_T \int_T \Psi \left(\frac{|Y(s) - Y(t)|}{p(\rho(s, t))} \right) ds dt < \infty. \quad (54)$$

Then for all $s, t \in T$, we have

$$|Y(s) - Y(t)| \leq 8 \max_{z \in \{s, t\}} \int_0^{\rho(s, t)} \Psi^{-1} \left(\frac{4K}{\lambda_N(B_\rho(z, u))^2} \right) \tilde{p}(du), \quad (55)$$

where $\tilde{p}(u) = p(4u)$ for all $u \in \mathbf{R}_+$.

Applying Theorem 4.1, we prove the following theorem on the modulus of continuity of an anisotropic Gaussian random field.

Theorem 4.2. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a centered Gaussian field in \mathbf{R} satisfying Condition (C1). Then, almost surely, there exists a random variable A depending on N and (H_1, \dots, H_N) only such that A has finite moments of all orders and for all $s, t \in I$,*

$$|X(s) - X(t)| \leq A \rho(s, t) \sqrt{\log(1 + \rho(s, t)^{-1})}. \tag{56}$$

Proof. In Theorem 4.1, let $T = I$ and we choose the functions $\Psi(x) = \exp\left(\frac{x^2}{4c_{2,3}}\right) - 1$ and $p(x) = x$, where $c_{2,3} > 0$ is the constant in (40). It follows from Condition (C1) that the random variable K in (54) has finite moments of all orders and

$$\begin{aligned} E(K) &= E \int_I \int_I \Psi\left(\frac{|X(s) - X(t)|}{\rho(s, t)}\right) ds dt \\ &\leq \int_I \int_I E\Psi(c|\xi|) ds dt = c_{4,1} < \infty. \end{aligned} \tag{57}$$

In the above ξ is a standard normal random variable. Note that $\Psi^{-1}(u) = \sqrt{4c_{2,3} \log(1 + u)}$ and $\lambda_N(B_\rho(z, u)) \asymp u^Q$ is independent of z . Hence by Theorem 4.1 we have

$$\begin{aligned} |X(s) - X(t)| &\leq c \int_0^{\rho(s,t)} \sqrt{\log\left(1 + \frac{4K}{u^Q}\right)} du \\ &\leq A \rho(s, t) \sqrt{\log(1 + \rho(s, t)^{-1})}, \end{aligned} \tag{58}$$

where A is a random variable depending on K and we can choose it so that $A \leq c \max\{1, \log K\}$. Thus all moments of A are finite. This finishes the proof of Theorem 4.2. \square

Let $X = \{X(t), t \in T\}$ be a stochastic process defined on a separable metric space (T, d) and let ψ be a Young function [that is, ψ is strictly increasing, convex and $\psi(0) = 0$]. Recently, Kwapien and Rosinski [60] investigated the following problem: When can one find an appropriate metric τ on T such that the implication

$$\sup_{s,t \in T} E\psi\left(\frac{|X(s) - X(t)|}{d(s, t)}\right) < \infty \quad \Rightarrow \quad \sup_{s,t \in T} \frac{|X(s) - X(t)|}{\tau(s, t)} < \infty, \quad \text{a.s.} \tag{59}$$

holds? Their results can be applied to derive sharp modulus of continuity for a large class of stochastic processes including Gaussian random fields [but not stable random fields].

Recall from [60] that a probability measure m on T is called a weakly majorizing measure relative to ψ and the metric d if for all $s, t \in T$,

$$\int_0^{d(s,t)} \psi^{-1}\left(\frac{1}{m(B_d(s, r))}\right) dr < \infty, \tag{60}$$

where ψ^{-1} denotes the inverse function of ψ and $B_d(s, r) = \{t \in T : d(t, s) \leq r\}$. For every weakly majorizing measure m , the ‘‘minorizing metric’’ $\tau = \tau_{\psi, d, m}$ on T relative to ψ , d and m is defined as

$$\tau(s, t) = \max \left\{ \int_0^{d(s,t)} \psi^{-1} \left(\frac{1}{m(B_d(s, r))} \right) dr, \int_0^{d(t,s)} \psi^{-1} \left(\frac{1}{m(B_d(t, r))} \right) dr \right\}. \tag{61}$$

The following theorem of [60] gives a sufficient condition for (59) to hold.

Theorem 4.3. *Let ψ be a Young function satisfying the following growth condition:*

$$\psi(x)\psi(y) \leq \psi(c_{4,2}(x + y)) \quad \text{for all } x, y \geq 0, \tag{62}$$

where $c_{4,2} > 0$ is a constant. Let m be a weakly majorizing measure relative to ψ and d on T . Then there exists a positive constant $c_{4,3}$ depending only on ψ such that for every stochastic process $X = \{X(t), t \in T\}$,

$$\mathbb{E}\psi \left(c_{4,3} \sup_{s,t \in T} \frac{|X(s) - X(t)|}{\tau(s, t)} \right) \leq 1 + \sup_{s,t \in T} \mathbb{E}\psi \left(\frac{|X(s) - X(t)|}{d(s, t)} \right), \tag{63}$$

where τ is the minorizing metric relative to ψ , d and m .

Note that, for any $\alpha > 0$, the function $\psi(x) = x^\alpha$ does not satisfy the growth condition (62), hence Theorem 4.3 is not applicable to stable random fields.

By applying Theorem 4.3 to the metric space (I, ρ) in our setting, we can provide more information about the random variable A in Theorem 4.2.

Corollary 4.4. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a centered Gaussian field in \mathbf{R} satisfying Condition (C1). Then there exists a constant $c_{4,4} > 0$ such that*

$$\mathbb{E} \exp \left(c_{4,4} \sup_{s,t \in I} \frac{|X(s) - X(t)|^2}{\rho^2(s, t) \log(1 + \rho(s, t)^{-1})} \right) < \infty. \tag{64}$$

Proof. This can be verified by showing that the Lebesgue measure on I is a weakly majorizing measure relative to the Young function $\psi(x) = e^{x^2} - 1$ and the metric ρ ; and the corresponding minorizing metric $\tau(s, t)$ satisfies

$$c_{4,5} \rho(s, t) \sqrt{\log(1 + \rho(s, t)^{-1})} \leq \tau(s, t) \leq c_{4,6} \rho(s, t) \sqrt{\log(1 + \rho(s, t)^{-1})}, \tag{65}$$

for all $s, t \in I$. We leave the details to an interested reader. □

As a third method, we mention that it is also possible to obtain a uniform modulus of continuity for a Gaussian random field satisfying Condition (C1) by using the Gaussian isoperimetric inequality [85, Lemma 2.1]. To this end, we introduce an auxiliary Gaussian random field $Y = \{Y(s, t) : t \in I, s \in [0, h]\}$ defined by $Y(t, s) = X(t + s) - X(t)$, where $h \in (0, 1)^N$. Then the canonical metric d on $T := I \times [0, h]$ associated with Y satisfies the following inequality:

$$d((t, s), (t', s')) \leq c \min \{ \rho(0, s) + \rho(0, s'), \rho(s, s') + \rho(t, t') \}. \quad (66)$$

Denote the d -diameter of T by D . It follows from (66) that $D \leq c_{4,7} \sum_{j=1}^N h_j^{H_j} = c_{4,7} \rho(0, h)$, and the d -covering number of T satisfies

$$N_d(T, \varepsilon) \leq c \left(\frac{1}{\varepsilon} \right)^Q \prod_{j=1}^N \left(\frac{h_j}{\varepsilon^{1/H_j}} \right) \leq c_{4,8} \varepsilon^{-2Q}.$$

One can verify that

$$\int_0^D \sqrt{\log N_d(T, \varepsilon)} d\varepsilon \leq c_{4,9} \rho(0, h) \sqrt{\log(1 + \rho(0, h)^{-1})}. \quad (67)$$

It follows from Lemma 2.1 in [85] that for all $u \geq 2c_{4,9} \rho(0, h) \sqrt{\log(1 + \rho(0, h)^{-1})}$,

$$\mathbb{P} \left\{ \sup_{(t,s) \in T} |X(t+s) - X(t)| \geq u \right\} \leq \exp \left(- \frac{u^2}{D^2} \right). \quad (68)$$

By using (68) and a standard Borel-Cantelli argument, we can prove that

$$\limsup_{|h| \rightarrow 0} \frac{\sup_{t \in I, s \in [0, h]} |X(t+s) - X(t)|}{\rho(0, h) \sqrt{\log(1 + \rho(0, h)^{-1})}} \leq c_{4,10}, \quad (69)$$

where $c_{4,10} > 0$ is a finite constant depending on $c_{2,4}$, I and H only.

We believe that, for Gaussian random fields satisfying (C1), the rate function in (56) is sharp. This has been partly verified by Meerschaert, Wang and Xiao [68] who proved that, if a Gaussian field X satisfies Conditions (C1) and (C3), then

$$c_{4,11} \leq \limsup_{|h| \rightarrow 0} \frac{\sup_{t \in I, s \in [0, h]} |X(s) - X(t)|}{\rho(0, h) \sqrt{\log(1 + \rho(0, h)^{-1})}} \leq c_{4,12}, \quad (70)$$

where $c_{4,11}$ and $c_{4,12}$ are positive constants depending on $c_{2,3}$, $c_{2,4}$, I and H only.

On the other hand, we can also use the above metric entropy method to prove that, for all $t_0 \in I$ and $u > 0$ large enough,

$$\mathbb{P} \left\{ \sup_{s \in [0, h]} |X(t_0 + s) - X(t_0)| \geq \rho(0, h) u \right\} \leq \exp(-c_{4,13} u^2), \quad (71)$$

where $c_{4,13}$ is a positive constant depending on $c_{2,4}$, I and H only.

By using (71) and the Borel-Cantelli lemma, we derive the following local modulus of continuity for Gaussian random fields satisfying (C1): There exists a positive constant $c_{4,14}$ such that for every $t_0 \in I$,

$$\limsup_{|h| \rightarrow 0} \frac{\sup_{s \in [0, h]} |X(t_0 + s) - X(t_0)|}{\rho(0, h) \sqrt{\log \log(1 + \rho(0, h)^{-1})}} \leq c_{4,14}, \quad \text{a.s.} \quad (72)$$

Under certain mild conditions, it can be shown that (72) is sharp. For example, Meerschaert, Wang and Xiao [68] proved that, if X is a Gaussian random field with stationary increments and satisfies (C1), then for every $t_0 \in I$,

$$\limsup_{|h| \rightarrow 0} \frac{\sup_{s \in [0, h]} |X(t_0 + s) - X(t_0)|}{\rho(0, h) \sqrt{\log \log(1 + \rho(0, h)^{-1})}} = c_{4,15}, \quad \text{a.s.}, \quad (73)$$

where $c_{4,15}$ is a positive constant.

We should mention that one can also study the uniform and local moduli of continuity in terms of the increments of X over intervals. Related results of this type for fractional Brownian sheets have been obtained by Wang [95].

In the special case when X is a direct sum of independent fractional Brownian motions of indices H_1, \dots, H_N , that is,

$$X(t) = X_1(t_1) + \dots + X_N(t_N), \quad \forall t = (t_1, \dots, t_N) \in \mathbf{R}^N, \quad (74)$$

where X_1, \dots, X_N are independent fractional Brownian motions in \mathbf{R} of indices H_1, \dots, H_N , respectively, Kôno [58] established integral tests for the uniform and local upper and lower classes. It is natural to ask whether his results hold for more general anisotropic Gaussian random fields.

5 Small Ball Probabilities

In recent years, there has been much interest in studying the small ball probabilities of Gaussian processes [62; 63]. Small ball properties of fractional Brownian sheets have been considered by Dunker [32], Mason and Shi [66], Belinski and Linde [9].

The small ball behavior of operator-scaling Gaussian random fields with stationary increments and the solution to the stochastic heat equation is different, as shown by the following general result.

Theorem 5.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be a centered Gaussian field in \mathbf{R} satisfying Conditions (C1) and (C3') on $I = [0, 1]^N$. Then there exist positive constants $c_{5,1}$ and $c_{5,2}$ such that for all $\varepsilon > 0$,*

$$\exp\left(-\frac{c_{5,1}}{\varepsilon^Q}\right) \leq \mathbb{P}\left\{\max_{t \in [0,1]^N} |X(t)| \leq \varepsilon\right\} \leq \exp\left(-\frac{c_{5,2}}{\varepsilon^Q}\right), \quad (75)$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$.

In order to prove the lower bound in (75), we will make use of the following general result of [84], see also [61, p. 257].

Lemma 5.2. *Let $Y = \{Y(t), t \in T\}$ be a real-valued Gaussian process with mean zero and let d be the canonical metric on T defined by*

$$d(s, t) = \{E(|Y(s) - Y(t)|^2)\}^{1/2}, \quad s, t \in T, \tag{76}$$

and denote by $N_d(T, \varepsilon)$ the smallest number of d -balls of radius $\varepsilon > 0$ needed to cover T . Assume that there is a nonnegative function ψ on \mathbf{R}_+ such that $N_d(T, \varepsilon) \leq \psi(\varepsilon)$ for $\varepsilon > 0$ and such that

$$c_{5,3}\psi(\varepsilon) \leq \psi\left(\frac{\varepsilon}{2}\right) \leq c_{5,4}\psi(\varepsilon) \tag{77}$$

for some constants $1 < c_{5,3} \leq c_{5,4} < \infty$ and all $\varepsilon > 0$. Then there is a constant $c_{5,5} > 0$ such that

$$P\left\{\sup_{t,s \in T} |Y(t) - Y(s)| \leq \varepsilon\right\} \geq \exp(-c_{5,5}\psi(\varepsilon)). \tag{78}$$

Proof of Theorem 5.1. It follows from (C1) that for all $\varepsilon \in (0, 1)$,

$$N_\rho(I, \varepsilon) \leq c\varepsilon^{-Q} := \psi(\varepsilon). \tag{79}$$

Clearly $\psi(\varepsilon)$ satisfies the condition (77). Hence the lower bound in (75) follows from Lemma 5.2.

The proof of the upper bound in (75) is based on Condition (C3') and a conditioning argument in [71]. For any integer $n \geq 2$, we divide $[0, 1]^N$ into n^Q rectangles of side-lengths n^{-1/H_j} ($j = 1, \dots, N$). We denote the lower-left vertices of these rectangles (in any order) by $t_{n,k}$ ($k = 1, 2, \dots, n^Q$). Then

$$P\left\{\max_{t \in [0, 1]^N} |X(t)| \leq \varepsilon\right\} \leq P\left\{\max_{1 \leq k \leq n^Q} |X(t_{n,k})| \leq \varepsilon\right\}. \tag{80}$$

It follows from Condition (C3') that for every $1 \leq k \leq n^Q$,

$$\text{Var}(X(t_{n,k}) \mid X(t_{n,i}), 1 \leq i \leq k-1) \geq cn^{-1}. \tag{81}$$

This and Anderson's inequality for Gaussian measures imply the following upper bound for the conditional probabilities

$$P\left\{|X(t_{n,k})| \leq \varepsilon \mid X(t_{n,j}), 1 \leq j \leq k-1\right\} \leq \Phi(c\varepsilon n), \tag{82}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. It follows from (80) and (82) that

$$P\left\{\max_{t \in [0, 1]^N} |X(t)| \leq \varepsilon\right\} \leq [\Phi(c\varepsilon n)]^{n^Q}. \tag{83}$$

By taking $n = (c\varepsilon)^{-1}$, we obtain the upper bound in (75).

Remark 5.3. If $H_1 = H_2 = \dots = H_N$, then (75) is of the same form as the small ball probability estimates for multiparameter fractional Brownian motion [62; 85].

Among many applications, Theorem 5.1 can be applied to establish Chung-type laws of the iterated logarithm for anisotropic Gaussian random fields. Moreover, it would also be interesting to investigate the small ball probabilities of X in other norms such as the L^2 or Hölder norms; see [62; 63] for information on Gaussian processes.

6 Hausdorff and Packing Dimensions of the Range and Graph

In this section, we study the Hausdorff and packing dimensions of the range $X([0, 1]^N) = \{X(t) : t \in [0, 1]^N\}$ and the graph $\text{Gr}X([0, 1]^N) = \{(t, X(t)) : t \in [0, 1]^N\}$ of a Gaussian random field X satisfying Condition (C1) on $[0, 1]^N$.

Hausdorff dimension and Hausdorff measure have been extensively used in describing thin sets and fractals. For any set $E \subseteq \mathbf{R}^d$ and $\gamma > 0$, we will denote the Hausdorff dimension and the γ -dimensional Hausdorff measure of E by $\dim_{\mathcal{H}} E$ and $\mathcal{H}^\gamma(E)$, respectively [36; 46; 67]. More generally, for any nondecreasing, right continuous function $\varphi : [0, 1] \rightarrow [0, \infty)$ with $\varphi(0) = 0$, one can define the Hausdorff measure of E with respect to φ and denoted it by $\mathcal{H}^\varphi(E)$. We say that a function φ is an exact Hausdorff measure function for E if $0 < \mathcal{H}^\varphi(E) < \infty$.

Now we recall briefly the definition of capacity and its connection to Hausdorff dimension. A kernel κ is a measurable function $\kappa : \mathbf{R}^d \times \mathbf{R}^d \rightarrow [0, \infty]$. For a Borel measure μ on \mathbf{R}^d , the energy of μ with respect to the kernel κ (in short, κ energy of μ) is defined by

$$\mathcal{E}_\kappa(\mu) = \iint \kappa(x, y) \mu(dx) \mu(dy). \quad (84)$$

For any Borel set $E \subseteq \mathbf{R}^d$, the *capacity of E with respect to κ* , denoted by $\mathcal{C}_\kappa(E)$, is defined by

$$\mathcal{C}_\kappa(E) = \left[\inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_\kappa(\mu) \right]^{-1}, \quad (85)$$

where $\mathcal{P}(E)$ is the family of probability measures carried by E , and, by convention, $\infty^{-1} = 0$. Note that $\mathcal{C}_\kappa(E) > 0$ if and only if there is a probability measure μ on E with finite κ -energy. We will mostly consider the case when $\kappa(x, y) = f(|x - y|)$, where f is a non-negative and non-increasing function on \mathbf{R}_+ . In particular, if

$$f(r) = \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log\left(\frac{e}{r \wedge 1}\right) & \text{if } \alpha = 0, \end{cases} \quad (86)$$

then the corresponding $\mathcal{E}_\kappa(\mu)$ and $\mathcal{C}_\kappa(E)$ will be denoted by $\mathcal{E}_\alpha(\mu)$ and $\mathcal{C}_\alpha(E)$ respectively; and the former will be called the α -energy of μ and the latter will be called the *Bessel-Riesz capacity of E of order α* . The capacity dimension of E is defined by

$$\dim_c(E) = \sup\{\alpha > 0 : \mathcal{C}_\alpha(E) > 0\}. \tag{87}$$

The well-known Frostman’s theorem (see [46, p. 133] or [49]) states that $\dim_{\mathcal{H}} E = \dim_c(E)$ for every compact set E in \mathbf{R}^d . This result provides a very useful analytic way for the lower bound calculation of Hausdorff dimension. That is, for $E \subseteq \mathbf{R}^d$ in order to show $\dim_{\mathcal{H}} E \geq \alpha$, one only needs to find a measure μ on E such that the α -energy of μ is finite. For many deterministic and random sets such as self-similar sets or the range of a stochastic process, there are natural choices of μ . This argument is usually referred to as the *capacity argument*.

Packing dimension and packing measure were introduced by Tricot [92] and Taylor and Tricot [89] as dual concepts to Hausdorff dimension and Hausdorff measure. We only recall briefly a definition of packing dimension, which will be denoted by $\dim_{\mathcal{P}}$. For any $\varepsilon > 0$ and any bounded set $F \subseteq \mathbf{R}^d$, let $N(F, \varepsilon)$ be the smallest number of balls of radius ε needed to cover F . Then the *upper box-counting dimension* of F is defined as

$$\overline{\dim}_{\mathcal{B}} F = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \tag{88}$$

The packing dimension of F can be defined by

$$\dim_{\mathcal{P}} F = \inf \left\{ \sup_n \overline{\dim}_{\mathcal{B}} F_n : F \subseteq \bigcup_{n=1}^{\infty} F_n \right\}. \tag{89}$$

It is known that for any bounded set $F \subseteq \mathbf{R}^d$,

$$\dim_{\mathcal{H}} F \leq \dim_{\mathcal{P}} F \leq \overline{\dim}_{\mathcal{B}} F \leq d. \tag{90}$$

Further information on packing dimension and packing measure can be found in [36; 67]. We mention that various tools from fractal geometry have been applied to studying sample path properties of stochastic processes since 1950’s. The survey papers [88] and [106] summarize various fractal properties of random sets related to sample paths of Markov processes.

Throughout the rest of this paper, we will assume that

$$0 < H_1 \leq \dots \leq H_N < 1. \tag{91}$$

Theorem 6.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian field satisfying Condition (C1) on $I = [0, 1]^N$. Then, with probability 1,*

$$\dim_{\mathcal{H}} X([0, 1]^N) = \dim_{\mathcal{D}} X([0, 1]^N) = \min \left\{ d; \sum_{j=1}^N \frac{1}{H_j} \right\} \quad (92)$$

and

$$\begin{aligned} \dim_{\mathcal{H}} \text{Gr}X([0, 1]^N) &= \dim_{\mathcal{D}} \text{Gr}X([0, 1]^N) \\ &= \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - H_k)d, 1 \leq k \leq N; \sum_{j=1}^N \frac{1}{H_j} \right\} \\ &= \begin{cases} \sum_{j=1}^N H_j^{-1}, & \text{if } \sum_{j=1}^N H_j^{-1} \leq d, \\ \sum_{j=1}^k (H_k/H_j) + N - k + (1 - H_k)d, & \text{if } \sum_{j=1}^{k-1} H_j^{-1} \leq d < \sum_{j=1}^k H_j^{-1}, \end{cases} \end{aligned} \quad (93)$$

where $\sum_{j=1}^0 H_j^{-1} := 0$.

The last equality in (93) is verified by the following lemma, whose proof is elementary and is omitted. Denote

$$\kappa := \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - H_k)d, 1 \leq k \leq N; \sum_{j=1}^N \frac{1}{H_j} \right\}. \quad (94)$$

Lemma 6.2. *Assume (91) holds. We have*

- (i) *If $d \geq \sum_{j=1}^N H_j^{-1}$, then $\kappa = \sum_{j=1}^N H_j^{-1}$.*
- (ii) *If $\sum_{j=1}^{\ell-1} H_j^{-1} \leq d < \sum_{j=1}^{\ell} H_j^{-1}$ for some $1 \leq \ell \leq N$, then*

$$\kappa = \sum_{j=1}^{\ell} \frac{H_{\ell}}{H_j} + N - \ell + (1 - H_{\ell})d \quad (95)$$

and $\kappa \in (N - \ell + d, N - \ell + d + 1]$.

Because of (90) we can divide the proof of Theorem 6.1 into proving the upper bounds for the upper box dimensions and the lower bounds for the Hausdorff dimensions separately. The proofs are similar to those in [7] for fractional Brownian sheets. In the following, we first show that the upper bounds for $\overline{\dim}_{\mathcal{D}} X([0, 1]^N)$ and $\overline{\dim}_{\mathcal{D}} \text{Gr}X([0, 1]^N)$ follow from Theorem 4.2 and a covering argument.

Proof of the upper bounds in Theorem 6.1. In order to prove the upper bound in (92), we note that clearly $\overline{\dim}_{\mathcal{D}} X([0, 1]^N) \leq d$ a.s., so it suffices to prove the following inequality:

$$\overline{\dim}_{\mathcal{D}} X([0, 1]^N) \leq \sum_{j=1}^N \frac{1}{H_j}, \quad \text{a.s.} \quad (96)$$

For any constants $0 < \gamma_j < H_j$ ($1 \leq j \leq N$), it follows from Theorem 4.2 that there is a random variable A of finite moments of all orders such that for almost all $\omega \in \Omega$,

$$\sup_{s,t \in [0,1]^N} \frac{|X(s, \omega) - X(t, \omega)|}{\sum_{j=1}^N |s_j - t_j|^{\gamma_j}} \leq A(\omega). \tag{97}$$

We fix an ω such that (97) holds and then suppress it. For any integer $n \geq 2$, we divide $[0, 1]^N$ into m_n sub-rectangles $\{R_{n,i}\}$ with sides parallel to the axes and side-lengths n^{-1/H_j} ($j = 1, \dots, N$), respectively. Then

$$m_n \leq c_{6,1} n^{\sum_{j=1}^N (1/H_j)} \tag{98}$$

and $X([0, 1]^N)$ can be covered by $X(R_{n,i})$ ($1 \leq i \leq m_n$). By (97), we see that the diameter of the image $X(R_{n,i})$ satisfies

$$\text{diam}X(R_{n,i}) \leq c_{6,2} n^{-1+\delta}, \tag{99}$$

where $\delta = \max\{(H_j - \gamma_j)/H_j, 1 \leq j \leq N\}$. Consequently, for $\varepsilon_n = c_{6,2} n^{-1+\delta}$, $X([0, 1]^N)$ can be covered by at most m_n balls in \mathbf{R}^d of radius ε_n . That is,

$$N(X([0, 1]^N), \varepsilon_n) \leq c_{6,1} n^{\sum_{j=1}^N (1/H_j)}. \tag{100}$$

This implies

$$\overline{\dim}_{\mathscr{H}} X([0, 1]^N) \leq \frac{1}{1-\delta} \sum_{j=1}^N \frac{1}{H_j}, \quad \text{a.s.} \tag{101}$$

By letting $\gamma_j \uparrow H_j$ along rational numbers, we have $\delta \downarrow 0$ and (96) follows from (101).

Now we turn to the proof of the upper bound in (93). We will show that there are several different ways to cover $\text{Gr}X([0, 1]^N)$ by balls in \mathbf{R}^{N+d} of the same radius, each of which leads to an upper bound for $\overline{\dim}_{\mathscr{H}} \text{Gr}X([0, 1]^N)$.

For each fixed integer $n \geq 2$, we have

$$\text{Gr}X([0, 1]^N) \subseteq \bigcup_{i=1}^{m_n} R_{n,i} \times X(R_{n,i}). \tag{102}$$

It follows from (99) and (102) that $\text{Gr}X([0, 1]^N)$ can be covered by m_n balls in \mathbf{R}^{N+d} with radius $c_{6,2} n^{-1+\delta}$ and the same argument as the above yields

$$\overline{\dim}_{\mathscr{H}} \text{Gr}X([0, 1]^N) \leq \sum_{j=1}^N \frac{1}{H_j}, \quad \text{a.s.} \tag{103}$$

We fix an integer $1 \leq k \leq N$. Observe that each $R_{n,i} \times X(R_{n,i})$ can be covered by $\ell_{n,k}$ balls (or cubes) in \mathbf{R}^{N+d} of radius (or side-length) $n^{-(1/H_k)}$, where by (97) we have

$$\ell_{n,k} \leq c n^{\sum_{j=k}^N (H_k^{-1} - H_j^{-1})} \times n^{(H_k^{-1} - 1 + \delta)d}, \quad \text{a.s.} \quad (104)$$

Hence $\text{Gr}X([0, 1]^N)$ can be covered by $m_n \times \ell_{n,k}$ balls in \mathbf{R}^{N+d} with radius $n^{-(1/H_k)}$. Consequently,

$$\overline{\dim}_{\mathscr{B}} \text{Gr}X([0, 1]^N) \leq \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - H_k + \delta H_k)d, \quad \text{a.s.} \quad (105)$$

Letting $\gamma_j \uparrow H_j$ along rational numbers, we derive that for every $k = 1, \dots, N$,

$$\overline{\dim}_{\mathscr{B}} \text{Gr}X([0, 1]^N) \leq \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - H_k)d. \quad (106)$$

Combining (103) and (106) yields the upper bound in (93).

For proving the lower bounds in Theorem 6.1, we will make use of the following elementary Lemmas 6.3 and 6.4. The former is proved in [110, p. 212] which will be used to derive a lower bound for $\dim_{\mathscr{X}} X([0, 1]^N)$; the latter is proved in [7] which will be needed for determining a lower bound for $\dim_{\mathscr{X}} \text{Gr}X([0, 1]^N)$. Both lemmas will be useful in the proof of Theorem 7.1 in Section 7.

Lemma 6.3. *Let $0 < \alpha < 1$ and $\varepsilon > 0$ be given constants. Then for any constants $\delta > 2\alpha$, $M > 0$ and $p > 0$, there exists a positive and finite constant $c_{6,3}$, depending on ε , δ , p and M only, such that for all $0 < A \leq M$,*

$$\int_{\varepsilon}^1 ds \int_{\varepsilon}^1 \frac{dt}{(A + |s - t|^{2\alpha})^p} \leq c_{6,3} (A^{-(p-\frac{1}{\delta})} + 1). \quad (107)$$

Lemma 6.4. *Let α , β and η be positive constants. For $A > 0$ and $B > 0$, let*

$$J := J(A, B) = \int_0^1 \frac{dt}{(A + t^\alpha)^\beta (B + t)^\eta}. \quad (108)$$

Then there exist finite constants $c_{6,4}$ and $c_{6,5}$, depending on α , β , η only, such that the following hold for all real numbers $A, B > 0$ satisfying $A^{1/\alpha} \leq c_{6,3} B$:

(i) *If $\alpha\beta > 1$, then*

$$J \leq c_{6,5} \frac{1}{A^{\beta-\alpha^{-1}} B^\eta}; \quad (109)$$

(ii) *If $\alpha\beta = 1$, then*

$$J \leq c_{6,5} \frac{1}{B^\eta} \log(1 + BA^{-1/\alpha}); \quad (110)$$

(iii) If $0 < \alpha\beta < 1$ and $\alpha\beta + \eta \neq 1$, then

$$J \leq c_{6,5} \left(\frac{1}{B^{\alpha\beta + \eta - 1}} + 1 \right). \tag{111}$$

Proof of the lower bounds in Theorem 6.1. First we prove the lower bound in (92). Note that for any $\varepsilon \in (0, 1)$, $\dim_{\mathscr{H}} X([0, 1]^N) \geq \dim_{\mathscr{H}} X([\varepsilon, 1]^N)$. It is sufficient to show that $\dim_{\mathscr{H}} X([\varepsilon, 1]^N) \geq \gamma$ a.s. for every $0 < \gamma < \min\{d, \sum_{j=1}^N \frac{1}{H_j}\}$.

Let μ_X be the image measure of the Lebesgue measure on $[\varepsilon, 1]^N$ under the mapping $t \mapsto X(t)$. Then the energy of μ_X of order γ can be written as

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{\mu_X(dx) \mu_X(dy)}{|x - y|^\gamma} = \int_{[\varepsilon, 1]^N} \int_{[\varepsilon, 1]^N} \frac{ds dt}{|X(s) - X(t)|^\gamma}.$$

Hence by Frostman’s theorem [46, Chapter 10], it is sufficient to show that for every $0 < \gamma < \min\{d, \sum_{j=1}^N \frac{1}{H_j}\}$,

$$\mathcal{E}_\gamma = \int_{[\varepsilon, 1]^N} \int_{[\varepsilon, 1]^N} \mathbb{E} \left(\frac{1}{|X(s) - X(t)|^\gamma} \right) ds dt < \infty. \tag{112}$$

Since $0 < \gamma < d$, we have $0 < \mathbb{E}(|\Xi|^{-\gamma}) < \infty$, where Ξ is a standard d -dimensional normal vector. Combining this fact with Condition (C1), we have

$$\mathcal{E}_\gamma \leq c \int_\varepsilon^1 ds_1 \int_\varepsilon^1 dt_1 \cdots \int_\varepsilon^1 ds_N \int_\varepsilon^1 \frac{dt_N}{\left(\sum_{j=1}^N |s_j - t_j|^{2H_j}\right)^{\gamma/2}}. \tag{113}$$

We choose positive constants $\delta_2, \dots, \delta_N$ such that $\delta_j > 2H_j$ for each $2 \leq j \leq N$ and

$$\frac{1}{\delta_2} + \cdots + \frac{1}{\delta_N} < \frac{\gamma}{2} < \frac{1}{2H_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_N}. \tag{114}$$

This is possible since $\gamma < \sum_{j=1}^N (1/H_j)$. By applying Lemma 6.3 to (113) with

$$A = \sum_{j=1}^{N-1} |s_j - t_j|^{2H_j} \quad \text{and} \quad p = \gamma/2, \tag{115}$$

we find that \mathcal{E}_γ is at most $c_{6,6}$ plus $c_{6,6}$ times

$$\int_\varepsilon^1 ds_1 \cdots \int_\varepsilon^1 ds_{N-1} \int_\varepsilon^1 \frac{dt_{N-1}}{\left(\sum_{j=1}^{N-1} |s_j - t_j|^{2H_j}\right)^{(\gamma/2) - (1/\delta_N)}}. \tag{116}$$

By repeatedly using Lemma 6.3 to the integral in (116) for $N - 2$ steps, we deduce that

$$\mathcal{E}_\gamma \leq c_{6,7} + c_{6,7} \int_\varepsilon^1 ds_1 \int_\varepsilon^1 \frac{dt_1}{(|s_1 - t_1|^{2H_1})^{(\gamma/2) - ((1/\delta_2) + \dots + (1/\delta_N))}}. \quad (117)$$

Since the δ_j 's satisfy (114), we have $2H_1[\gamma/2 - (\delta_2^{-1} + \dots + \delta_N^{-1})] < 1$. Thus the integral in the right hand side of (117) is finite. This proves (112), and (92) follows.

Now we prove the lower bound in (93). Since $\dim_{\mathscr{H}} \text{Gr}X([0, 1]^N) \geq \dim_{\mathscr{H}} X([0, 1]^N)$ always holds, we only need to consider the case when

$$\sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^k \frac{1}{H_j} \quad \text{for some } 1 \leq k \leq N. \quad (118)$$

Here and in the sequel, $\sum_{j=1}^0 (1/H_j) = 0$.

Let $0 < \varepsilon < 1$ and $0 < \gamma < \sum_{j=1}^k (H_k/H_j) + N - k + (1 - H_k)d$ be fixed, but arbitrary, constants. By Lemma 6.2, we may and will assume $\gamma \in (N - k + d, N - k + d + 1)$. In order to prove $\dim_{\mathscr{H}} \text{Gr}X([\varepsilon, 1]^N) \geq \gamma$ a.s., again by Frostman's theorem, it is sufficient to show

$$\mathcal{G}_\gamma = \int_{[\varepsilon, 1]^N} \int_{[\varepsilon, 1]^N} \text{E} \left[\frac{1}{(|s - t|^2 + |X(s) - X(t)|^2)^{\gamma/2}} \right] ds dt < \infty. \quad (119)$$

Since $\gamma > d$, we note that for a standard normal vector Ξ in \mathbf{R}^d and any number $a \in \mathbf{R}$,

$$\text{E} \left[\frac{1}{(a^2 + |\Xi|^2)^{\gamma/2}} \right] \leq c_{6,8} a^{-(\gamma-d)}, \quad (120)$$

see [46, p. 279]. Consequently, we derive that

$$\mathcal{G}_\gamma \leq c_{6,8} \int_{[\varepsilon, 1]^N} \int_{[\varepsilon, 1]^N} \frac{ds dt}{\sigma(s, t)^d |s - t|^{\gamma-d}}, \quad (121)$$

where $\sigma^2(s, t) = \text{E}[(X_1(s) - X_1(t))^2]$. By Condition (C1) and a change of variables, we have

$$\mathcal{G}_\gamma \leq c_{6,9} \int_0^1 dt_N \cdots \int_0^1 \frac{dt_1}{\left(\sum_{j=1}^N t_j^{H_j}\right)^d \left(\sum_{j=1}^N t_j\right)^{\gamma-d}}. \quad (122)$$

In order to show the integral in (122) is finite, we will integrate $[dt_1], \dots, [dt_k]$ iteratively. Furthermore, we will assume $k > 1$ in (118) [If $k = 1$, we can use (111) to obtain (126) directly].

We integrate $[dt_1]$ first. Since $H_1 d > 1$, we can use (109) of Lemma 6.4 with $A = \sum_{j=2}^N t_j^{H_j}$ and $B = \sum_{j=2}^N t_j$ to get

$$\mathcal{G}_\gamma \leq c_{6,10} \int_0^1 dt_N \cdots \int_0^1 \frac{dt_2}{\left(\sum_{j=2}^N t_j^{H_j}\right)^{d-(1/H_1)} \left(\sum_{j=2}^N t_j\right)^{\gamma-d}}. \quad (123)$$

We can repeat this procedure for integrating dt_2, \dots, dt_{k-1} . Note that if $d = \sum_{j=1}^{k-1} (1/H_j)$, then we need to use (110) to integrate $[dt_{k-1}]$ and obtain

$$\mathcal{G}_\gamma \leq c_{6,11} \int_0^1 dt_N \cdots \int_0^1 \frac{1}{(\sum_{j=k}^N t_j)^{\gamma-d}} \log \left(1 + \frac{1}{\sum_{j=k}^N t_j} \right) dt_k < \infty. \tag{124}$$

Note that the last integral is finite since $\gamma - d < N - k + 1$. On the other hand, if $d > \sum_{j=1}^{k-1} (1/H_j)$, then (109) gives

$$\mathcal{G}_\gamma \leq c_{6,12} \int_0^1 dt_N \cdots \int_0^1 \frac{dt_k}{(\sum_{j=k}^N t_j)^{d-\sum_{j=1}^{k-1} (1/H_j)} (\sum_{j=k}^N t_j)^{\gamma-d}}. \tag{125}$$

We integrate $[dt_k]$ in (125) and by using (111), we see that \mathcal{G}_γ is bounded above by

$$c_{6,13} \left[\int_0^1 dt_N \cdots \int_0^1 \frac{dt_{k+1}}{(\sum_{j=k+1}^N t_j)^{\gamma-d+H_k(d-\sum_{j=1}^{k-1} (1/H_j)) - 1}} + 1 \right] < \infty, \tag{126}$$

since $\gamma - d + H_k(d - \sum_{j=1}^{k-1} (1/H_j)) - 1 < N - k$. Combining (124) and (126) yields (119). This completes the proof of Theorem 6.1.

There are several possible ways to strengthen and extend Theorem 6.1. For example, it would be interesting to determine the exact Hausdorff and packing measure functions for the range $X([0, 1]^N)$ and graph $\text{Gr}X([0, 1]^N)$ for anisotropic Gaussian random fields. When X is the Brownian sheet or a fractional Brownian motion, the problems on exact Hausdorff measure functions have been considered by Ehm [34], Talagrand [85; 86], Xiao [99; 101; 102]. Here is a summary of the known results:

- (i) Let $X^\alpha = \{X^\alpha(t), t \in \mathbf{R}^N\}$ be an (N, d) -fractional Brownian motion of index α . If $N < \alpha d$, then $\varphi_1(r) = r^{N/\alpha} \log \log 1/r$ is an exact Hausdorff measure function for the range and graph of X^α . If $N > \alpha d$, then $X^\alpha([0, 1]^N)$ a.s. has positive Lebesgue measure and interior points; and $\varphi_2(r) = r^{N+(1-\alpha)d} (\log \log 1/r)^{\frac{\alpha d}{N}}$ is an exact Hausdorff measure function for the graph of X^α . If $N = \alpha d$, then $\mathcal{H}^{\varphi_3}(X^\alpha([0, 1]^N))$ is σ -finite almost surely, where $\varphi_3(r) = r^d \log(1/r) \log \log 1/r$. In the latter case the same is also true for the Hausdorff measure of the graph set of $X^\alpha(t)$. However, the lower bound problems for the Hausdorff measure of the range and graph remain open.
- (ii) Let $W = \{W(t), t \in \mathbf{R}_+^N\}$ be the Brownian sheet in \mathbf{R}^d . If $2N < d$, then $\varphi_4(r) = r^{2N} (\log \log 1/r)^N$ is an exact Hausdorff measure function for the range and graph of W . If $2N > d$, then $W([0, 1]^N)$ a.s. has interior points and $\varphi_5(r) = r^{N+\frac{d}{2}} (\log \log 1/r)^{\frac{d}{2}}$ is an exact Hausdorff measure function for the graph of W . When $2N = d$, the problems for finding exact Hausdorff measure functions for the range and graph of W are completely open.

It is interesting to notice the subtle differences in the exact Hausdorff functions for the range and graph sets of fractional Brownian motion and the Brownian sheet, respectively. I believe the differences are a reflection of the two different types of local nondeterminism that they satisfy.

We remark that the methods in the aforementioned references rely respectively on specific properties of the Brownian sheet and fractional Brownian motion, and it is not clear whether these methods are applicable to Gaussian random fields satisfying (C3) or (C3'). It would be interesting to develop general methods that are applicable to larger classes of (Gaussian) random fields.

The problems on exact packing measure functions for $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ are related to the liminf properties of the occupation measures of X and are more difficult to study. When X is an (N, d) -fractional Brownian motion of index α and $N < \alpha d$, Xiao [100; 105] proved that $\varphi_6(r) = r^{N/\alpha}(\log \log 1/r)^{-N/(2\alpha)}$ is an exact packing measure function for $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$. For all the other Gaussian fields including the Brownian sheet, the corresponding problems remain to be open.

On the other hand, it is a natural question to find $\dim_{\mathcal{H}} X(E)$ when $E \subseteq \mathbf{R}^N$ is an arbitrary Borel set, say a fractal set. It is not hard to see that, due to the anisotropy of X , the Hausdorff dimension of $X(E)$ can not be determined by $\dim_{\mathcal{H}} E$ and the index H alone, as shown by Example 6.6 below. This is in contrast with the cases of fractional Brownian motion or the Brownian sheet.

We start with the following Proposition 6.5 which determines $\dim_{\mathcal{H}} X(E)$ when E belongs to a special class of Borel sets in \mathbf{R}^N . Since the proof is almost the same as that of Proposition 3.1 in [97], we omit the proof.

Proposition 6.5. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian random field satisfying Condition (C1) on $I = [0, 1]^N$ with parameters (H_1, \dots, H_N) . Assume that E_j ($j = 1, \dots, N$) are Borel subsets of $(0, 1)$ satisfying the following property: $\exists \{j_1, \dots, j_{N-1}\} \subseteq \{1, \dots, N\}$ such that $\dim_{\mathcal{H}} E_{j_k} = \dim_{\mathcal{H}} E_{j_k}$ for $k = 1, \dots, N - 1$. Let $E = E_1 \times \dots \times E_N \subseteq \mathbf{R}^N$, then we have*

$$\dim_{\mathcal{H}} X(E) = \min \left\{ d; \sum_{j=1}^N \frac{\dim_{\mathcal{H}} E_j}{H_j} \right\}, \quad \text{a.s.} \quad (127)$$

The following simple example illustrates that, in general, $\dim_{\mathcal{H}} E$ alone is not enough to determine the Hausdorff dimension of $X(E)$.

Example 6.6. Let $X = \{X(t), t \in \mathbf{R}^2\}$ be a $(2, d)$ -Gaussian field with index $H = (H_1, H_2)$ and $H_1 < H_2$. Let $E = E_1 \times E_2$ and $F = E_2 \times E_1$, where $E_1 \subseteq (0, 1)$ satisfies $\dim_{\mathcal{H}} E_1 = \dim_{\mathcal{H}} E_1$ and $E_2 \subseteq (0, 1)$ is arbitrary. It is well known that

$$\dim_{\mathcal{H}} E = \dim_{\mathcal{H}} E_1 + \dim_{\mathcal{H}} E_2 = \dim_{\mathcal{H}} F. \quad (128)$$

See [36, p. 94]. However, by Proposition 6.5 we have

$$\dim_{\mathcal{H}} X(E) = \min \left\{ d; \frac{\dim_{\mathcal{H}} E_1}{H_1} + \frac{\dim_{\mathcal{H}} E_2}{H_2} \right\}, \tag{129}$$

and

$$\dim_{\mathcal{H}} X(F) = \min \left\{ d; \frac{\dim_{\mathcal{H}} E_2}{H_1} + \frac{\dim_{\mathcal{H}} E_1}{H_2} \right\}. \tag{130}$$

We see that $\dim_{\mathcal{H}} X(E) \neq \dim_{\mathcal{H}} X(F)$ in general unless $\dim_{\mathcal{H}} E_1 = \dim_{\mathcal{H}} E_2$.

Example 6.6 shows that for determining $\dim_{\mathcal{H}} X(E)$, we need to have more information about the geometry of E than its Hausdorff dimension.

In order to solve the problem for finding the Hausdorff dimension of the image $B^H(E)$ of fractional Brownian sheet B^H , Wu and Xiao [97] applied a measure-theoretic approach and introduced a notion of *Hausdorff dimension contour* for finite Borel measures and Borel sets.

Recall that the Hausdorff dimension of a Borel measure μ on \mathbf{R}^N (or lower Hausdorff dimension as it is sometimes called) is defined by

$$\dim_{\mathcal{H}} \mu = \inf \{ \dim_{\mathcal{H}} F : \mu(F) > 0 \text{ and } F \subseteq \mathbf{R}^N \text{ is a Borel set} \}. \tag{131}$$

Hu and Taylor [44] proved the following characterization of $\dim_{\mathcal{H}} \mu$: If μ is a finite Borel measure on \mathbf{R}^N , then

$$\dim_{\mathcal{H}} \mu = \sup \left\{ \gamma \geq 0 : \limsup_{r \rightarrow 0^+} \frac{\mu(B(t, r))}{r^\gamma} = 0 \text{ for } \mu\text{-a.e. } t \in \mathbf{R}^N \right\}, \tag{132}$$

where $B(t, r) = \{s \in \mathbf{R}^N : |s - t| \leq r\}$. It can be verified that for every Borel set $E \subseteq \mathbf{R}^N$, we have

$$\dim_{\mathcal{H}} E = \sup \{ \dim_{\mathcal{H}} \mu : \mu \in \mathcal{M}_c^+(E) \}, \tag{133}$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures on E with compact support in E .

From (132), we note that $\dim_{\mathcal{H}} \mu$ only describes the local behavior of μ in an isotropic way and is not quite informative if μ is highly anisotropic. To overcome this difficulty, Wu and Xiao [97] introduce the following notion of “dimension” for $E \subseteq (0, \infty)^N$ that is natural for studying $X(E)$.

Definition 6.7. Given a Borel probability measure μ on \mathbf{R}^N , we define the set $A_\mu \subseteq \mathbf{R}_+^N$ by

$$A_\mu = \left\{ \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbf{R}_+^N : \limsup_{r \rightarrow 0^+} \frac{\mu(R(t, r))}{r^{(\lambda, H^{-1})}} = 0 \text{ for } \mu\text{-a.e. } t \in \mathbf{R}^N \right\},$$

where $R(t, r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$ and $H^{-1} = (H_1^{-1}, \dots, H_N^{-1})$.

The following lemma is proved in [97], which summarizes some basic properties of Λ_μ . Recall that $H_1 = \min\{H_j : 1 \leq j \leq N\}$.

Lemma 6.8. Λ_μ has the following properties:

(i) The set Λ_μ is bounded:

$$\Lambda_\mu \subseteq \left\{ \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbf{R}_+^N : \langle \lambda, H^{-1} \rangle \leq \frac{N}{H_1} \right\}. \quad (134)$$

(ii) For all $\beta \in (0, 1]^N$ and $\lambda \in \Lambda_\mu$, the Hadamard product of β and λ , $\beta \circ \lambda = (\beta_1 \lambda_1, \dots, \beta_N \lambda_N) \in \Lambda_\mu$.

(iii) Λ_μ is convex; i.e., $b\lambda + (1-b)\eta \in \Lambda_\mu$ for all $\lambda, \eta \in \Lambda_\mu$ and $0 < b < 1$.

(iv) For every $a \in (0, \infty)^N$, $\sup_{\lambda \in \Lambda_\mu} \langle \lambda, a \rangle$ is achieved on the boundary of Λ_μ .

We call the boundary of Λ_μ , denoted by $\partial\Lambda_\mu$, the *Hausdorff dimension contour* of μ . See [97] for some examples for determining $\partial\Lambda_\mu$.

For any Borel set $E \subseteq \mathbf{R}^N$, we define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_c^+(E)} \Lambda_\mu. \quad (135)$$

Similar to the case for measures, we call the set $\bigcup_{\mu \in \mathcal{M}_c^+(E)} \partial\Lambda_\mu$ the *Hausdorff dimension contour* of E . It follows from Lemma 6.8 that, for every $a \in (0, \infty)^N$, the supremum $\sup_{\lambda \in \Lambda(E)} \langle \lambda, a \rangle$ is determined by the Hausdorff dimension contour of E .

The same proof of Theorem 3.10 in [97] yields the following result.

Theorem 6.9. Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian random field satisfying Condition (C1) on $I = [0, 1]^N$. Then for every Borel set $E \subseteq [0, 1]^N$,

$$\dim_{\mathcal{H}} X(E) = \min\{d, s(H, E)\} \quad \text{a.s.}, \quad (136)$$

where $s(H, E) = \sup_{\lambda \in \Lambda(E)} \langle \lambda, H^{-1} \rangle = \sup_{\mu \in \mathcal{M}_c^+(E)} s_\mu(E)$.

In the following, we give a more direct approach. Our results yield more geometric information about the quantity $s(H, E)$ as well.

For an arbitrary vector $(H_1, \dots, H_N) \in (0, 1)^N$, we consider the metric space (\mathbf{R}^N, ρ) , where ρ is defined by (37). For any $\beta > 0$ and $E \subseteq \mathbf{R}^N$, define the β -dimensional Hausdorff measure [in the metric ρ] of E by

$$\mathcal{H}_\rho^\beta(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : E \subseteq \bigcup_{n=1}^{\infty} B_\rho(r_n), r_n \leq \delta \right\}. \quad (137)$$

This is a metric outer measure and all Borel sets are \mathcal{H}_ρ^β -measurable. The corresponding Hausdorff dimension of E is defined by

$$\dim_{\mathcal{H}}^\rho E = \inf \{ \beta > 0 : \mathcal{H}_\rho^\beta(E) = 0 \}. \quad (138)$$

In some special cases, Hausdorff measure and dimension of this type have been applied by Kaufman [48], Hawkes [42], Taylor and Watson [90], and Testard [91] to study the hitting probability of space-time processes of Brownian motion and other processes.

Note that the metric space (\mathbf{R}^N, ρ) is complete and separable. Hence the following generalized Frostman’s lemma is a consequence of Theorem 8.17 in [67] and a remark on page 117 of the same reference. It can also be proved by using a 1977 result of Assouad [46, p. 137] on the quasi-helix and the classical Frostman’s lemma; see [91, p. 4] for a special case.

Lemma 6.10. *For any Borel set $E \subseteq \mathbf{R}^N$, $\mathcal{H}_\rho^\beta(E) > 0$ if and only if there exist a Borel probability measure on E and a positive constant c such that $\mu(B_\rho(x, r)) \leq cr^\beta$ for all $x \in \mathbf{R}^N$ and $r > 0$.*

We can now prove an equivalent result to Theorem 6.9, which extends the Hausdorff dimension result for the range of X in Theorem 6.1.

Theorem 6.11. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian random field satisfying Condition (C1) on $I = [0, 1]^N$. Then for every Borel set $E \subseteq [0, 1]^N$,*

$$\dim_{\mathcal{H}} X(E) = \min\{d; \dim_{\mathcal{H}}^\rho E\} \quad \text{a.s.} \tag{139}$$

Proof. Since the idea for proving (139) is quite standard, we only give a sketch of it. For any $\gamma > \dim_{\mathcal{H}}^\rho E$, there is a covering $\{B_\rho(r_n), n \geq 1\}$ of E such that $\sum_{n=1}^\infty (2r_n)^\gamma \leq 1$. Note that $X(E) \subseteq \cup_{n=1}^\infty X(B_\rho(r_n))$ and the uniform modulus of continuity of X implies that the diameter of $X(B_\rho(r_n))$ is at most $cr_n^{1-\delta}$, where $\delta \in (0, 1)$ is a constant. We can show that $\dim_{\mathcal{H}} X(E) \leq \gamma/(1-\delta)$ almost surely. The desired upper bound follows from the arbitrariness of γ and δ .

To prove the lower bound, let $\gamma \in (0, \min\{d; \dim_{\mathcal{H}}^\rho E\})$ be fixed. Then by using the generalized Frostman’s lemma [Lemma 6.10] one can show that there exists a probability measure μ on E such that

$$\int \int \frac{1}{\rho(s, t)^\gamma} \mu(ds) \mu(dt) < \infty. \tag{140}$$

This and Condition (C1) immediately imply

$$\mathbb{E} \int \int \frac{\mu(ds) \mu(dt)}{|X(s) - X(t)|^\gamma} < \infty. \tag{141}$$

Hence $\dim_{\mathcal{H}} X(E) \geq \min\{d; \dim_{\mathcal{H}}^\rho E\}$ almost surely. □

Combining Theorems 6.9 and 6.11, the invariance properties of $\dim_{\mathcal{H}}^\rho E$ and $s(H, E)$, we can derive the following alternative expression for $s(H, E)$. Of course, this can also be proved directly by using measure-theoretic methods.

Corollary 6.12. *For every Borel set $E \subseteq \mathbf{R}^N$, we have $\dim_{\mathcal{H}}^\rho E = s(H, E)$.*

As in the case of fractional Brownian sheets considered by Wu and Xiao [97], the image $X(E)$ has rich Fourier analytic and topological properties. For example, by modifying the proofs in [97], one can prove that if X is a Gaussian random field with stationary increments and spectral density satisfying (43) then $X(E)$ is a Salem set [46; 67] whenever $\dim_{\mathcal{H}}^\rho E \leq d$, and $X(E)$ has interior points whenever $\dim_{\mathcal{H}}^\rho E > d$ [It is an exercise to work out the details].

Finally, we consider the special case when $H = \langle \alpha \rangle$. Theorem 6.11 implies that for every Borel set $E \subseteq [0, 1]^N$,

$$\dim_{\mathcal{H}} X(E) = \min \left\{ d, \frac{1}{\alpha} \dim_{\mathcal{H}} E \right\} \quad \text{a.s.} \quad (142)$$

The following theorem gives us a uniform version of (142).

Theorem 6.13. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be as in Theorem 6.11 with $H = \langle \alpha \rangle$. If $N \leq \alpha d$ and X satisfies either Condition (C3) or (C3'), then with probability 1*

$$\dim_{\mathcal{H}} X(E) = \frac{1}{\alpha} \dim_{\mathcal{H}} E \text{ for all Borel sets } E \subseteq I, \quad (143)$$

and

$$\dim_{\mathcal{D}} X(E) = \frac{1}{\alpha} \dim_{\mathcal{D}} E \text{ for all Borel sets } E \subseteq I. \quad (144)$$

The proof of Theorem 6.13 is reminiscent to those in [51; 70; 97]. The key step is to apply Condition (C3) or (C3') to prove the following lemma. For simplicity, assume $I = [0, 1]^N$.

Lemma 6.14. *Suppose the assumptions of Theorem 6.13 hold, and let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$ be given constants. Then with probability 1, for all integers n large enough, there do not exist more than $2^{n\delta}$ distinct points of the form $t^j = 4^{-n} k^j$, where $k^j \in \{1, 2, \dots, 4^n\}^N$, such that*

$$|X(t^i) - X(t^j)| < 3 \cdot 2^{-n\beta} \quad \text{for } i \neq j. \quad (145)$$

Proof. A proof of Lemma 6.14 under Condition (C3) is given in [97]; see also [51]. The proof under (C3') is similar and is left to the reader as an exercise. \square

Both (142) and Theorem 6.13 imply that sometimes one can determine the packing dimension of the image $X(E)$ by the packing dimension of E . However, it is known that the conclusion is false if $N > \alpha d$ [87]. The method in [103] shows that if $X = \{X(t), t \in \mathbf{R}^N\}$ is a Gaussian random field satisfying (C1) with $H = \langle \alpha \rangle$ then for every Borel set $E \subseteq I$,

$$\dim_{\mathcal{D}} X(E) = \frac{1}{\alpha} \text{Dim}_{\alpha d} E \quad \text{a.s.}, \quad (146)$$

where $\text{Dim}_s E$ is the packing dimension profile of E defined by Falconer and Howroyd [37]. However, the analogous problem for general anisotropic Gaussian random fields has not been settled.

7 Hausdorff Dimension of the Level Sets and Hitting Probabilities

Under Conditions (C1) and (C2), we can study fractal properties of the level set $L_x = \{t \in I : X(t) = x\}$ ($x \in \mathbf{R}^d$) and the hitting probabilities of Gaussian random field X .

The following result determines the Hausdorff and packing dimensions of the level set.

Theorem 7.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian random field satisfying Conditions (C1) and (C2) on $I = [\varepsilon, 1]^N$.*

- (i) *If $\sum_{j=1}^N H_j^{-1} < d$, then for every $x \in \mathbf{R}^d$, $L_x = \emptyset$ a.s.*
- (ii) *If $\sum_{j=1}^N H_j^{-1} > d$, then for every $x \in \mathbf{R}^d$,*

$$\begin{aligned} \dim_{\mathcal{H}} L_x &= \dim_{\mathcal{P}} L_x \\ &= \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d, 1 \leq k \leq N \right\} \\ &= \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d \quad \text{if} \quad \sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^k \frac{1}{H_j}, \end{aligned} \tag{147}$$

with positive probability.

Remark 7.2. In the critical case when $\sum_{j=1}^N H_j^{-1} = d$, it is believed that $L_x = \emptyset$ a.s. In the Brownian sheet case, this was proved by Orey and Pruitt [76, Theorem 3.4]. It also follows from a potential theoretic result of [50]. If X is a fractional Brownian motion of index $\alpha \in (0, 1)$, then an argument of [86] can be modified to show $L_x = \emptyset$ a.s. However, the problem whether $L_x = \emptyset$ a.s. for more general Gaussian random fields remains open. A proof would require Condition (C3) or (C3') and some extra conditions on the function $E(|X_1(t) - X_1(s)|^2)$.

Proof of Theorem 7.1. Similar to the proof of Theorem 5 in [7], we divide the proof of Theorem 7.1 into two steps. In Step one, we prove (i) and the upper bound for $\dim_{\mathcal{P}} L_x$ in (147); and in Step two we prove the lower bound for $\dim_{\mathcal{H}} L_x$ by constructing a random measure on L_x and using a capacity argument. Moreover, the last equality in (147) follows from Lemma 6.2.

First we prove

$$\overline{\dim}_{\mathcal{P}} L_x \leq \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d, 1 \leq k \leq N \right\} \quad \text{a.s.} \tag{148}$$

and $L_x = \emptyset$ a.s. whenever the right hand side of (148) is negative. It can be verified that the latter is equivalent to $\sum_{j=1}^N H_j^{-1} < d$.

For an integer $n \geq 1$, divide the interval $[\varepsilon, 1]^N$ into m_n sub-rectangles $R_{n,\ell}$ of side lengths n^{-1/H_j} ($j = 1, \dots, N$). Then $m_n \leq c n^{\sum_{j=1}^N (1/H_j)}$. Let $0 < \delta < 1$ be fixed and let $\tau_{n,\ell}$ be the lower-left vertex of $R_{n,\ell}$. Then

$$\begin{aligned} \mathbb{P}\{x \in X(R_{n,\ell})\} &\leq \mathbb{P}\left\{\max_{s,t \in R_{n,\ell}} |X(s) - X(t)| \leq n^{-(1-\delta)}; x \in X(R_{n,\ell})\right\} \\ &\quad + \mathbb{P}\left\{\max_{s,t \in R_{n,\ell}} |X(s) - X(t)| > n^{-(1-\delta)}\right\} \quad (149) \\ &\leq \mathbb{P}\left\{|X(\tau_{n,\ell}) - x| \leq n^{-(1-\delta)}\right\} + e^{-c n^{2\delta}} \\ &\leq c n^{-(1-\delta)d}. \end{aligned}$$

In the above we have applied Lemma 2.1 in [85] to get the second inequality. If $\sum_{j=1}^N H_j^{-1} < d$, we choose $\delta > 0$ such that $(1-\delta)d > \sum_{j=1}^N H_j^{-1}$. Let N_n be the number of rectangles $R_{n,\ell}$ such that $x \in X(R_{n,\ell})$. It follows from (149) that

$$\mathbb{E}(N_n) \leq c n^{\sum_{j=1}^N (1/H_j)} n^{-(1-\delta)d} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (150)$$

Since the random variables N_n are integer-valued, (150) and Fatou's lemma imply that a.s. $N_n = 0$ for infinitely many integers $n \geq 1$. Therefore $L_x = \emptyset$ almost surely.

Now we assume $\sum_{j=1}^N H_j^{-1} > d$ and define a covering $\{R'_{n,\ell}\}$ of L_x by $R'_{n,\ell} = R_{n,\ell}$ if $x \in X(R_{n,\ell})$ and $R'_{n,\ell} = \emptyset$ otherwise. We will show that there are N different ways to cover L_x by using cubes of the same side-lengths and each of these ways leads to an upper bound for $\overline{\dim}_{\emptyset} L_x$.

For every $1 \leq k \leq N$, $R'_{n,\ell}$ can be covered by $n^{\sum_{j=k+1}^N (H_k^{-1} - H_j^{-1})}$ cubes of side-length n^{-1/H_k} . Thus we can cover the level set L_x by a sequence of cubes of side-length n^{-1/H_k} . Denote the number of such cubes by $M_{n,k}$. Using (149) again, we have

$$\begin{aligned} \mathbb{E}(M_{n,k}) &\leq c_{7,1} n^{\sum_{j=1}^N H_j^{-1}} n^{-(1-\delta)d} \cdot n^{\sum_{j=k+1}^N (H_k^{-1} - H_j^{-1})} \\ &= c_{7,1} n^{(N-k)H_k^{-1} + \sum_{j=1}^k H_j^{-1} - (1-\delta)d}. \end{aligned} \quad (151)$$

Now let η be the constant defined by

$$\eta = (N-k)H_k^{-1} + \sum_{j=1}^k H_j^{-1} - (1-2\delta)d. \quad (152)$$

We consider the sequence of integers $n_i = 2^i$ ($i \geq 1$). Then by (151), the Markov inequality and the Borel-Cantelli lemma we see that almost surely $M_{n_i,k} \leq c n_i^\eta$ for all i large enough. This implies that $\overline{\dim}_{\emptyset} L_x \leq H_k \eta$ almost surely. Letting $\delta \downarrow 0$ along rational numbers, we have

$$\overline{\dim}_{\emptyset} L_x \leq \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d \quad \text{a.s.} \quad (153)$$

Optimizing (153) over $k = 1, \dots, N$ yields (148).

In order to prove the lower bound for $\dim_{\mathscr{H}} L_x$ in (147), we assume that $\sum_{j=1}^{k-1} H_j^{-1} \leq d < \sum_{j=1}^k H_j^{-1}$ for some $1 \leq k \leq N$. Let $\delta > 0$ be a small constant such that

$$\gamma := \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k(1 + \delta)d > N - k. \tag{154}$$

This is possible by Lemma 6.2. Note that if we can prove that there is a constant $c_{\gamma,2} > 0$, independent of δ , such that

$$\mathbb{P}\{\dim_{\mathscr{H}} L_x \geq \gamma\} \geq c_{\gamma,2}, \tag{155}$$

then the lower bound in (147) will follow by letting $\delta \downarrow 0$.

Our proof of (155) is based on the capacity argument due to Kahane [46]. Similar methods have been used by Adler [1], Testard [91], and Xiao [98] to certain Gaussian and stable random fields.

Let \mathscr{M}_γ^+ be the space of all non-negative measures on \mathbf{R}^N with finite γ -energy [recall (84)]. It is known [1] that \mathscr{M}_γ^+ is a complete metric space under the metric

$$\|\mu\|_\gamma^2 = \iint \frac{\mu(dt)\mu(ds)}{|t-s|^\gamma}. \tag{156}$$

We define a sequence of random positive measures $\mu_n := \mu_n(x, \bullet)$ on the Borel sets C of $[\varepsilon, 1]^N$ by

$$\begin{aligned} \mu_n(C) &= \int_C (2\pi n)^{d/2} \exp\left(-\frac{n|X(t)-x|^2}{2}\right) dt \\ &= \int_C \int_{\mathbf{R}^d} \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, X(t)-x \rangle\right) d\xi dt. \end{aligned} \tag{157}$$

It follows from [46, p. 206] or [91, p. 17] that if there exist positive and finite constants $c_{\gamma,3}$, $c_{\gamma,4}$ and $c_{\gamma,5}$ such that

$$\mathbb{E}(\|\mu_n\|) \geq c_{\gamma,3}, \quad \mathbb{E}(\|\mu_n\|^2) \leq c_{\gamma,4}, \tag{158}$$

$$\mathbb{E}(\|\mu_n\|_\gamma) \leq c_{\gamma,5}, \tag{159}$$

where $\|\mu_n\| = \mu_n([\varepsilon, 1]^N)$ denotes the total mass of μ_n , then there is a subsequence of $\{\mu_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \rightarrow \mu$ in \mathscr{M}_γ^+ and μ is strictly positive with probability $\geq c_{\gamma,3}^2/(2c_{\gamma,4})$. In this case, it follows from (157) that μ has its support in L_x almost surely. Moreover, (159) and the monotone convergence theorem together imply that the γ -energy of μ is finite. Hence Frostman's theorem yields (155) with $c_{\gamma,2} = c_{\gamma,3}^2/(2c_{\gamma,4})$.

It remains to verify (158) and (159). By Fubini's theorem we have

$$\begin{aligned}
 E(\|\mu_n\|) &= \int_{[\varepsilon,1]^N} \int_{\mathbf{R}^d} e^{-i\langle \xi, x \rangle} \exp\left(-\frac{|\xi|^2}{2n}\right) E \exp\left(i\langle \xi, X(t) \rangle\right) d\xi dt \\
 &= \int_{[\varepsilon,1]^N} \int_{\mathbf{R}^d} e^{-i\langle \xi, x \rangle} \exp\left(-\frac{1}{2}(n^{-1} + \sigma^2(t))|\xi|^2\right) d\xi dt \\
 &= \int_{[\varepsilon,1]^N} \left(\frac{2\pi}{n^{-1} + \sigma^2(t)}\right)^{d/2} \exp\left(-\frac{|x|^2}{2(n^{-1} + \sigma^2(t))}\right) dt \\
 &\geq \int_{[\varepsilon,1]^N} \left(\frac{2\pi}{1 + \sigma^2(t)}\right)^{d/2} \exp\left(-\frac{|x|^2}{2\sigma^2(t)}\right) dt := c_{7,3}.
 \end{aligned} \tag{160}$$

Denote by I_{2d} the identity matrix of order $2d$ and $\text{Cov}(X(s), X(t))$ the covariance matrix of the random vector $(X(s), X(t))$. Let $\Gamma = n^{-1}I_{2d} + \text{Cov}(X(s), X(t))$ and $(\xi, \eta)'$ be the transpose of the row vector (ξ, η) . Then $E(\|\mu_n\|^2)$ is equal to

$$\begin{aligned}
 &\int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{-i\langle \xi + \eta, x \rangle} \exp\left(-\frac{1}{2}(\xi, \eta) \Gamma (\xi, \eta)'\right) d\xi d\eta ds dt \\
 &= \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{(2\pi)^d}{\sqrt{\det \Gamma}} \exp\left(-\frac{1}{2}(x, x) \Gamma^{-1} (x, x)'\right) ds dt \\
 &\leq \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{(2\pi)^d}{[\det \text{Cov}(X_0(s), X_0(t))]^{d/2}} ds dt.
 \end{aligned} \tag{161}$$

It follows from Conditions (C1), (C2) and (53) that for all $s, t \in [\varepsilon, 1]^N$,

$$\det \text{Cov}(X_0(s), X_0(t)) \geq c_{7,6} \sum_{j=1}^N |s_j - t_j|^{2H_j}. \tag{162}$$

We combine (161), (162) and then apply Lemma 6.3, repeatedly, to obtain

$$E(\|\mu_n\|^2) \leq c_{7,7} \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{ds dt}{\left[\sum_{j=1}^N |s_j - t_j|^{2H_j}\right]^{d/2}} := c_{7,4} < \infty. \tag{163}$$

This is similar to (113)–(117) in the proof of Theorem 6.1. Thus we have shown (158) holds.

Similar to (161), we find that $E(\|\mu_n\|_\gamma)$ is equal to

$$\begin{aligned}
 &\int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{ds dt}{|s - t|^\gamma} \\
 &\quad \times \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{-i\langle \xi + \eta, x \rangle} \exp\left(-\frac{1}{2}(\xi, \eta) \Gamma (\xi, \eta)'\right) d\xi d\eta \\
 &\leq c_{7,8} \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{ds dt}{\left(\sum_{j=1}^N |s_j - t_j|\right)^\gamma \left(\sum_{j=1}^N |s_j - t_j|^{2H_j}\right)^{d/2}} \\
 &\leq c_{7,9} \int_0^1 dt_N \cdots \int_0^1 \frac{dt_1}{\left(\sum_{j=1}^N t_j^{H_j}\right)^d \left(\sum_{j=1}^N t_j\right)^\gamma},
 \end{aligned} \tag{164}$$

where the two inequalities follow from (162) and a change of variables. Note that the last integral in (164) is similar to (122) and can be estimated by using Lemma 6.4 in the same way as in the proof of (123)–(126). Moreover, we can take δ small enough, say $\delta < \delta_0$, so that the γ defined in (154) is bounded away from $N - k$ and $N - k + 1$. This implies that $E(\|\mu_n\|_\gamma) \leq c_{7,9}$ which is independent of δ . This proves (159) and hence Theorem 7.1.

In light of Theorem 7.1, it is of interest to further study the following question about fractal properties of the level sets.

Question 7.3. Determine the exact Hausdorff and packing measure functions for the level set L_x .

Questions 7.3 is closely related to the regularity properties such as the laws of the iterated logarithm of the local times of X . The latter will be considered in Section 8. When X is an (N, d) -fractional Brownian motion with index α , Xiao [102] proved that $\varphi_\gamma(r) = r^{N-\alpha d}(\log \log 1/r)^{\alpha d/N}$ is an exact Hausdorff measure function for L_x . In Theorem 8.11 we give a partial result [i.e., lower bound] for the exact Hausdorff measure of the level set L_x . It seems that the method in [102] may be modified to determine an exact Hausdorff measure function for the level sets of Gaussian random fields satisfying (C3) or (C3').

So far no exact packing measure results have been proved for the level sets of fractional Brownian motion or the Brownian sheet. These problems are related to the liminf behavior of the local times of X which are more difficult to study.

More general than level sets, one can consider the following questions:

Question 7.4. Given a Borel set $F \subseteq \mathbf{R}^d$, when is $P\{X(I) \cap F \neq \emptyset\}$ positive?

Question 7.5. If $P\{X(I) \cap F \neq \emptyset\} > 0$, what are the Hausdorff and packing dimensions of the inverse image $X^{-1}(F) \cap I$?

Question 7.4 is an important question in potential theory of random fields. Complete answer has only been known for a few types of random fields with certain Markov structures. We mention that Khoshnevisan and Shi [50] proved that if X is an (N, d) -Brownian sheet, then for every Borel set $F \subseteq \mathbf{R}^d$,

$$P\{X(I) \cap F \neq \emptyset\} > 0 \iff \mathcal{C}_{d-2N}(F) > 0. \quad (165)$$

Recall that \mathcal{C}_α denotes the Bessel-Riesz capacity of order α . Dalang and Nualart [26] have recently extended the methods of [50] and proved similar results for the solution of a system of d nonlinear hyperbolic stochastic partial differential equations with two variables. In this context, we also mention that Khoshnevisan and Xiao [52; 53; 54; 56] and Khoshnevisan, Xiao, and Zhong [57] have established systematic potential theoretical results for additive Lévy processes in \mathbf{R}^d . The arguments in the aforementioned work rely on the multiparameter martingale theory [49].

For random fields with general dependence structures, it is more difficult to solve Question 7.4 completely. Instead, one can look for sufficient conditions and necessary conditions on F so that $P\{X(I) \cap F \neq \emptyset\} > 0$. For example, when X is an (N, d) -fractional Brownian motion, Testard [91] and Xiao [104] proved the following results:

$$\mathcal{C}_{d-N/\alpha}(F) > 0 \Rightarrow P\{X(I) \cap F \neq \emptyset\} > 0 \Rightarrow \mathcal{H}^{d-N/\alpha}(F) > 0. \quad (166)$$

Similar results for the solution to a non-linear stochastic heat equation with multiplicative noise have been proved recently by Dalang, Khoshnevisan and Nualart [23].

The following theorem is an analogue of (166) for all Gaussian random fields X satisfying Conditions (C1) and (C2).

Theorem 7.6. *Assume that an (N, d) -Gaussian random field $X = \{X(t), t \in \mathbf{R}^N\}$ satisfies Conditions (C1) and (C2) on I and $d > Q$. Then for every Borel set $F \subseteq \mathbf{R}^d$,*

$$c_{7,10} \mathcal{C}_{d-Q}(F) \leq P\{X(I) \cap F \neq \emptyset\} \leq c_{7,11} \mathcal{H}_{d-Q}(F), \quad (167)$$

where $Q = \sum_{j=1}^N H_j^{-1}$, $c_{7,10}$ and $c_{7,11}$ are positive constants depending on I, F and H only.

Remark 7.7. When $d < Q$, Theorem 7.1 tells us that X hits points, hence (167) holds automatically. When $d = Q$, our proof below shows that the lower bound in (167) remains to be true provided \mathcal{C}_0 means the logarithmic capacity [see (86)]. This can be seen by estimating the integral in (173) when $d = Q$. However, if $\mathcal{C}_0(F) > 0$, then the upper bound in (167) becomes ∞ , thus not informative.

Proof of Theorem 7.6. The lower bound in (167) can be proved by using a second moment argument. In fact one can follow the method in [26; 50; 91] to prove the lower bound in (167).

In the following, we provide an alternative proof for the lower bound which is similar to that of Theorem 7.1. For any Borel probability measure κ on F with $\mathcal{E}_{d-Q}(\kappa) < \infty$ and for all integers $n \geq 1$, we consider a family of random measures ν_n on I defined by

$$\begin{aligned} \int_I f(t) \nu_n(dt) &= \int_I \int_{\mathbf{R}^d} (2\pi n)^{d/2} \exp(-n|X(t) - x|^2) f(t) \kappa(dx) dt \\ &= \int_I \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, X(t) - x \rangle\right) f(t) d\xi \kappa(dx) dt, \end{aligned} \quad (168)$$

where f is an arbitrary measurable function on I . We claim that the following two inequalities hold:

$$E(\|\nu_n\|) \geq c_{7,12}, \quad E(\|\nu_n\|^2) \leq c_{7,13} \mathcal{E}_{d-Q}(\kappa), \quad (169)$$

where the constants $c_{7,12}$ and $c_{7,13}$ are independent of κ and n .

Since the proof of the first inequality in (169) is very similar to (160) in the proof of Theorem 7.1, we only prove the second inequality in (169).

Denote by I_{2d} the identity matrix of order $2d$ and $\text{Cov}(X(s), X(t))$ the covariance matrix of the random vector $(X(s), X(t))$. Let $\Gamma_n = n^{-1}I_{2d} + \text{Cov}(X(s), X(t))$ and $(\xi, \eta)'$ be the transpose of the row vector (ξ, η) . Since Γ_n is positive definite, we have

$$\begin{aligned} \mathbb{E}(\|\nu_n\|^2) &= \int_I \int_I \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} e^{-i(\langle \xi, x \rangle + \langle \eta, y \rangle)} \\ &\quad \times \exp\left(-\frac{1}{2}(\xi, \eta) \Gamma_n (\xi, \eta)'\right) d\xi d\eta \kappa(dx) \kappa(dy) ds dt \quad (170) \\ &= \int_I \int_I \int_{\mathbf{R}^{2d}} \frac{(2\pi)^d}{\sqrt{\det \Gamma_n}} \exp\left(-\frac{1}{2}(x, y) \Gamma_n^{-1} (x, y)'\right) \kappa(dx) \kappa(dy) ds dt. \end{aligned}$$

By modifying an argument from [91], we can prove that, under conditions (C1) and (C2), we have

$$\frac{(2\pi)^d}{\sqrt{\det \Gamma_n}} \exp\left(-\frac{1}{2}(x, y) \Gamma_n^{-1} (x, y)'\right) \leq \frac{c_{7,14}}{\max\{\rho^d(s, t), |x - y|^d\}} \quad (171)$$

for all $s, t \in I$ and $x, y \in \mathbf{R}^d$; see [16] for details. Hence, it follows from (170) and (171) that

$$\mathbb{E}(\|\nu_n\|^2) \leq c_{7,14} \int_I \int_I \int_{\mathbf{R}^{2d}} \frac{1}{\max\{\rho^d(s, t), |x - y|^d\}} \kappa(dx) \kappa(dy) ds dt. \quad (172)$$

We can verify that for all $x, y \in \mathbf{R}^d$,

$$\int_I \int_I \frac{ds dt}{\max\{\rho^d(s, t), |x - y|^d\}} \leq c_{7,15} |x - y|^{-(d-Q)}, \quad (173)$$

where $c_{7,15} > 0$ is a constant. This can be done by breaking the integral in (173) over the regions $\{(s, t) \in I \times I : \rho(s, t) \leq |x - y|\}$ and $\{(s, t) \in I \times I : \rho(s, t) > |x - y|\}$, and estimate them separately. It is clear that (170), (173) and Fubini's theorem imply the second inequality in (169).

By using (169) and the Paley-Zygmund inequality [46], one can verify that there is a subsequence of $\{\nu_n, n \geq 1\}$ that converges weakly to a finite measure ν which is positive with positive probability [depending on $c_{7,12}$ and $c_{7,13}$ only] and ν also satisfies (169). Since ν is supported on $X^{-1}(F) \cap I$, we use the Paley-Zygmund inequality again to derive

$$\mathbb{P}\{X(I) \cap F \neq \emptyset\} \geq \mathbb{P}\{\|\nu\| > 0\} \geq \frac{[\mathbb{E}(\|\nu\|)]^2}{\mathbb{E}[\|\nu\|^2]} \geq \frac{c_{7,16}}{\mathcal{E}_{d-Q}(\kappa)}, \quad (174)$$

where $c_{7,16} = c_{7,12}^2 / c_{7,13}$. This implies the lower bound in (167).

Our proof of the upper bound in (167) relies on the following lemma on the hitting probability of X , whose proof will be deferred to the end of this section.

Lemma 7.8. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian random field satisfying Conditions (C1) and (C2) on I . Then there exists a constant $c_{7.17} > 0$ such that for all $x \in I$ and $y \in \mathbf{R}^d$,*

$$\mathbb{P}\left\{\inf_{t \in B_\rho(x,r)} |X(t) - y| \leq r\right\} \leq c_{7.17} r^d. \tag{175}$$

Now we proceed to prove the upper bound in (167) by using a simple covering argument. Choose and fix an arbitrary constant $\gamma > \mathcal{H}_{d-Q}(F)$. By the definition of $\mathcal{H}_{d-Q}(F)$, there is a sequence of balls $\{B(y_j, r_j), j \geq 1\}$ in \mathbf{R}^d such that

$$F \subseteq \bigcup_{j=1}^{\infty} B(y_j, r_j) \quad \text{and} \quad \sum_{j=1}^{\infty} (2r_j)^{d-Q} \leq \gamma. \tag{176}$$

Clearly we have

$$\{F \cap X(I) \neq \emptyset\} \subseteq \bigcup_{j=1}^{\infty} \{B(y_j, r_j) \cap X(I) \neq \emptyset\}. \tag{177}$$

For every $j \geq 1$, we divide the interval I into cr_j^{-Q} intervals of side-lengths r_j^{-1/H_ℓ} ($\ell = 1, \dots, N$). Hence I can be covered by at most cr_j^{-Q} many balls of radius r_j in the metric ρ . It follows from Lemma 7.8 that

$$\mathbb{P}\{B(y_j, r_j) \cap X(I) \neq \emptyset\} \leq cr_j^{d-Q}. \tag{178}$$

Combining (177) and (178) we derive that $\mathbb{P}\{F \cap X(I) \neq \emptyset\} \leq c\gamma$. Since $\gamma > \mathcal{H}_{d-Q}(F)$ is arbitrary, the upper bound in (167) follows.

The following are some further remarks and open questions related to Theorem 7.6.

Remark 7.9. For any Borel set $F \subseteq \mathbf{R}^d$, Theorem 7.6 provides a sufficient condition and a necessary condition for $\mathbb{P}\{X^{-1}(F) \cap I \neq \emptyset\} > 0$. It is interesting to determine the Hausdorff and packing dimensions of $X^{-1}(F)$ when it is not empty. Recently, Biermé, Lacaux and Xiao (2007) determined the Hausdorff dimension of $X^{-1}(F)$. Namely, they proved that if $\dim_{\mathcal{H}} F > d - \sum_{\ell=1}^N (1/H_\ell)$, then

$$\begin{aligned} & \|\dim_{\mathcal{H}}(X^{-1}(F) \cap I)\|_{L^\infty(\mathbb{P})} \\ &= \min_{1 \leq k \leq N} \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\mathcal{H}} F) \right\}, \end{aligned} \tag{179}$$

where, for all nonnegative random variables Y ,

$$\|Y\|_{L^\infty(\mathbb{P})} = \sup \{\theta : Y \geq \theta \text{ on an event } E \text{ with } \mathbb{P}(E) > 0\}. \tag{180}$$

However, except for the special case of $F = \{x\}$, there have been no results on the packing dimension of $X^{-1}(F)$ for a given Borel set $F \subseteq \mathbf{R}^d$.

Remark 7.10. Note that the event on which (179) holds depends on F . Motivated by the results in [70], we may ask the following natural question: When $\sum_{\ell=1}^N H_\ell^{-1} > d$, is it possible to have a single event $\Omega_1 \subseteq \Omega$ of positive probability such that on Ω_1 (179) holds for all Borel sets $F \subseteq \mathbf{R}^d$?

Here are some partial answers. If in addition to Conditions (C1) and (C2), we also assume Condition (C3) or (C3') holds and $H_1 = H_2 = \dots = H_N$, then one can modify the proofs in [70] to show that the answer to the above question is affirmative. In general, it can be proved that, when $\sum_{\ell=1}^N H_\ell^{-1} > d$, the upper bound in (179) holds almost surely for all Borel sets $F \subseteq \mathbf{R}^d$. But it is likely that the lower bound may not hold uniformly due to the anisotropy of X .

Remark 7.11. The method for proving Theorem 7.6 may be extended to provide necessary conditions and sufficient conditions for $P\{X(E) \cap F \neq \emptyset\} > 0$, where $E \subseteq (0, \infty)^N$ and $F \subseteq \mathbf{R}^d$ are arbitrary Borel sets. Some difficulties arise when both E and F are fractal sets. Testard [91] obtained some partial results for fractional Brownian motion and, for every fixed Borel set $E \subseteq (0, \infty)^N$ (or $F \subseteq \mathbf{R}^d$), Xiao [104] characterized the “smallest” set F (or E) such that $P\{X(E) \cap F \neq \emptyset\} > 0$. No such results on anisotropic Gaussian random fields have been proved.

Finally, let us prove Lemma 7.8. There are two ways to prove (175). One is to use the argument in the proof of Proposition 4.4 of [22] and the other is reminiscent to the proof of Lemma 3.1 in [100]. While the former method is slightly simpler, the latter can be adapted to establish hitting probability estimates of the form (194) below for anisotropic Gaussian random fields. Hence we will use an argument similar to that in [100].

Proof of Lemma 7.8. For every integer $n \geq 1$, let $\varepsilon_n = r \exp(-2^{n+1})$ and denote by $N_n = N_\rho(B_\rho(x, r), \varepsilon_n)$ the minimum number of ρ -balls of radius ε_n that are needed to cover $B_\rho(x, r)$. Note that $N_n \leq c \exp(Q2^{n+1})$ [recall that $Q = \sum_{\ell=1}^N (1/H_\ell)$].

Let $\{t_i^{(n)}, 1 \leq i \leq N_n\}$ be a set of the centers of open balls with radius ε_n that cover $B_\rho(x, r)$. Denote

$$r_n = \beta \varepsilon_n 2^{(n+1)/2}, \tag{181}$$

where $\beta \geq c_{4,10}$ is a constant to be determined later. Here $c_{4,10}$ is the constant in (69).

For all integers $n \geq 1, 1 \leq j \leq n$ and $1 \leq i \leq N_n$, we consider the following events

$$A_i^{(j)} = \left\{ |X(t_i^{(j)}) - y| \leq r + \sum_{k=j}^\infty r_k \right\}, \tag{182}$$

$$A^{(n)} = \bigcup_{j=1}^n \bigcup_{i=1}^{N_j} A_i^{(j)} = A^{(n-1)} \cup \bigcup_{i=1}^{N_n} A_i^{(n)}.$$

Then by a chaining argument, the triangle inequality and (69), we have

$$\mathbb{P}\left\{\inf_{t \in \mathcal{B}_\rho(x,r)} |X(t) - y| \leq r\right\} \leq \lim_{n \rightarrow \infty} \mathbb{P}(A^{(n)}). \quad (183)$$

By (182), we have

$$\mathbb{P}(A^{(n)}) \leq \mathbb{P}(A^{(n-1)}) + \mathbb{P}(A^{(n)} \setminus A^{(n-1)}) \quad (184)$$

and

$$\mathbb{P}(A^{(n)} \setminus A^{(n-1)}) \leq \sum_{i=1}^{N_n} \mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}), \quad (185)$$

where i' is chosen so that $\rho(t_i^{(n)}, t_{i'}^{(n-1)}) < \varepsilon_{n-1}$. Note that

$$\begin{aligned} & \mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}) \\ &= \mathbb{P}\left\{|X(t_i^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k, |X(t_{i'}^{(n-1)}) - y| > r + \sum_{k=n-1}^{\infty} r_k\right\} \\ &\leq \mathbb{P}\left\{|X(t_i^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k, |X(t_i^{(n)}) - X(t_{i'}^{(n-1)})| \geq r_{n-1}\right\}. \end{aligned} \quad (186)$$

By the elementary properties of Gaussian random variables, we can write

$$\frac{X(t_i^{(n)}) - X(t_{i'}^{(n-1)})}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)})} = \eta \frac{X(t_i^{(n)})}{\sigma(t_i^{(n)})} + \sqrt{1 - \eta^2} \Xi, \quad (187)$$

where

$$\eta = \frac{\mathbb{E}\left[\left(X_1(t_i^{(n)}) - X_1(t_{i'}^{(n-1)})\right) X_1(t_i^{(n)})\right]}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)}) \sigma(t_i^{(n)})} \quad (188)$$

and where Ξ is a centered Gaussian random vector in \mathbf{R}^d with the identity matrix as its covariance matrix and, moreover, Ξ is independent of $X(t_i^{(n)})$.

We observe that

$$r + \sum_{k=n}^{\infty} r_k \leq r + \sum_{k=0}^{\infty} r_k \leq \left(1 + c \int_0^{\infty} \exp(-\alpha x^2) dx\right) r := c_{7,18} r. \quad (189)$$

It follows from Condition (C1) that (186) is bounded by

$$\begin{aligned} & \mathbb{P}\left\{|X(t_i^{(n)}) - y| \leq c_{7,18} r, |\Xi| \geq \frac{1}{\sqrt{1 - \eta^2}} \left[\frac{r_{n-1}}{\rho(t_i^{(n)}, t_{i'}^{(n-1)})} - \eta \frac{X(t_i^{(n)})}{\sigma(t_i^{(n)})}\right]\right\} \\ &\leq \mathbb{P}\left\{|X(t_i^{(n)}) - y| \leq c_{7,18} r, |\Xi| \geq \frac{\beta d}{2} 2^{n/2}\right\} \\ &\quad + \mathbb{P}\left\{|X(t_i^{(n)}) - y| \leq c_{7,18} r, \eta \frac{|X(t_i^{(n)})|}{\sigma(t_i^{(n)})} \geq \frac{\beta d}{2} 2^{n/2}\right\} \\ &:= I_1 + I_2. \end{aligned} \quad (190)$$

By the independence of Ξ and $X(t_i^{(n)})$, we have

$$\begin{aligned}
 I_1 &= \mathbb{P} \left\{ |X(t_i^{(n)}) - y| \leq c_{7,18} r \right\} \cdot \mathbb{P} \left\{ |\Xi| \geq \beta d 2^{-1+(n/2)} \right\} \\
 &\leq c_{7,19} r^d \exp\left(-\frac{(\beta d)^2}{16} 2^n\right).
 \end{aligned}
 \tag{191}$$

On the other hand,

$$\begin{aligned}
 I_2 &\leq \int_{\{|u-y| \leq c_{7,18} r, |u| \geq \beta d 2^{-1+(n/2)} \sigma(t_i^{(n)})\}} \left(\frac{1}{2\pi}\right)^{d/2} \frac{1}{\sigma^d(t_i^{(n)})} \exp\left(-\frac{|u|^2}{2\sigma^2(t_i^{(n)})}\right) du \\
 &\leq c_{7,20} \int_{\{|u-y| \leq c_{7,18} r\}} \frac{1}{\sigma^d(t_i^{(n)})} \exp\left(-\frac{|u|^2}{4\sigma^2(t_i^{(n)})}\right) du \cdot \exp\left(-\frac{(\beta d)^2}{16} 2^n\right) \\
 &\leq c_{7,21} r^d \exp\left(-\frac{(\beta d)^2}{16} 2^n\right).
 \end{aligned}
 \tag{192}$$

Combining (184) through (192) and choosing $\beta \geq c_{4,10}$ satisfying $(\beta d)^2 > 32$, we obtain

$$\begin{aligned}
 \mathbb{P}(A^{(n)}) &\leq \mathbb{P}(A^{(n-1)}) + c_{7,22} N_n r^d \exp\left(-\frac{(\beta d)^2}{16} 2^n\right) \\
 &\leq c_{7,23} \left[\sum_{k=0}^{\infty} N_k \exp\left(-\frac{(\beta d)^2}{16} 2^k\right) \right] r^d \\
 &\leq c_{7,24} r^d.
 \end{aligned}
 \tag{193}$$

Therefore, (175) follows from (183) and (193).

When X is an (N, d) -fractional Brownian motion of index α , Xiao [104] proved the following hitting probability result: If $N < \alpha d$, then there exist positive and finite constants $c_{7,25}$ and $c_{7,26}$, depending only on N, d and α , such that for any $r > 0$ small enough and any $y \in \mathbf{R}^d$ with $|y| \geq r$, we have

$$\begin{aligned}
 c_{7,25} \left(\frac{r}{|y|}\right)^{d-\frac{N}{\alpha}} &\leq \mathbb{P} \left\{ \exists t \in \mathbf{R}^N \text{ such that } |X(t) - y| < r \right\} \\
 &\leq c_{7,26} \left(\frac{r}{|y|}\right)^{d-\frac{N}{\alpha}}.
 \end{aligned}
 \tag{194}$$

It would be interesting and useful to establish analogous results for all Gaussian random fields satisfying Conditions (C1) and (C2). Such an estimate will be useful in studying the escape rate and exact packing measure of the sample paths of Gaussian random fields; see [105] for the special case of fractional Brownian motion.

8 Local Times and Their Joint Continuity

We start by briefly recalling some aspects of the theory of local times. For excellent surveys on local times of random and deterministic vector fields, we refer to [31; 41].

Let $X(t)$ be a Borel vector field on \mathbf{R}^N with values in \mathbf{R}^d . For any Borel set $T \subseteq \mathbf{R}^N$, the occupation measure of X on T is defined as the following Borel measure on \mathbf{R}^d :

$$\mu_T(\bullet) = \lambda_N\{t \in T : X(t) \in \bullet\}. \quad (195)$$

If μ_T is absolutely continuous with respect to the Lebesgue measure λ_d , we say that $X(t)$ has a *local time* on T , and define its local time, $L(\bullet, T)$, as the Radon–Nikodým derivative of μ_T with respect to λ_d , i.e.,

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x) \quad \forall x \in \mathbf{R}^d. \quad (196)$$

In the above, x is the so-called *space variable*, and T is the *time variable*. Sometimes, we write $L(x, t)$ in place of $L(x, [0, t])$. It is clear that if X has local times on T , then for every Borel set $S \subseteq T$, $L(x, S)$ also exists.

By standard martingale and monotone class arguments, one can deduce that the local times of X have a version, still denoted by $L(x, T)$, such that it is a kernel in the following sense:

- (i) For each fixed $S \subseteq T$, the function $x \mapsto L(x, S)$ is Borel measurable in $x \in \mathbf{R}^d$.
- (ii) For every $x \in \mathbf{R}^d$, $L(x, \cdot)$ is Borel measure on $\mathcal{B}(T)$, the family of Borel subsets of T .

Moreover, $L(x, T)$ satisfies the following *occupation density formula*: For every Borel set $T \subseteq \mathbf{R}^N$ and for every measurable function $f : \mathbf{R}^d \rightarrow \mathbf{R}_+$,

$$\int_T f(X(t)) dt = \int_{\mathbf{R}^d} f(x) L(x, T) dx. \quad (197)$$

See [41, Theorems 6.3 and 6.4].

Suppose we fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i]$ in \mathcal{A} . Then, whenever we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \dots, t_N) \in \mathbf{R}^d \times \prod_{i=1}^N [0, h_i]$, X is said to have a *jointly continuous local time* on T . When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X_T^{-1}(x) = \{t \in T : X(t) = x\}; \quad (198)$$

see [1] for details. In other words, local times often act as a natural measure on the level sets of X for applying the capacity argument. As such, they are useful in studying the various fractal properties of level sets and inverse images of the vector field X [13; 34; 70; 81; 102].

First we consider the existence of the local times of Gaussian random fields.

Theorem 8.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian field defined by (36) and suppose Condition (C1) is satisfied on I . Then X has a local time $L(x, I) \in L^2(\mathbf{P} \times \lambda_d)$ if and only if $d < \sum_{j=1}^N (1/H_j)$. In the latter case, $L(x, I)$ admits the following L^2 representation:*

$$L(x, I) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-i\langle y, x \rangle} \int_I e^{i\langle y, X(s) \rangle} ds dy, \quad \forall x \in \mathbf{R}^d. \tag{199}$$

Proof. The Fourier transform of the occupation measure μ_I is

$$\widehat{\mu}_I(\xi) = \int_I e^{i\langle \xi, X(t) \rangle} dt. \tag{200}$$

By applying Fubini’s theorem twice, we have

$$\mathbb{E} \int_{\mathbf{R}^d} |\widehat{\mu}(\xi)|^2 d\xi = \int_I ds \int_I dt \int_{\mathbf{R}^d} \mathbb{E} \exp\left(i\langle \xi, X(s) - X(t) \rangle\right) d\xi. \tag{201}$$

We denote the right hand side of (201) by $\mathcal{J}(I)$. It follows from the Plancherel theorem that X has a local time $L(\cdot, I) \in L^2(\mathbf{P} \times \lambda_d)$ if and only if $\mathcal{J}(I) < \infty$; see [41, Theorem 21.9] or [46]. Hence it is sufficient to prove

$$\mathcal{J}(I) < \infty \quad \text{if and only if} \quad d < \sum_{j=1}^N \frac{1}{H_j}. \tag{202}$$

For this purpose, we use the independence of the coordinate processes of X and Condition (C1) to deduce

$$\begin{aligned} \mathcal{J}(I) &= \int_I \int_I \frac{ds dt}{[\mathbb{E}(X_0(s) - X_0(t))^2]^{d/2}} \\ &\asymp \int_I \int_I \frac{ds dt}{(\sum_{j=1}^N |s_j - t_j|^{2H_j})^{d/2}}. \end{aligned} \tag{203}$$

By using Lemma 8.6 below, it is elementary to verify that the last integral in (203) is finite if and only if $d < \sum_{j=1}^N (1/H_j)$. This proves (202), and hence Theorem 8.1. □

The following result on the joint continuity of the local times is similar to those proved by Xiao and Zhang [110], Ayache, Wu and Xiao [6] for fractional Brownian sheets.

Theorem 8.2. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an (N, d) -Gaussian field defined by (36) and we assume Conditions (C1) and (C3’) are satisfied on I . If $d < \sum_{j=1}^N (1/H_j)$, then X has a jointly continuous local time on I .*

Remark 8.3. The conclusion of Theorem 8.2 can also be proved to hold for all Gaussian random fields satisfying Conditions (C1) and (C3). The proof follows a similar line, but some modifications are needed in order to prove analogous estimates in Lemmas 8.4 and 8.8. This is left to the reader as an exercise.

To prove Theorem 8.2 we will, similar to [6; 34; 102], first use the Fourier analytic arguments to derive estimates on the moments of the local times [see Lemmas 8.4 and 8.8 below] and then apply a multiparameter version of Kolmogorov continuity theorem [49]. It will be clear that Condition (C3') plays an essential role in the proofs of Lemmas 8.4 and 8.8.

Our starting points is the following identities about the moments of the local time and its increments. It follows from [41, (25.5) and (25.7)] or [77] that for all $x, y \in \mathbf{R}^d$, $T \in \mathcal{A}$ and all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}[L(x, T)^n] &= (2\pi)^{-nd} \int_{T^n} \int_{\mathbf{R}^{nd}} e^{-i \sum_{j=1}^n \langle u^j, x \rangle} \\ &\quad \times \mathbb{E} \left[e^{i \sum_{j=1}^n \langle u^j, X(t^j) \rangle} \right] d\bar{u} d\bar{t} \end{aligned} \tag{204}$$

and for all even integers $n \geq 2$,

$$\begin{aligned} &\mathbb{E}[(L(x, T) - L(y, T))^n] \\ &= (2\pi)^{-nd} \int_{T^n} \int_{\mathbf{R}^{nd}} \prod_{j=1}^n \left[e^{-i \langle u^j, x \rangle} - e^{-i \langle u^j, y \rangle} \right] \mathbb{E} \left[e^{i \sum_{j=1}^n \langle u^j, X(t^j) \rangle} \right] d\bar{u} d\bar{t}, \end{aligned} \tag{205}$$

where $\bar{u} = (u^1, \dots, u^n)$, $\bar{t} = (t^1, \dots, t^n)$, and each $u^j \in \mathbf{R}^d$, $t^j \in T \subseteq (0, \infty)^N$. In the coordinate notation we then write $u^j = (u_1^j, \dots, u_d^j)$.

Lemma 8.4. *Suppose the assumptions of Theorem 8.2 hold. Let $\tau \in \{1, \dots, N\}$ be the integer such that*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}, \tag{206}$$

then there exists a positive and finite constant $c_{s,1}$, depending on N , d , H and I only, such that for all hypercubes $T = [a, a + \langle r \rangle] \subseteq I$ with side-length $r \in (0, 1)$, all $x \in \mathbf{R}^d$ and all integers $n \geq 1$,

$$\mathbb{E}[L(x, T)^n] \leq c_{s,1}^n n! r^{n\beta_\tau}, \tag{207}$$

where $\beta_\tau = \sum_{\ell=1}^{\tau} (H_\tau/H_\ell) + N - \tau - H_\tau d$.

Remark 8.5. (i) It is important to note that, when (206) holds, $\beta_\tau = \sum_{\ell=1}^{\tau} (H_\tau/H_\ell) + N - \tau - H_\tau d$ is the Hausdorff dimension of the level set L_x ; see Theorem 7.1. Combining (207) with the upper density theorem of [80], one can obtain some information on the exact Hausdorff measure of L_x [see Corollary 8.11 below].

(ii) We point out that the upper bound in (207) is not sharp, and one may be able to prove the following inequality:

$$E[L(x, T)^n] \leq c_{8,2}^n (n!)^{N-\beta_\tau} r^{n\beta_\tau}. \tag{208}$$

If this is indeed true, then one can conjecture that the function $\varphi_8(r) = r^{\beta_\tau} (\log \log 1/r)^{N-\beta_\tau}$ is an exact Hausdorff measure function for L_x .

For proving Lemma 8.4, we will make use of the following elementary lemma [which is stronger than Lemma 6.3].

Lemma 8.6. *Let α, β and A be positive constants. Then*

$$\int_0^1 \frac{1}{(A+t^\alpha)^\beta} dt \asymp \begin{cases} A^{-(\beta-\frac{1}{\alpha})} & \text{if } \alpha\beta > 1, \\ \log(1+A^{-1/\alpha}) & \text{if } \alpha\beta = 1, \\ 1 & \text{if } \alpha\beta < 1. \end{cases} \tag{209}$$

Proof of Lemma 8.4. Since X_1, \dots, X_d are independent copies of X_0 , it follows from (204) that for all integers $n \geq 1$,

$$E[L(x, T)^n] \leq (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^d \left\{ \int_{\mathbf{R}^n} e^{-\frac{1}{2}\text{Var}(\sum_{j=1}^n u_k^j X_0(t^j))} dU_k \right\} d\bar{t}, \tag{210}$$

where $U_k = (u_k^1, \dots, u_k^n) \in \mathbf{R}^n$. It is clear that in order to bound the integral in $d\bar{t}$ it is sufficient to consider only the integral over $T_\neq^n = \{\bar{t} \in T^n : t^1, \dots, t^n \text{ are distinct}\}$ [the set of $\bar{t} \in \mathbf{R}^{Nn}$ having $t^i = t^j$ for some $i \neq j$ has (Nn) -dimensional Lebesgue measure 0]. It follows from Lemma 3.4 that for every $\bar{t} \in T_\neq^n$, the covariance matrix of $X_0(t^1), \dots, X_0(t^n)$ is invertible. We denote the inverse matrix by $R(t^1, \dots, t^n)$, and let (Z_1, \dots, Z_n) be the Gaussian vector with mean zero and the covariance matrix $R(t^1, \dots, t^n)$. Then the density function of (Z_1, \dots, Z_n) is given by

$$\frac{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}U \text{Cov}(X_0(t^1), \dots, X_0(t^n))U'}, \tag{211}$$

where $U = (u^1, \dots, u^n) \in \mathbf{R}^n$ and U' is the transpose of U . Hence for each $1 \leq k \leq d$,

$$\int_{\mathbf{R}^n} e^{-\frac{1}{2}\text{Var}(\sum_{j=1}^n u_k^j X_0(t^j))} dU_k = \frac{(2\pi)^{n/2}}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{1/2}}. \tag{212}$$

Combining (210) and (212), we derive

$$E[L(x, T)^n] \leq (2\pi)^{-nd/2} \int_{T^n} \frac{1}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{d/2}} d\bar{t}. \tag{213}$$

It follows from Condition (C3') and (53) that

$$\begin{aligned} \det \text{Cov}(X_0(t^1), \dots, X_0(t^n)) &= \prod_{j=1}^n \text{Var}(X_0(t^j) | X_0(t^i), j < i \leq n) \\ &\geq c_{8,3}^n \prod_{j=1}^n \min_{j < i \leq n+1} \rho(t^j, t^i)^2, \end{aligned} \tag{214}$$

where $t^{n+1} = 0$. This and (213) together imply that

$$\mathbb{E}[L(x, T)^n] \leq c_{8,4}^n \int_{T^n} \prod_{j=1}^n \frac{1}{\left[\min_{j < i \leq n+1} \rho(t^j, t^i) \right]^d} d\bar{t}. \tag{215}$$

We will estimate the integral in (215) by integrating in the order dt^1, dt^2, \dots, dt^n . Considering first the integral in dt^1 , we have

$$\begin{aligned} \int_T \frac{1}{\left[\min_{1 < i \leq n+1} \rho(t^j, t^i) \right]^d} dt^1 &\leq \sum_{i=2}^{n+1} \int_T \frac{1}{[\rho(t^j, t^i)]^d} dt^1 \\ &\leq cn \int_{[0,r]^N} \frac{ds_1 \cdots ds_N}{\left[\sum_{k=1}^N s_k^{H_k} \right]^d}, \end{aligned} \tag{216}$$

where the last inequality follows from a change of variables. Integrating the last integral in the order ds_1, \dots, ds_N and applying (209) in Lemma 6.3, we can show that, because of (206), the last integrand in (216) only affects the integral in ds_1, \dots, ds_τ which contributes [up to a constant] the factor $r^{\sum_{\ell=1}^\tau (H_\tau/H_\ell) - H_\tau d}$; and the integral in $ds_{\tau+1}, \dots, ds_N$ contributes the factor $r^{N-\tau}$. In other words, we have

$$\int_{[0,r]^N} \frac{ds_1 \cdots ds_N}{\left[\sum_{k=1}^N s_k^{H_k} \right]^d} \leq c_{8,5} r^{\sum_{\ell=1}^\tau (H_\tau/H_\ell) + N - \tau - H_\tau d}. \tag{217}$$

This and (216) imply

$$\int_T \frac{dt^1}{\left[\min_{1 < i \leq n+1} \rho(t^j, t^i) \right]^d} \leq c_{8,6} n r^{\sum_{\ell=1}^\tau (H_\tau/H_\ell) + N - \tau - H_\tau d}. \tag{218}$$

Repeating the same procedure for integrating in dt^2, \dots, dt^n in (215) yields (207). This proves Lemma 8.4.

Remark 8.7. In the proof of Lemma 8.4, we have assumed T is a hypercube $T = [a, a + \langle r \rangle]$. This is only for convenience and one can consider arbitrary

closed intervals $T = \prod_{\ell=1}^N [a_\ell, a_\ell + r_\ell] \subseteq I$. The argument is the same as above, but (216) becomes

$$\int_T \frac{dt^1}{\left[\min_{1 < i \leq n+1} \rho(t^j, t^i) \right]^d} \leq cn \int_{\prod_{k=1}^N [0, r_k]} \frac{ds_1 \cdots ds_N}{\left[\sum_{k=1}^N s_k^{H_k} \right]^d}. \tag{219}$$

Choose N positive numbers $p_1, \dots, p_N \in (0, 1)$ defined by

$$p_k = \frac{H_k^{-1}}{\sum_{i=1}^N H_i^{-1}} \quad (k = 1, \dots, N). \tag{220}$$

Then $\sum_{k=1}^N p_k = 1$. By using the following inequality

$$\sum_{k=1}^N s_k^{H_k} \geq \sum_{k=1}^N p_k s_k^{H_k} \geq \prod_{k=1}^N s_k^{p_k H_k} \quad \forall s \in (0, \infty)^N, \tag{221}$$

one can verify that

$$\int_{\prod_{k=1}^N [0, r_k]} \frac{ds_1 \cdots ds_N}{\left[\sum_{k=1}^N s_k^{H_k} \right]^d} \leq c \lambda_N(T)^{1-\nu}, \tag{222}$$

where $\nu = d / (\sum_{i=1}^N H_i^{-1}) \in (0, 1)$. Combining (215), (219) and (222) we derive that

$$\mathbb{E}[L(x, T)^n] \leq c_{s, \tau}^n n! \lambda_N(T)^{n(1-\nu)} \tag{223}$$

holds for every interval $T \subseteq I$. We will apply this inequality in the proof of Theorem 8.2 below.

Lemma 8.4 implies that for all $n \geq 1$, $L(x, T) \in L^n(\mathbf{R}^d)$ a.s. [41, p. 42]. Our next lemma estimates the moments of the increments of $L(x, T)$ in the space variable x .

Lemma 8.8. *Assume (206) holds for some $\tau \in \{1, \dots, N\}$. Then there exists a positive and finite constant $c_{s, s}$, depending on N, d, H and I only, such that for all hypercubes $T = [a, a + \langle r \rangle] \subseteq I$, $x, y \in \mathbf{R}^d$ with $|x - y| \leq 1$, all even integers $n \geq 1$ and all $\gamma \in (0, 1)$ small enough,*

$$\mathbb{E} \left[(L(x, T) - L(y, T))^n \right] \leq c_{s, s}^n (n!)^{(1+\gamma)} |x - y|^{n\gamma} r^{n(\beta_\tau - 2H_\tau\gamma)}. \tag{224}$$

In order to prove Lemma 8.8, we will make use of the following lemma essentially due to Cuzick and DuPreez [19]; see also [55].

Lemma 8.9. Let Z_1, \dots, Z_n be mean zero Gaussian variables which are linearly independent, then for any nonnegative function $g: \mathbf{R} \rightarrow \mathbf{R}_+$,

$$\begin{aligned} \int_{\mathbf{R}^n} g(v_1) e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n v_j Z_j)} dv_1 \cdots dv_n \\ = \frac{(2\pi)^{(n-1)/2}}{[\det \text{Cov}(Z_1, \dots, Z_n)]^{1/2}} \int_{-\infty}^{\infty} g\left(\frac{v}{\sigma_1}\right) e^{-v^2/2} dv, \end{aligned} \quad (225)$$

where $\sigma_1^2 = \text{Var}(Z_1 | Z_2, \dots, Z_n)$ denotes the conditional variance of Z_1 given Z_2, \dots, Z_n .

Proof of Lemma 8.8. Let $\gamma \in (0, 1)$ be a constant whose value will be determined later. Note that by the elementary inequalities

$$|e^{iu} - 1| \leq 2^{1-\gamma} |u|^\gamma \quad \text{for all } u \in \mathbf{R} \quad (226)$$

and $|u + v|^\gamma \leq |u|^\gamma + |v|^\gamma$, we see that for all $u^1, \dots, u^n, x, y \in \mathbf{R}^d$,

$$\prod_{j=1}^n \left| e^{-i\langle u^j, x \rangle} - e^{-i\langle u^j, y \rangle} \right| \leq 2^{(1-\gamma)n} |x - y|^{n\gamma} \sum' \prod_{j=1}^n |u_{k_j}^j|^\gamma, \quad (227)$$

where the summation \sum' is taken over all the sequences $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$.

It follows from (205) and (227) that for every even integer $n \geq 2$,

$$\begin{aligned} \mathbb{E} \left[(L(x, T) - L(y, T))^n \right] &\leq (2\pi)^{-nd} 2^{(1-\gamma)n} |x - y|^{n\gamma} \\ &\quad \times \sum' \int_{T^n} \int_{\mathbf{R}^{nd}} \prod_{m=1}^n |u_{k_m}^m|^\gamma \mathbb{E} \left[e^{-i \sum_{j=1}^n \langle u^j, X(t^j) \rangle} \right] d\bar{u} d\bar{t} \\ &\leq c_{s,9}^n |x - y|^{n\gamma} \sum' \int_{T^n} d\bar{t} \\ &\quad \times \prod_{m=1}^n \left\{ \int_{\mathbf{R}^{nd}} |u_{k_m}^m|^{n\gamma} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \langle u^j, X(t^j) \rangle)} d\bar{u} \right\}^{1/n}, \end{aligned} \quad (228)$$

where the last inequality follows from the generalized Hölder inequality.

Now we fix a vector $\bar{k} = (k_1, \dots, k_n) \in \{1, \dots, d\}^n$ and n distinct points $t^1, \dots, t^n \in T$ [the set of such points has full (nN) -dimensional Lebesgue measure]. Let $\mathcal{M} = \mathcal{M}(\bar{k}, \bar{t}, \gamma)$ be defined by

$$\mathcal{M} = \prod_{m=1}^n \left\{ \int_{\mathbf{R}^{nd}} |u_{k_m}^m|^{n\gamma} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \langle u^j, X(t^j) \rangle)} d\bar{u} \right\}^{1/n}. \quad (229)$$

Note that X_ℓ ($1 \leq \ell \leq N$) are independent copies of X_0 . By Lemma 3.4, the random variables $X_\ell(t^j)$ ($1 \leq \ell \leq N, 1 \leq j \leq n$) are linearly independent. Hence Lemma 8.9 gives

$$\begin{aligned}
 & \int_{\mathbf{R}^{nd}} |u_{k_m}^m|^{n\gamma} e^{-\frac{1}{2}\text{Var}(\sum_{j=1}^n \langle u^j, X(t^j) \rangle)} d\bar{u} \\
 &= \frac{(2\pi)^{(nd-1)/2}}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{d/2}} \int_{\mathbf{R}} \left(\frac{v}{\sigma_m}\right)^{n\gamma} e^{-v^2/2} dv \quad (230) \\
 &\leq \frac{c_{s,10}^n (n!)^\gamma}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{d/2}} \frac{1}{\sigma_m^{n\gamma}},
 \end{aligned}$$

where σ_m^2 is the conditional variance of $X_{k_m}(t^m)$ given $X_i(t^j)$ ($i \neq k_m$ or $i = k_m$ but $j \neq m$), and the last inequality follows from Stirling's formula.

Combining (229) and (230) we obtain

$$\mathcal{M} \leq \frac{c_{s,11}^n (n!)^\gamma}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}. \quad (231)$$

The second product in (231) will be treated as a ‘‘perturbation’’ factor and will be shown to be small when integrated. For this purpose, we use again the independence of the coordinate processes of X and Condition (C3') to derive

$$\begin{aligned}
 \sigma_m^2 &= \text{Var}(X_{k_m}(t^m) \mid X_{k_m}(t^j), \mathbf{j} \neq m) \\
 &\geq c_{s,12}^2 \min \{ \rho(t^m, t^j)^2 : j = 0 \text{ or } j \neq m \}. \quad (232)
 \end{aligned}$$

Now we define a permutation π of $\{1, \dots, n\}$ such that

$$\rho(t^{\pi(1)}, 0) = \min_{1 \leq j \leq n} \rho(t^j, 0). \quad (233)$$

and once $t^{\pi(m-1)}$ has been defined, we choose $t^{\pi(m)}$ such that

$$\begin{aligned}
 & \rho(t^{\pi(m)}, t^{\pi(m-1)}) \\
 &= \min \left\{ \rho(t^j, t^{\pi(m-1)}), j \in \{1, \dots, n\} \setminus \{\pi(1), \dots, \pi(m-1)\} \right\}. \quad (234)
 \end{aligned}$$

By this definition, we see that for every $m = 1, \dots, n$,

$$\begin{aligned}
 & \min \left\{ \rho(t^{\pi(m)}, t^j) : j = 0 \text{ or } j \neq \pi(m) \right\} \\
 &= \min \left\{ \rho(t^{\pi(m)}, t^{\pi(m-1)}), \rho(t^{\pi(m+1)}, t^{\pi(m)}) \right\}. \quad (235)
 \end{aligned}$$

It follows from (232), (235) and (53) that

$$\begin{aligned}
 \prod_{m=1}^n \frac{1}{\sigma_m^\gamma} &\leq c_{8,12}^{-n\gamma} \prod_{m=1}^n \frac{1}{\min \{ \rho(t^{\pi(m)}, t^j)^\gamma : j = 0 \text{ or } j \neq \pi(m) \}} \\
 &\leq c_{8,13}^n \prod_{m=1}^n \frac{1}{\min \{ \rho(t^{\pi(m)}, t^{\pi(m-1)})^\gamma, \rho(t^{\pi(m+1)}, t^{\pi(m)})^\gamma \}} \\
 &\leq c_{8,13}^n \prod_{m=1}^n \frac{1}{\rho(t^{\pi(m)}, t^{\pi(m-1)})^{2\gamma}} \tag{236} \\
 &\leq c_{8,14}^n \prod_{m=1}^n \frac{1}{[\text{Var}(X_0(t^{\pi(m)})|X_0(t^{\pi(i)}), i = 1, \dots, m-1)]^\gamma} \\
 &= \frac{c_{8,15}^n}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^\gamma}.
 \end{aligned}$$

Combining (231) and (236), we obtain

$$\begin{aligned}
 \mathcal{M} &\leq \frac{c_{8,16}^n (n!)^\gamma}{[\det \text{Cov}(X_0(t^1), \dots, X_0(t^n))]^{\frac{d}{2} + \gamma}} \\
 &\leq \frac{c_{8,17}^n (n!)^\gamma}{\prod_{j=1}^n \left[\min_{j < i \leq n+1} \rho(t^j, t^i) \right]^{d+2\gamma}}, \tag{237}
 \end{aligned}$$

where the last step follows from Condition (C3') and (53).

It follows from (228), (229), (231) and (237) that

$$\begin{aligned}
 &\text{E} \left[(L(x+y, T) - L(x, T))^n \right] \\
 &\leq c_{8,18}^n |y|^{n\gamma} (n!)^\gamma \int_{T^n} \prod_{j=1}^n \frac{1}{\left[\min_{j < i \leq n+1} \rho(t^j, t^i) \right]^{d+2\gamma}} d\bar{t}. \tag{238}
 \end{aligned}$$

Note that the last integral in (238) is similar to that in (215) and can be estimated by integrating in the order dt^1, dt^2, \dots, dt^n . To this end, we take $\gamma \in (0, 1)$ small such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d + 2\gamma < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{239}$$

Then, similar to (216)–(218), we derive

$$\begin{aligned}
 &\int_{T^n} \prod_{j=1}^n \frac{1}{\left[\min_{j < i \leq n+1} \rho(t^j, t^i) \right]^{d+2\gamma}} d\bar{t} \\
 &\leq c_{8,19}^n (n!)^{1+\gamma} r^n \left(\sum_{\ell=1}^{\tau} (H_\tau / H_\ell) + N - \tau - H_\tau (d+2\gamma) \right). \tag{240}
 \end{aligned}$$

It is now clear that (224) follows from (238) and (240). This proves Lemma 8.8.

Now we are ready to prove Theorem 8.2.

Proof of Theorem 8.2. It follows from Lemma 8.8 and the multiparameter version of Kolmogorov’s continuity theorem [49] that, for every fixed interval $T \in \mathcal{A}$ such that $T \subseteq I$, X has almost surely a local time $L(x, T)$ that is continuous for all $x \in \mathbf{R}^d$.

To prove the joint continuity, observe that for all $x, y \in \mathbf{R}^d$, $s, t \in I$ and all even integers $n \geq 1$, we have

$$\begin{aligned} & \mathbf{E} [(L(x, [\varepsilon, s]) - L(y, [\varepsilon, t]))^n] \\ & \leq 2^{n-1} \{ \mathbf{E} [(L(x, [\varepsilon, s]) - L(x, [\varepsilon, t]))^n] + \mathbf{E} [(L(x, [\varepsilon, t]) - L(y, [\varepsilon, t]))^n] \}. \end{aligned} \tag{241}$$

Since the difference $L(x, [\varepsilon, s]) - L(x, [\varepsilon, t])$ can be written as a sum of finite number (only depends on N) of terms of the form $L(x, T_j)$, where each $T_j \in \mathcal{A}$ is a closed subinterval of I with at least one edge length $\leq |s - t|$, we can use Lemma 8.4 and Remark 8.7, to bound the first term in (241). On the other hand, the second term in (241) can be dealt with using Lemma 8.8 as above. Consequently, for some $\gamma \in (0, 1)$ small, the right hand side of (241) is bounded by $c_{8,20}^n (|x - y| + |s - t|)^{n\gamma}$, where $n \geq 2$ is an arbitrary even integer. Therefore the joint continuity of the local times $L(x, t)$ follows again from the multiparameter version of Kolmogorov’s continuity theorem. This finishes the proof of Theorem 8.2.

The proof of Theorem 8.2 also provides some information about the modulus of continuity of $L(x, t)$ as a function of x and t . It would be interesting to establish sharp local and uniform Hölder conditions for the local time, because such results bear rich information about the irregular properties of the sample functions of X [1; 13; 41; 102].

By applying Lemma 8.4 and the Borel-Cantelli lemma, one can easily derive the following law of the iterated logarithm for the local time $L(x, \cdot)$: There exists a positive constant $c_{8,21}$ such that for every $x \in \mathbf{R}^d$ and $t \in (0, 1)^N$,

$$\limsup_{r \rightarrow 0} \frac{L(x, U(t, r))}{\varphi_9(r)} \leq c_{8,21}, \tag{242}$$

where $U(t, r)$ denotes the open or closed ball [in the Euclidean metric] centered at t with radius r and $\varphi_9(r) = r^{\beta\tau} \log \log(1/r)$. It follows from Fubini’s theorem that, with probability one, (242) holds for λ_N -almost all $t \in I$. Now we prove a stronger version of this result, which is useful in determining the exact Hausdorff measure of the level set L_x .

Theorem 8.10. *Let X be an (N, d) -Gaussian random field defined by (36). We assume Conditions (C1) and (C3’) are satisfied on I and $d < \sum_{j=1}^N (1/H_j)$. Let $\tau \in \{1, \dots, N\}$ be the integer so that (206) holds and let L be the jointly*

continuous local time of X . Then, there is a finite constant $c_{8,22}$ such that with probability one,

$$\limsup_{r \rightarrow 0} \frac{L(x, U(t, r))}{\varphi_9(r)} \leq c_{8,22} \tag{243}$$

holds for $L(x, \cdot)$ -almost all $t \in I$.

Proof. The proof is similar to that of Theorem 4.1 in [6]. See also [102].

For every integer $k \geq 1$, we consider the random measure $\mu_k := \mu_k(x, \bullet)$ on the Borel subsets C of I defined by (157) [where the integer n is replaced by k]. Then, by the occupation density formula (197) and the continuity of the function $y \mapsto L(y, C)$, one can verify that almost surely $\mu_k(C) \rightarrow L(x, C)$ as $k \rightarrow \infty$ for every Borel set $C \subseteq I$.

For every integer $m \geq 1$, denote $f_m(t) = L(x, U(t, 2^{-m}))$. From the proof of Theorem 8.2 we can see that almost surely the functions $f_m(t)$ are continuous and bounded. Hence we have almost surely, for all integers $m, n \geq 1$,

$$\int_I [f_m(t)]^n L(x, dt) = \lim_{k \rightarrow \infty} \int_I [f_m(t)]^n \mu_k(dt). \tag{244}$$

It follows from (244), (157) and the proof of Proposition 3.1 of [77] that for every positive integer $n \geq 1$,

$$\begin{aligned} & \mathbb{E} \int_I [f_m(t)]^n L(x, dt) \\ &= \left(\frac{1}{2\pi}\right)^{(n+1)d} \int_I \int_{U(t, 2^{-m})^n} \int_{\mathbf{R}^{(n+1)d}} e^{-i \sum_{j=1}^{n+1} \langle x, u^j \rangle} \\ & \quad \times \mathbb{E} \exp \left(i \sum_{j=1}^{n+1} \langle u^j, X(s^j) \rangle \right) d\bar{u} d\bar{s}, \end{aligned} \tag{245}$$

where $\bar{u} = (u^1, \dots, u^{n+1}) \in \mathbf{R}^{(n+1)d}$ and $\bar{s} = (t, s^1, \dots, s^n)$. Similar to the proof of (207) we have that the right hand side of Eq. (245) is at most

$$\begin{aligned} & c_{8,23}^n \int_I \int_{U(t, 2^{-m})^n} \frac{d\bar{s}}{[\det \text{Cov}(X_0(t), X_0(s^1), \dots, X_0(s^n))]^{d/2}} \\ & \leq c_{8,24}^n n! 2^{-mn\beta_\tau}, \end{aligned} \tag{246}$$

where $c_{8,24}$ is a positive and finite constant depending on N, d, H , and I only.

Let $\gamma > 0$ be a constant whose value will be determined later. We consider the random set

$$I_m(\omega) = \{t \in I : f_m(t) \geq \gamma \varphi_9(2^{-m})\}.$$

Denote by ν_ω the restriction of the random measure $L(x, \cdot)$ on I , that is, $\nu_\omega(E) = L(x, E \cap I)$ for every Borel set $E \subseteq \mathbf{R}_+^N$. Now we take $n = \lfloor \log m \rfloor$,

where $\lfloor y \rfloor$ denotes the integer part of y . Then by applying (246) and Stirling's formula, we have

$$\begin{aligned} \mathbb{E}[\nu_\omega(I_m)] &\leq \frac{\mathbb{E} \int_I [f_m(t)]^n L(x, dt)}{[\gamma \varphi_9(2^{-m})]^n} \\ &\leq \frac{c_{8,25}^n n! 2^{-mn} \beta_\tau}{\gamma^n 2^{-mn} \beta_\tau (\log m)^n} \leq m^{-2}, \end{aligned} \tag{247}$$

provided $\gamma > 0$ is chosen large enough, say, $\gamma \geq c_{8,25} e^2 := c_{8,26}$. This implies that

$$\mathbb{E} \left(\sum_{m=1}^\infty \nu_\omega(I_m) \right) < \infty. \tag{248}$$

Therefore, with probability 1 for ν_ω almost all $t \in I$, we have

$$\limsup_{m \rightarrow \infty} \frac{L(x, U(t, 2^{-m}))}{\varphi_9(2^{-m})} \leq c_{8,26}. \tag{249}$$

Finally, for any $r > 0$ small enough, there exists an integer m such that $2^{-m} \leq r < 2^{-m+1}$ and (249) is applicable. Since the function $\varphi_9(r)$ is increasing near $r = 0$, (243) follows from (249) and a monotonicity argument. \square

Since $L(x, \cdot)$ is a random Borel measure supported on the level set $L_x = \{t \in I : X(t) = x\}$, Theorem 8.10 and the upper density theorem of [80] imply the following partial result on the exact Hausdorff measure of L_x .

Corollary 8.11. *Assume the conditions of Theorem 8.10 are satisfied. Then there exists a positive constant $c_{8,27}$ such that for every $x \in \mathbf{R}^d$, we have*

$$\mathcal{H}^{\varphi_9}(L_x) \geq c_{8,27} L(x, I), \quad a.s. \tag{250}$$

Proof. The proof is left to the reader as an exercise. \square

We should mention that the estimates in Lemmas 8.4 and 8.8 are not sharp and it would be interesting to improve them. In the rest of this section, we consider the special case when $H = \langle \alpha \rangle$ and $\alpha \in (0, 1)$. Many sample path properties of such Gaussian random fields have been studied in [1; 49; 83; 102; 108]. By applying Lemma 2.3 in [102] in place of (216), we prove the following sharp estimates.

Lemma 8.12. *Let X be an (N, d) -Gaussian random field satisfying the conditions (C1) and (C3') with $H = \langle \alpha \rangle$. We assume that $N > \alpha d$. Then there exists a positive and finite constant $c_{8,28}$, depending on N, d, α and I only, such that for all intervals $T = [a, a + \langle r \rangle] \subseteq I$ with edge length $r \in (0, 1)$, all $x \in \mathbf{R}^d$ and all integers $n \geq 1$,*

$$\mathbb{E} [L(x, T)^n] \leq c_{8,28}^n (n!)^{\alpha d/N} r^{n(N-\alpha d)} \tag{251}$$

and for any $0 < \gamma < \min\{1, (N/\alpha - d)/2\}$, there exists a positive and finite constant $c_{8,29}$ such that

$$\begin{aligned} \mathbb{E} [(L(x, T) - L(y, T))^n] \\ \leq c_{8,29}^n (n!)^{2\gamma + \alpha(d+\gamma)/N} |x - y|^{n\gamma} r^{n(N-\alpha(d+\gamma))}. \end{aligned} \tag{252}$$

Note that, for a Gaussian random field X satisfying the assumptions of Lemma 8.12, Eq. (251) allows us to improve the results in Theorem 8.10 and Corollary 8.11 by replacing the Hausdorff measure function $\varphi_9(r)$ by $\varphi_7(r) = r^{N-\alpha d}(\log \log 1/r)^{\alpha d/N}$. Moreover, by combining Lemma 8.12 and the argument in [102], one can establish the following sharp local and uniform Hölder conditions for the maximum local time $L^*(\bullet)$ of X defined by

$$L^*(T) = \sup_{x \in \mathbb{R}^d} L(x, T) \quad \forall T \subseteq I. \tag{253}$$

Theorem 8.13. *Let X be an (N, d) -Gaussian random field satisfying the conditions (C1) and (C3') with $H = \langle \alpha \rangle$ and $N > \alpha d$. The following statements hold:*

(i) *There exists a positive and finite constant $c_{8,30}$ such that for every $t \in I$,*

$$\limsup_{r \rightarrow 0} \frac{L^*(U(t, r))}{\varphi_7(r)} \leq c_{8,30} \quad a.s., \tag{254}$$

where $U(t, r) = \{s \in I : |s - t| \leq r\}$.

(ii) *There exists a positive and finite constant $c_{8,31}$ such that*

$$\limsup_{r \rightarrow 0} \sup_{t \in I} \frac{L^*(U(t, r))}{\varphi_{10}(r)} \leq c_{8,31} \quad a.s., \tag{255}$$

where $\varphi_{10}(r) = r^{N-\alpha d}(\log 1/r)^{\alpha d/N}$.

Proof. The proofs of (254) and (255) are based on Lemma 8.12 and a chaining argument, which is the same as those of Theorems 1.1 and 1.2 in [102]; see also [34]. We leave the details to the reader. \square

Similar to [102; 108], one can apply Lemma 8.12 and Theorem 8.13 to derive further results, such as the Chung-type laws of the iterated logarithm, modulus of nowhere differentiability, tail probability of the local times, for (N, d) -Gaussian random fields satisfying the conditions (C1) and (C3') with $H = \langle \alpha \rangle$. These are left to the reader as exercises.

The following is our final remark.

Remark 8.14. Both Conditions (C3) and (C3') are useful in studying the existence and regularity of the self-intersection local times of X which, in turn, provide information on the fractal dimensions of the sets of multiple points and multiple times of X . When X is an (N, d) -fractional Brownian sheet, these problems have been studied in [69]. It is expected that analogous results also hold for Gaussian random fields satisfying Conditions (C1) and (C3').

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