NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
May 16-20, 2022
The University of Texas Rio Grande Valley

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May 17, 2022

LECTURE 4 (under construction)
CRANK AND RANK CONGRUENCES - PART 2
(Includes joint work with John Streese)

MODULAR FORMS MODULO ℓ

CONGRUENCES FOR HALF-INTEGER WEIGHT MOUDLAR FORMS

RANK MOMENTS MOD ℓ

- THE SECOND RANK MOMENT
- THE $(\ell - 1)$TH RANK MOMENT

CRANK MOMENTS MOD ℓ

- RANK AND CRANK DIFFERENCES MOD 13

REMARK To extend a partition congruence of the the form

$$p(An+b) \equiv \pmod{\ell}$$

in terms of the rank or crank, it is NOT necessary to have

$$M(r,\ell,An+b) \equiv 0 \pmod{\ell}$$

or

$$N(r,\ell,An+b) \equiv 0 \pmod{\ell}$$

for EVERY $0 \leq r \leq \ell - 1$.

PROBLEM Find A, B so that

$$M(0,\ell,An+b) \equiv \cdots \equiv M(\ell-1,\ell,An+B) \pmod{\ell}$$

or

$$N(0,\ell,An+b) \equiv \cdots \equiv N(\ell-1,\ell,An+B) \pmod{\ell}$$

ASSUMPTION THROUGHOUT: $\ell > 3$ is a fixed prime

MODULAR FORMS MODULO ℓ

Let M_k be the space of entire modular forms of weight k. For any subring R of $\mathbb{C}$ let

$$M_k(R) = \{f \in M_k : f = \sum_{n=0}^{\infty} a(n)q^n \in R[[q]]\}.$$

It is well known that $M_k(\mathbb{Z})$ has an upper-triangular $\mathbb{Z}$-basis

$$\left\{E_4^a \Delta^b \;:, 4a + 12b = k\right\} \qquad \text{if } k \equiv 0 \pmod 4$$

$$\left\{E_4^a E_6 \Delta^b \;:, 4a + 12b = k - 6\right\} \qquad \text{if } k \equiv 2 \pmod 4$$

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We call a rational number $\frac{m}{n}(m, n \in \mathbb{Z}, (m,n)=1)$ $\boxed{\underline{\ell\text{-integral}}}$ if $\ell \nmid n$ and let $\mathbb{Z}_{(\ell)}$ be the ring of ℓ-integral rational numbers. For $\alpha \in \mathbb{Z}_{(\ell)}$ we denote $\overline{\alpha}$ to be the reduction of $\alpha \pmod{\ell}$ and define

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$$\overline{\mathcal{M}_k} = \{\overline{f} = \sum_{n=0}^{\infty} \overline{a(n)} q^n \in \mathbb{F}_\ell[[q]] : f = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\mathbb{Z}_{(\ell)})\}.$$

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Theorem

As elements of $\mathbb{F}_\ell[[q]]$

1. $\overline{E_{\ell-1}} = 1$,
2. $\overline{E_{\ell+1}} = E_2$,
3. $\overline{\partial E_{\ell-1}} = \overline{E_{\ell+1}}$,
4. $\overline{\partial E_{\ell+1}} = -\overline{E_4 E_{\ell-1}}$,

where $\partial := 12q\frac{d}{dq} - kE_2$ *and* $\partial : \mathcal{M}_k \to \mathcal{M}_{k+2}$.

We call a polynomial $F(E_4, E_6) \in R[E_4, E_6]$ isobaric of weight k if

$$F(E_4, E_6) = \sum_{4a+6b=k} c_{a,b} E_4^a E_6^b.$$

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Theorem

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_\ell$.

1. *The irreducible isobaric elements of $\mathbb{F}[E_4, E_6]$ are $E_4, E_6, E_4^3 - cE_6^2$ where $c \in \mathbb{F}$.*
2. *Any isobaric element of $\mathbb{F}[E_4, E_6]$ can be uniquely written as a product of irreducible isobaric elements.*
3. *There are polynomials $A(E_4, E_6), B(E_4, E_6) \in \mathbb{Z}_{(\ell)}[E_4, E_6]$ such that*
$$A(E_4, E_6) = E_{\ell-1}, \quad B(E_4, E_6) = E_{\ell+1}. \tag{1}$$
4. *$E_4^3 - E_6^2$ does not divide A.*
5. *A has no multiple irreducible factor and is relatively prime to B.*
6. *$\overline{\mathcal{M}}$ is naturally isomorphic to $\mathbb{F}_\ell[E_4, E_6]/(\overline{A} - 1)$.*

<u>NOTE</u> $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{1}{1728}(E_4^3 - E_6^2)$.
Since $\overline{E_{\ell-1}} = 1$ we have

$$\cdots \subset \overline{\mathcal{M}_k} \subset \overline{\mathcal{M}_{k+(\ell-1)}} \subset \cdots \subset \overline{\mathcal{M}_{k+n(\ell-1)}} \subset \ldots$$

For $\alpha \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ we define

$$\overline{\mathcal{M}}^{\alpha} := \bigcup_{k\in\alpha} \overline{\mathcal{M}}_k.$$

Theorem

$$\overline{\mathcal{M}} = \bigoplus_{\alpha\in\mathbb{Z}/(\ell-1)\mathbb{Z}} \overline{\mathcal{M}}^{\alpha}$$

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Let f be a graded element of $\overline{\mathcal{M}}$ i.e $f \in \overline{\mathcal{M}}^{\alpha}$ for some $\alpha \in \mathcal{Z}/(\ell-1)\mathbb{Z}$. The **filtration of f** is denoted and defined by

$$w(f) := \inf\{k : \overline{f} \in \overline{\mathcal{M}}_k\}.$$

DEFINE $\Theta := q\frac{d}{dq}$ and $U = U(\ell)$ is the operator

$$f \mid U(\ell) = \sum_{n=0}^{\infty} a(\ell n) q^n$$

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Theorem

Let $f \in \overline{\mathcal{M}}$. Then

1. $w(\Theta f) \leq w(f) + \ell + 1$ *with equality if and only iff* $w(f) \not\equiv 0 \pmod{\ell}$.
2. *If* $w(f) \equiv 0 \pmod{\ell}$ *then* $w(\Theta f) \leq w(f) + 2$.
3. $w(f^i) = iw(f)$, *for* $i \geq 1$.
4. *If* $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$ *then* $w(f) < k$ *if and only if* $\overline{A}$ *divides* $\overline{F}$ *where* $f = F(E_4, E_6)$ *(*F *is an isobaric polynomial of weight* k*).*

Theorem

Suppose $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$. Then

i) $w(f \mid U(\ell)) \leq \ell + \frac{1}{\ell}(w(f) - 1)$

ii) *If* $w(f) = \ell - 1$ *then* $w(f|U(\ell)) = \ell - 1$

iii) $\overline{\Theta f} \in \overline{\mathcal{M}}_{k+(\ell+1)}$

iv) $\overline{\Theta^j f} \in \overline{\mathcal{M}}_{k+j(\ell+1)}$ *for* $j \geq 1$.

CONGRUENCES FOR HALF-INTEGER WEIGHT MOUDLAR FORMS

DEFINE $\Theta^* = 24\Theta - 1$ so that

$$\Theta^* f = \sum_{n=0}^{\infty} (24n-1)a(n)q^n \quad \text{if } f = \sum_{n=0}^{\infty} a(n)q^n$$

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For $f = \sum_{n=0}^{\infty} a(n)q^n$ DEFINE

$$f \mid U^* = U^*(f) = U_\ell^*(f) = \sum_{n=0}^{\infty} a(\ell n + \beta_\ell)q^n$$

where $1 < \beta_\ell < \ell$ satisfies and $24\beta_\ell \equiv 1 \pmod{\ell}$.

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Lemma

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$$q^{\beta_\ell}(f|U^*)^\ell \equiv f - (\Theta^*)^{\ell-1}f \pmod{\ell}$$

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Proof Let $f = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}_{(\ell)}[[q]]$.

$$(f|U^*)^\ell \equiv (\sum_{n=0}^{\infty} a(\ell n + \beta_\ell)q^n)^\ell \equiv \sum_{n=0}^{\infty} a(\ell n + \beta_\ell)q^{n\ell} \pmod{\ell}$$

$$(\Theta^*)^{\ell-1} f = \sum_{n=0}^{\infty}(24n-1)^{\ell-1} a(n)q^n$$

$$= \sum_{n \not\equiv \beta_\ell \pmod{\ell}} (24n-1)^{\ell-1} a(n)q^n + \sum_{n \equiv \beta_\ell \pmod{\ell}} (24n-1)^{\ell-1} a(n)q^n$$

$$\equiv \sum_{n \not\equiv \beta_\ell \pmod{\ell}} a(n)q^n \pmod{\ell}$$

$$\equiv \sum_{n=0} a(n)q^n - \sum_{n \equiv \beta_\ell \pmod{\ell}} a(n)q^n$$

$$\equiv f - \sum_{n=0}^{\infty} a(\ell n + \beta_\ell)q^{\ell n + \beta_\ell}$$

$$q^{\beta_\ell} \sum_{n=0}^{\infty} a(\ell n + \beta_\ell) q^{\ell n} \equiv f - (\Theta^*)^{\ell-1} f,$$

$$q^{\beta_\ell} (f|U^*)^{\ell} \equiv f - (\Theta^*)^{\ell-1} f \pmod{\ell}.$$

Lemma

If $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$ then

$$\Theta^*(Pf) \equiv Pg \pmod{\ell},$$

for some $g \in \overline{\mathcal{M}}_{k+\ell+1}$.

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Proof Let $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$.

$$\begin{aligned}\Theta^*(Pf) &= (24\Theta - 1)(Pf)\\ &= 24\Theta(Pf) - Pf = 24P\Theta(f) + 24f\Theta(P) - Pf\\ &= 24P\Theta(f) + f(24\Theta - 1)(P)\\ &= 24P\Theta(f) + f\Theta^*(P) = 24P\Theta(f) - PE_2f\\ &= P\big(24\Theta - E_2\big)(f)\\ &= P\big(2(12\Theta - kE_2)(f)\big)(f) + P(2k-1)E_2f\\ &\equiv Pg \pmod{\ell}\end{aligned}$$

where

$$g = 2\partial f + (2k-1)E_2 f \in \overline{\mathcal{M}}_{k+\ell+1}$$

since $\partial f \in \overline{\mathcal{M}}_{k+2} \subset \overline{\mathcal{M}}_{k+2+(\ell-1)} = \overline{\mathcal{M}}_{k+\ell+1}$ and $E_2 \equiv E_{\ell+1}$ $\pmod{\ell}$.
□

Theorem

Let $\ell > 3$ be prime, $1 \le \beta_\ell < \ell$ such that $24\beta_\ell \equiv 1 \pmod{\ell}$, and let

$$r_\ell = \frac{24\beta_\ell - 1}{\ell}, \quad \lambda_\ell = \frac{\ell^2 + 24\beta_\ell - 1}{24\ell} = \frac{1}{24}(\ell + r_\ell).$$

Suppose $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$. Then

$$(Pf)|U^* \equiv E(q)^{r_\ell} g \pmod{\ell},$$

for some $g \in \overline{\mathcal{M}}_{k+\ell-1-12\lambda_\ell}$ and $w(g) \le \frac{k+\ell^2-1-12\beta_\ell}{\ell}$.

Proof Then by a Theorem from LECTURE 3,

$$(Pf)|U^* \equiv E(q)g \pmod{\ell}$$

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So

$$\begin{aligned}\left((Pf)|U^*\right)^\ell &\equiv E(q)^{r_\ell \ell} g^\ell \pmod{\ell}\\ &\equiv PE(q)^{24\beta_\ell} g^\ell \pmod{\ell},\end{aligned}$$

and

$$q^{\beta_\ell}\left((Pf)|U^*\right)^\ell \equiv P\Delta^{\beta_\ell} g^\ell \pmod{\ell}.$$

Let $w(g) = k'$ so that for some isboaric polynomial $Q[E_4, E_6]$ of weight k' in $\mathbb{F}_\ell[E_4, E_6]$,

$$\overline{g} = \overline{Q}(E_4, E_6).$$

Since $w(g) = k'$, $\overline{A} \nmid \overline{g}$. Also $E_4^3 - E_6^2 \nmid \overline{A}$, $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$ is a irreducible element of $\mathbb{F}_\ell[E_4, E_6]$. $\gcd(\overline{\Delta}, \overline{A}) = 1$, $\overline{A} \nmid \overline{\Delta^{\beta_\ell} g^\ell}$ so that

$$w(\Delta^{\beta_\ell} g^\ell) = 12\beta_\ell + \ell k'.$$

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By a previous LEMMA

$$\begin{aligned} q^{\beta_\ell}(Pf|U^*)^\ell &\equiv f - (\Theta^*)^{\ell-1}Pf \pmod{\ell} \\ &\equiv Ph \pmod{\ell} \end{aligned}$$

for some $h \in \overline{\mathcal{M}}_{k+(\ell-1)(\ell+1)}$ by other previous LEMMA. Hence

$$w(\Delta^{\beta_\ell} g^\ell) = 12\beta_\ell + \ell k' \leq k + (\ell-1)(\ell+1),$$

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$$k' = w(g) \leq \frac{1}{\ell}(k + \ell^2 - 1 - 12\beta_\ell)$$

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NOTE: This Theorem is an improvement of a THEOREM in LECTURE 3.

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<u>NOTE</u>: This Theorem is an improvement of a THEOREM in LECTURE 3.

Theorem

Suppose $F \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$ *and*

$$\sum_{n=0}^{\infty} \alpha_{r_\ell}(F, n) q^n = E(q)^{r_\ell} F(q).$$

Then

$$\sum_{n=0}^{\infty} \alpha_{r_\ell}(F, \ell n + \ell - \lambda_\ell) q^n = E(q)^{23} g(q) \pmod{\ell}.$$

for some $g \in \overline{\mathcal{M}}_{k+12\lambda_\ell-12}$, *with* $w(g) \leq \ell + \frac{1}{\ell}(12\lambda_\ell + k - 1) - 12.$

Theorem

Suppose $f \in \mathcal{M}_k(\mathbb{Z}_{(\ell)})$, $j \geq 1$, *and*

$$g(q) = \sum_{n=0}^{\infty} \gamma(n) q^n = E(q)^{r_\ell} f(q).$$

THEN

$$\sum_{n=0}^{\infty} (n + \lambda_\ell)^j \gamma(n) q^n \equiv E(q)^{r_\ell} h(q) \pmod{\ell}$$

for some $h(q) \in \mathcal{M}_{k+j(\ell+1)}(\mathbb{Z})$.

PROOF OF RANK CRANK MOD 13

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There is a connection between rank and crank moments:

$$spt(n) = \frac{1}{2}(M_2(n) - N_2(n)),$$

for $n \geq 1$.

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$$\sum_{n=0}^{\infty} spt(\ell n + \beta_\ell) q^n \equiv E(q)^{r_\ell} L_1(q) \pmod{\ell}$$

where $L_1 \in \mathcal{M}_{\ell+1-12\lceil \frac{\ell}{24} \rceil} = \mathcal{M}_{\ell+1-\frac{1}{2}(\ell+r_\ell)}$.

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From a previous result we have

$$\sum_{n=0}^{\infty} p(\ell n + \beta_\ell) q^n \equiv E(q)^{r_\ell} L_2(q) \pmod{\ell}$$

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EXAMPLE $\ell = 13, r_\ell = 11,\ k = 0.$ Since $\dim M_2 = 0$ we find that

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We prove an improvement of THEOREM (LECTURE 3).
NOTATION:

$$w_1(a,b,\ell) = a(\ell+1) + 2b + (\ell-1) - \frac{1}{2}(\ell + r_\ell),$$
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For $2 \le k \le \frac{1}{2}(\ell-1)$,

$$\begin{aligned} \mathcal{S}_{k,\ell} = &\{(a,b,j) : j \in \{0,4,6,\ldots,\ell-3,\ell+1\}, 0 \le a \le k, 0 \le b \le k-a, \\ &w_1(a,b,\ell) \ge 0, j \le \mu(a,b,\ell), j \equiv w_1(a,b,\ell) \pmod{\ell-1}\}. \end{aligned}$$

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$$\overline{m}(j,k,\ell) = max\{\mu(a,b,\ell) : (a,b,\ell) \in \mathcal{S}_{k,\ell}\},$$
$$m(j,k,\ell) = j + (\ell-1)\lfloor \frac{\overline{m}(j,k,\ell) - j}{\ell - 1} \rfloor.$$

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Theorem

For $2 \le k \le \frac{1}{2}(\ell-3)$,

$$\sum_{n=0}^{\infty} R_{2k}(\ell n+\beta_\ell)q^n \equiv c_k \sum_{n=0}^{\infty} R_2(\ell n+\beta_\ell)q^n + E(q)^{r_\ell} G_{\ell,R,2k}(q) \pmod{\ell}$$

for some $c_k \in \mathbb{Z}$, where $G_{\ell,R,2k}(q) = \sum_{j \in \mathcal{J}_{k,\ell}} g_j(q)$ with each $g_j(q) \in \overline{\mathcal{M}}_{m(j,k,\ell)}$.

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EXAMPLE $\ell = 13$, $2 \leq k \leq 5$. It turns out that in this case:

k	$\mathcal{J}_{k,13}$
2	{0,4}
3	{0,4,6}
4	{0,4,6,8}
5	{0,4,6,8,10}

and $m(j, k, 13) = j$ for $j \in \mathcal{J}_{k,13}$.

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We find (after some calculation) that

$$\sum_{n=0}^{\infty} N_4(13n+6)q^n \equiv E(q)^{11}(6E_4+12) \pmod{13}$$

$$\sum_{n=0}^{\infty} N_6(13n+6)q^n \equiv E(q)^{11}(4+E_4+10E_6) \pmod{13}$$

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$$\sum_{n=0}^{\infty} N_{10}(13n+6)q^n \equiv E(q)^{11}(4+2E_4+10E_6+12E_{10}) \pmod{13}$$

THE $(\ell-1)$TH RANK MOMENT

Theorem

$$\sum_{n=0}^{\infty} R_{\ell-1}(\ell n+\beta_\ell)q^n \equiv c_{\ell-1}\sum_{n=0}^{\infty} N_2(\ell n+\beta_\ell)q^n + E(q)^{r_\ell} G_{\ell,R,\ell-1}(q) + \frac{1}{\ell}E(q)^{r_\ell}(H_{1,\ell}(z)-H_{2,\ell}(z)) \pmod{\ell},$$

for some integer $c_{\ell-1}$ where $G_{\ell,R,\ell-1}(q)$ is a sum of ℓ-integral modular forms $G_{\ell,R,\ell-1}(q)=\sum_{j\in\mathcal{J}_{\frac{1}{2}(\ell-1),\ell}} g_j(q)$ where each $g_j(q)\in\overline{\mathcal{M}}_{m(j,\frac{1}{2}(\ell-1),\ell)}$ and $H_{1,\ell}(z)-H_{2,\ell}(z)$ are as in LECTURE 3 THEOREM.

Conjecture

$$\sum_{n=0}^{\infty} R_{\ell-1}(\ell n+\beta_\ell)q^n$$
$$\equiv c_{\ell-1}\sum_{n=0}^{\infty} R_2(\ell n+\beta_\ell)q^n$$
$$+E(q)^{r_\ell}G_{\ell,R,\ell-1}(q)+\Phi_{\ell-2}\cdot h(q) \pmod{\ell}$$

where $h(q)\in\overline{\mathcal{M}}_{\ell-\frac{1}{2}(r_\ell+\ell)}$ *and*

$$\Phi_{\ell-2}=\sum_{n=1}^{\infty}\sigma_{\ell-2}(n)q^n.$$

EXAMPLE. $\ell = 13$. Proceeding as before we find that

$$U^*(F) \equiv E^{11}(2E_4 + 6E_6 + E_{10} + 4E_6^2 + 9\Delta) \pmod{13}.$$

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We find

$$\Delta^7 | T(13) = \Delta(z) H_1(z)$$

$H_1(z) \in \mathcal{M}_{72}(\mathbb{Z})$ and

$$\begin{aligned} H_1(z) \equiv & 11E_6^{12} + 46E_6^{10}\Delta + 116E_6^8\Delta^2 + 60E_6^6\Delta^3 \\ & + 16E_6^4\Delta^4 + 154E_6^2\Delta^5 + 93\Delta^6 \pmod{13^2}. \end{aligned}$$

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SIMILARLY

$$qU^*(E_{12}P) \equiv \frac{(\Delta^7 E_{12})|U}{E^{13}} \equiv \frac{(\Delta^7 E_{12})|T(13)}{E^{13}} \pmod{13^2}.$$

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$$B = 9E_6^{12} + E_6^{10}\Delta + 11E_6^{8}\Delta^2 + 3E_6^{6}\Delta^3 + 6E_6^{4}\Delta^4 + 9E_6^{2}\Delta^5 + 4\Delta^6,$$

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After CALCULATION we find that $w(A)=12$, $w(B)=0$ and

$$A \equiv E_6^2 + \Delta \pmod{13}$$
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THEREFORE

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CRANK MOMENTS MOD ℓ

Since

$$M_2(n) = \frac{1}{2}np(n)$$

the second crank moment is easier than the second rank moment and

$$\sum_{n=0}^{\infty} M_2(\ell n + \beta_\ell)q^n \equiv E(q)^{r_\ell} G_{\ell,C,2}(q) \pmod{\ell}$$

where $G_{\ell,R,2}(q) \in \overline{M}_{\ell-1-\frac{1}{2}(r_\ell+\ell)}$.

EXAMPLE $\ell = 13, r_\ell = 11$, $\ell - 1 - \frac{1}{2}(r_\ell + \ell) = 0$. we find that

$$\sum_{n=0}^{\infty} M_2(13n+6)q^n \equiv 4E(q)^{11} \pmod{13}.$$

The analogues of the rank moments mod ℓ hold for the crank moments mod ℓ except there is no extra term involving the second crank moment.

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$$\sum_{n=0}^{\infty} M_2(13n+6)q^n \equiv 2E(q)^{11} \pmod{13}$$

$$\sum_{n=0}^{\infty} M_4(13n+6)q^n \equiv 12E(q)^{11}(E_4+1) \pmod{13}$$

$$\sum_{n=0}^{\infty} M_6(13n+6)q^n \equiv E(q)^{11}(2E_4+9E_6+4) \pmod{13}$$

$$\sum_{n=0}^{\infty} M_8(13n+6)q^n \equiv 4E(q)^{11}(E_4+2E_6+E_8+2) \pmod{13}$$

$$\sum_{n=0}^{\infty} M_{10}(13n+6)q^n \equiv E(q)^{11}(9E_4E_6+4E_4+9E_6+2E_8+4) \pmod{13}$$

$$\sum_{n=0}^{\infty} M_{12}(13n+6)q^n \equiv E(q)^{11}(5E_6^2+4E_{10}+4E_4+8E_6 + 2E_8+9\Phi_{11}+12\Delta) \pmod{13}$$

RANK AND CRANK DIFFERENCES MOD 13

DEFINE

$$N_{r_1,r_2,\ell}(n) = N(r_1,\ell,n) - N(r_2,\ell,n)$$

and

$$r_{a,b}(d) = r_{a,b}(d,t) = \sum_{n=0}^{\infty} N_{a,b,t}(tn+d)q^n$$

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Rank differences mod ℓ can be expressed in terms of $p(n)$ and the rank moments $N_{2k}(n)$ for $1 \le k \le \frac{1}{2}(\ell - 3)$.

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DEFINE

$$M_{r_1,r_2,\ell}(n) = M(r_1,\ell,n) - M(r_2,\ell,n)$$

and

$$c_{a,b}(d) = c_{a,b}(d,t) = \sum_{n=0}^{\infty} M_{a,b,t}(tn+d)q^n$$

Crank differences mod ℓ can be expressed in terms of $p(n)$ and the rank moments $M_{2k}(n)$ for $1 \le k \le \frac{1}{2}(\ell-3)$.

DEFINE

$$M_{r_1,r_2,\ell}(n) = M(r_1,\ell,n) - M(r_2,\ell,n)$$

and

$$c_{a,b}(d) = c_{a,b}(d,t) = \sum_{n=0}^{\infty} M_{a,b,t}(tn+d)q^n$$

Crank differences mod ℓ can be expressed in terms of $p(n)$ and the rank moments $M_{2k}(n)$ for $1 \le k \le \frac{1}{2}(\ell-3)$.

Theorem

$$
\begin{aligned}
r_{0,1}(6) &\equiv E(q)^{11}(12E_{10} + 11E_4 + 2) \pmod{13},\\
r_{0,2}(6) &\equiv E(q)^{11}(9E_{10} + 2E_4 + 9E_6 + 6) \pmod{13},\\
r_{0,3}(6) &\equiv E(q)^{11}(4E_{10} + E_4 + E_6 + 7) \pmod{13},\\
r_{0,4}(6) &\equiv E(q)^{11}(10E_{10} + 4E_4 + 3E_6 + 10) \pmod{13},\\
r_{0,5}(6) &\equiv E(q)^{11}(E_{10} + 5E_4 + 12E_6 + 8) \pmod{13},\\
r_{0,6}(6) &\equiv E(q)^{11}(3E_{10} + 3E_4 + E_6 + 7) \pmod{13}.
\end{aligned}
$$

Theorem

$$c_{0,4}(6) \equiv E(q)^{11}(9E_{10} + 9E_4 + 6E_8 + 2) \pmod{13},$$
$$c_{0,4}(6) \equiv E(q)^{11}(10E_{10} + 4E_4 + 12E_6 + 7E_8 + 6) \pmod{13},$$
$$c_{0,4}(6) \equiv E(q)^{11}(3E_{10} + 2E_4 + 10E_6 + 4E_8 + 7) \pmod{13},$$
$$c_{0,4}(6) \equiv E(q)^{11}(E_{10} + 8E_4 + 4E_6 + 3E_8 + 10) \pmod{13},$$
$$c_{0,4}(6) \equiv E(q)^{11}(4E_{10} + 10E_4 + 3E_6 + 2E_8 + 8) \pmod{13},$$
$$c_{0,4}(6) \equiv E(q)^{11}(12E_{10} + 6E_4 + 10E_6 + 4E_8 + 7) \pmod{13}.$$

Theorem

$$N(0,13,13n+6) \equiv \cdots \equiv N(12,13,13n+6) \pmod{13} \qquad \text{(A)}$$

if and only if

$$M(0,13,13n+6) \equiv \cdots \equiv M(12,13,13n+6) \pmod{13}. \qquad \text{(B)}$$

Further, if either Congruence (A) or (B) holds then

$$N(0,13,13n+6) \equiv M(0,13,13n+6) \pmod{13}. \qquad \text{(C)}$$

PROOF DEFINE

$$F_0 = \sum_{n=0}^{\infty} f_0(n)q^n = E(q)^{11},$$

$$F_4 = \sum_{n=0}^{\infty} f_4(n)q^n = E(q)^{11}E_4,$$

$$F_6 = \sum_{n=0}^{\infty} f_6(n)q^n = E(q)^{11}E_6,$$

$$F_8 = \sum_{n=0}^{\infty} f_8(n)q^n = E(q)^{11}E_4^2,$$

$$F_{10} = \sum_{n=0}^{\infty} f_{10}(n)q^n = E(q)^{11}E_4E_6,$$

$$F_{12} = \sum_{n=0}^{\infty} f_{12}(n)q^n = E(q)^{11}(E_6^2 + \Delta).$$

$\boxed{\Rightarrow}$

Suppose (A) holds for a fixed n. Then consider the coefficient matrix

$$B = \begin{bmatrix} 2 & 11 & 0 & 12 \\ 6 & 2 & 9 & 9 \\ 7 & 1 & 1 & 4 \\ 10 & 4 & 3 & 10 \end{bmatrix}$$

The i-th row of B corresponds to the coefficient of F_0, F_4, F_6, F_{10} in equation for $N_{0,i,13}(13n+6)$. Therefore since $\det(B) \equiv 6 \not\equiv 0 \pmod{13}$ this implies that

$$f_0(n) \equiv f_4(n) \equiv f_6(n) \equiv f_{10}(n) \equiv 0 \pmod{13}$$

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We need to show that

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By qdq THEOREM

$$\sum_{n=0}^{\infty}(n+1)^5 f_{10}(n)q^n \equiv E(q)^{11}h(q) \pmod{13}$$

for some $h(q) \in M_{80}(\mathbb{Z})$. $80 \equiv 8 \pmod{12}$ and we find that $w(h) = 8$ and $h(q) \equiv E_8 \pmod{13}$, so that

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and $f_8(n) \equiv 0 \pmod{13}$ as required. This implies (B) holds.

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and $f_8(n) \equiv 0 \pmod{13}$ as required. This implies (B) holds.

$\boxed{\Leftarrow}$

The proof that (B) implies (A) is straightforward.

Now suppose (A) or (B) holds for a fixed n. Then

$$f_0(n) \equiv f_4(n) \equiv f_6(n) \equiv f_8(n) \equiv f_{10}(n) \equiv 0 \pmod{13}$$

$\Leftarrow$

The proof that (B) implies (A) is straightforward.
Now suppose (A) or (B) holds for a fixed n. Then

$$f_0(n) \equiv f_4(n) \equiv f_6(n) \equiv f_8(n) \equiv f_{10}(n) \equiv 0 \pmod{13}$$

We find that

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(0,13,13n+6) - M(0,13,13n+6))q^n \\ &\equiv (E(q))^{11}(4 + 2E_4 + 2E_6 + 2E_8 + 3E_{10} + 2\Delta) \pmod{13} \\ &\equiv (E(q))^{11}(2E_4 + 2E_6 + 2E_8 + 3E_{10} + 4(E_6^2 + \Delta)) \pmod{13}\end{aligned}$$

so that

$\Leftarrow$

The proof that (B) implies (A) is straightforward.
Now suppose (A) or (B) holds for a fixed n. Then

$$f_0(n) \equiv f_4(n) \equiv f_6(n) \equiv f_8(n) \equiv f_{10}(n) \equiv 0 \pmod{13}$$

We find that

$$\sum_{n=0}^{\infty}(N(0,13,13n+6) - M(0,13,13n+6))q^n$$
$$\equiv (E(q))^{11}(4 + 2E_4 + 2E_6 + 2E_8 + 3E_{10} + 2\Delta) \pmod{13}$$
$$\equiv (E(q))^{11}(2E_4 + 2E_6 + 2E_8 + 3E_{10} + 4(E_6^2 + \Delta)) \pmod{13}$$

so that

$$N(0,13,13n+6) - M(0,13,13n+6)$$
$$\equiv 2f_4(n) + 2f_6(n) + 2f_8(n) + 3f_{10}(n) + 4f_{12}(n)) \pmod{13}$$

$\boxed{\Leftarrow}$

The proof that (B) implies (A) is straightforward.
Now suppose (A) or (B) holds for a fixed n. Then

$$f_0(n) \equiv f_4(n) \equiv f_6(n) \equiv f_8(n) \equiv f_{10}(n) \equiv 0 \pmod{13}$$

We find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(N(0,13,13n+6) - M(0,13,13n+6))q^n \\
&\equiv (E(q))^{11}(4 + 2E_4 + 2E_6 + 2E_8 + 3E_{10} + 2\Delta) \pmod{13} \\
&\equiv (E(q))^{11}(2E_4 + 2E_6 + 2E_8 + 3E_{10} + 4(E_6^2 + \Delta)) \pmod{13}
\end{aligned}$$

so that

$$\begin{aligned}
&N(0,13,13n+6) - M(0,13,13n+6) \\
&\equiv 2f_4(n) + 2f_6(n) + 2f_8(n) + 3f_{10}(n) + 4f_{12}(n)) \pmod{13}
\end{aligned}$$

Applying qdq THEOREM and proceeding as before we find that

$$\begin{aligned} f_{12}(n) &\equiv 2(n+1)^9 f_6(n) + 12 f_{10}(n) \pmod{13} \\ &\equiv 0 \pmod{13}, \end{aligned}$$

as required and (C) follows.