CBMS Conference: Gaussian Random Fields, Fractals, SPDEs, and Extremes

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The following are the tentative titles and abstracts of the ten lectures.

1. *An introduction on random fields*

   We present an overview on univariate and multivariate random fields and introduce important statistical characteristics such as spectral representations, self-similarity, anisotropy of random fields. Important examples of random fields include Gaussian random fields with stationary increments and solutions to stochastic partial differential equations.

2. *General results on regularity of Gaussian random fields.*

   Regularity properties such as continuity and differentiability of the sample functions of Gaussian processes are important topics in probability theory and essential for statistical applications. Necessary and sufficient conditions for sample path continuity based on the metric entropy or majorizing measure were established by Dudley (1967), Fernique (1975), and Talagrand (1987).

   This lecture recalls briefly some general methods for studying the regularity properties of Gaussian random fields. These include bounds on expectations of the supremum of Gaussian processes in terms of metric entropy and majorizing measure, sample path continuity, and differentiability. These methods are applicable to the Gaussian random fields described in Lecture 1.

3. *Exact results on regularity of Gaussian random fields*

   The purpose of this module is to present methods for establishing exact uniform and local modulus of continuity results for Gaussian random fields. In particular, we prove four types of limit theorems: the law of the iterated logarithm, uniform modulus of continuity, Chung's law of the iterated logarithm, and the modulus of nondifferentiability.

   For this purpose, we formulate a framework that will be convenient for studying the solutions of stochastic partial differential equations. An important condition in this framework is the property of strong local nondeterminism.

4. *Properties of strong local nondeterminism*

   Sufficient conditions in terms of spectral measures or covariance functions for the properties of strong local nondeterminism for Gaussian random fields will be provided.

   A comparison theorem allows to study Gaussian random fields with stationary increments with discrete or singular spectral measures.

   Properties of strong local nondeterminism are proved for solutions to stochastic heat and wave equations with colored or fractional noises.

5. *Fractal properties of random fields.*

   Fractal geometry is important for studying random fields with non-differentiable sample functions. We introduce Hausdorff and packing measure and dimensions and main techniques for their computation.

   We determine the Hausdorff dimensions of various random sets generated by multivariate Gaussian random fields including the range, graph, level sets, and the set of multiple points.

   We determine the exact Hausdorff measure functions for the range, graph and level sets of a large class of Gaussian random fields.

6. *Local times of Gaussian random fields*

   Besides of the interest on their own, local times and self-intersections local times of Gaussian random fields are important for studying fractal properties of the level sets and the set of multiple points. The properties of strong local nondeterminism are applied to establish sharp regularity results on the local times.

7. *Hitting probabilities, polarity, and self-intersections.*

   The lecture is concerned with hitting probabilities of Gaussian random fields and their applications in studying the existence and Hausdorff dimensions of intersections.

   Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$. For any compact sets $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, we study conditions on E and F for $\mathbb{P}\big\{X(E) \cap F \neq \varnothing\big\} > 0$. Only in a few special cases, this hitting probability problem has been completely solved [e.g., Khoshnevisan and Shi (1999), Khoshnevisan and Xiao (2007, 2015)].

   We will present some necessary conditions and sufficient conditions that improve those in Biermé, Lacaux, and Xiao (2009). In some special cases such as F is a singleton, more precise results have been proven in Dalang, Mueller, and Xiao (2017). This method has been extended by Dalang, Lee, Mueller, and Xiao (2021) to study the existence of multiple points of Gaussian random fields.

8. *Properties of linear stochastic heat and wave equations*

We consider linear stochastic heat and wave equations with Gaussian noise (including "white-colored" and "fractional-colored" noises). By applying general methods for Gaussian random fields, we describe various regularity, singularity, and fractal properties of the solutions to these SPDEs that highlight the effects of the equations and noises on the behavior of the solutions.

Some of the techniques that are useful for analyzing SPDEs of higher orders and fractional SPDEs.

9. *Properties of nonlinear stochastic heat equation*

Let $(t\,,x) \mapsto u_t(x)$ denote the solution to the stochastic heat equation with multiplicative noise

$$\partial_t u = -\frac{1}{2}(-\Delta)^{\alpha/2} u_t(x) + \lambda \sigma(u_t(x))\dot{W},$$

where the variable x ranges over $\mathbb{R}$, $t > 0$, $-(-\Delta)^{\alpha/2}$ denotes the generator of a symmetric α-stable Lévy process on $\mathbb{R}$, and $\dot{W}$ denotes the space-time white noise on $(0\,,\infty) \times \mathbb{R}$. Based on a quantitative version of the intuitively-appealing statement that "locally, $t \mapsto u_t(x)$ behaves as a conditionally-Gaussian process", We derive a number of results about the local behavior of $t \mapsto u_t(x)$, where $x \in \mathbb{R}$ is fixed.

If time permits, we will also study the polarity of the solution to the a system of the stochastic heat equations with multiplicative noise.

10. *Extreme values of random fields.*

General bounds for the excursion probabilities of real-valued Gaussian random fields can obtained by applying the general methods (e.g., the metric entropy method). These results have found many applications in mathematics and statistics. However, studies on multivariate Gaussian random fields are scarce.

In this lecture, we consider a class of bivariate Gaussian random fields and apply the double sum method to establish more precise approximations to the excursion probabilities.