NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
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The University of Texas Rio Grande Valley

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May 16, 2022

LECTURE 1
RAMANUJAN'S PARTITON CONGRUENCES

THE PARTITION FUNCTION

RAMANUJAN'S PARTITION CONGRUENCES

DYSON'S RANK

DYSON's RANK CONJECTURE

A PAGE FROM THE LOST NOTEBOOK

DYSON'S CRANK CONJECTURE

THE PARTITION FUNCTION

THE PARTITION FUNCTION

$p(n)$

A *partition* of n is a nonincreasing sequence of positive integers whose sum is n.

$n = 1$

$$1$$

$n = 2$

$$2, \quad 1+1$$

$n = 3$

$$3, \quad 2+1, \quad 1+1+1$$

$n = 4$

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

$n = 5$

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1

n	$p(n) =$ the number of partitions of n
1	1
2	2
3	3
4	5
$\vdots$	
10	42
$\vdots$	
100	190569292
$\vdots$	
1000	24061467864032622473692149727991
$\vdots$	
10000	361672513256362939882047189095369549501603033931565042208186860588795256875406642059231055605290691643514

- HARDY - RAMANUJAN (1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad \text{(Hardy-Ramanujan 1918)}$$

- EULER:

$$\sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$$= \frac{1}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots}$$

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MACMAHON'S TABLE

1	1	21	792	41	44583	61	1121505
2	2	22	1002	42	53174	62	1300156
3	3	23	1255	43	63261	63	1505499
4	5	24	1575	44	75175	64	1741630
5	7	25	1958	45	89134	65	2012558
6	11	26	2436	46	105558	66	2323520
7	15	27	3010	47	124754	67	2679689
8	22	28	3718	48	147273	68	3087735
9	30	29	4565	49	173525	69	3554345
10	42	30	5604	50	204226	70	4087968
11	56	31	6842	51	239943	71	4697205
12	77	32	8349	52	281589	72	5392783
13	101	33	10143	53	329931	73	6185689
14	135	34	12310	54	386155	74	7089500
15	176	35	14883	55	451276	75	8118264
16	231	36	17977	56	526823	76	9289091
17	297	37	21637	57	614154	77	10619863
18	385	38	26015	58	715220	78	12132164
19	490	39	31185	59	831820	79	13848650
20	627	40	37338	60	966467	80	15796476

...contd.

*The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating the coefficients in the identity

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots)\sum_0^\infty p(n)x^n = 1.$$

We have verified the table by direct calculation up to $n = 158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

RAMANUJAN'S PARTITION CONGRUENCES

$$
\begin{array}{rlllllll}
(1) & p(4), & p(9), & p(14), & p(19), & \cdots & \equiv 0 & \pmod{5}, \\
(2) & p(5), & p(12), & p(19), & p(26), & \cdots & \equiv 0 & \pmod{7}, \\
(3) & p(6), & p(17), & p(28), & p(39), & \cdots & \equiv 0 & \pmod{11}, \\
(4) & p(24), & p(49), & p(74), & p(99), & \cdots & \equiv 0 & \pmod{25}, \\
(5) & p(19), & p(54), & p(89), & p(124), & \cdots & \equiv 0 & \pmod{35}, \\
(6) & p(47), & p(96), & p(145), & p(194), & \cdots & \equiv 0 & \pmod{49}, \\
(7) & p(39), & p(94), & p(149), & \cdots & & \equiv 0 & \pmod{55}, \\
(8) & p(61), & p(138), & \cdots & & & \equiv 0 & \pmod{77}, \\
(9) & p(116), & \cdots & & & & \equiv 0 & \pmod{121}, \\
(10) & p(99), & \cdots & & & & \equiv 0 & \pmod{125}.
\end{array}
$$

From these data I conjectured the truth of the following theorem: if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$ then

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\cdots \equiv 0 \pmod{\delta}.$$

This theorem is supported by all the available evidence; but I have not yet been able to find a general proof.

CORRECTION (Chowla)
"7^b" should be "$7^{\lfloor b/2 \rfloor + 1}$"

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RAMANUJAN'S SIMPLE PROOFS

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$

EULER

$$\prod_{n=1}^{\infty}(1-q^n) = 1 - q - q^2 + q^5 + q^7 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}(1+q^n)$$
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JACOBI

$$\prod_{n=1}^{\infty}(1-q^n)^3 = 1 - 3q + 5q^3 - 7q^6 + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

$$\equiv \frac{\prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^3}{\prod_{n=1}^{\infty}(1-q^{5n})} \pmod 5$$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$
$$= \frac{\prod_{n=1}^{\infty}(1-q^n)^4}{\prod_{n=1}^{\infty}(1-q^n)^5}$$
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$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \\ &= \frac{\prod_{n=1}^{\infty}(1-q^n)^4}{\prod_{n=1}^{\infty}(1-q^n)^5} \\ &\equiv \frac{\prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^3}{\prod_{n=1}^{\infty}(1-q^{5n})} \pmod 5\end{aligned}$$

$$\sum_{n=0}^{\infty} p(n)q^n$$

$$\equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2}$$

$$\times \sum_{k=0}^{\infty} p(k) q^{5k} \pmod 5$$

$$\equiv \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+m} p(k)(2m+1) q^{n(3n-1)/2+m(m+1)/2+5k} \pmod 5$$

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$$
\begin{aligned}
& n(3n-1)/2 + m(m+1)/2 + 5k \equiv 4 \pmod 5 \\
\Leftrightarrow \quad & 3n^2 - n + m^2 + m \equiv 3 \pmod 5 \\
\Leftrightarrow \quad & 3(n-1)^2 + (m-2)^2 \equiv 0 \pmod 5 \\
\Leftrightarrow \quad & n \equiv 1 \pmod 5 \quad \text{and} \quad m \equiv 2 \pmod 5
\end{aligned}
$$

$$p(5n+4) \equiv 0 \pmod 5$$

since $(2m+1) \equiv 0 \pmod 5$ when $m \equiv 2 \pmod 5$.

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Ramanujan's proof of

$$p(7n+5) \equiv 0 \pmod 7$$

is similar and uses

$$\prod_{n=1}^{\infty}(1-q^n)^6 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(2n+1)(2m+1)q^{n(n+1)/2+m(m+1)/2}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

EISENSTEIN SERIES

$$P = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

$$R = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}$$

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$$q\frac{dP}{dq} = \frac{1}{12}(P^2 - Q), \quad q\frac{dQ}{dq} = \frac{1}{3}(PQ - R), \quad q\frac{dR}{dq} = \frac{1}{2}(PR - Q^2)$$

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$$Q^3 - R^2 = 1728\, q \prod_{n=1}^{\infty} (1-q^n)^{24} := 1728 \sum_{n=1}^{\infty} \tau(n) q^n$$

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where

$$\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}, \qquad B_k \text{ is the } k\text{-th Bernoulli number}$$

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NOTE: E_{2k} is a modular form of weight $2k$ (for $k > 1$)

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THETA OPERATOR

$$\Theta := q\frac{d}{dq}$$

so that

$$\Theta^j\left(\sum a(n)q^n\right) = \sum n^j a(n)q^n$$

RAMANUJAN

$$\Theta^j(E_{2k}) = \sum_{2\ell+4m+6m=2k+2j} \alpha_{\ell,m,n} P^\ell Q^m R^n$$

Quasi-modular form of weight $2\ell + 4m + 6n = 2k + 2j$

RAMANUJAN

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Quasi-modular form of weight $2\ell + 4m + 6n = 2k + 2j$

SOME CALCULATIONS

$$E_{10} = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n \equiv 1 \pmod{11}$$

$$E_{10} = E_4 E_6 = QR, \quad \text{and} \quad QR - 1 \equiv 0 \pmod{11}$$

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$$E_{12} = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n$$

$$\equiv 1 + 9\sum_{n=1}^{\infty}\sigma_1(n)q^n \pmod{11} \equiv P \pmod{11}$$

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$$\begin{aligned} E_{12} &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \\ &\equiv 1 + 9 \sum_{n=1}^{\infty} \sigma_1(n) q^n \pmod{11} \equiv P \pmod{11} \end{aligned}$$

$$E_{12} = \frac{1}{691}\left(250R^2 + 441Q^3\right) \equiv 5Q^3 + 7R^2 \pmod{11}$$

so that

$$Q^3 - 3R^2 + 2P \equiv 0 \pmod{11}$$

SOME EISTENSTEIN CONGRUENCES

$$QR - 1 \equiv 0 \quad \text{and} \quad Q^3 - 3R^2 + 2P \equiv 0 \pmod{11}$$

A LITTLE MAPLE

```
> with(Groebner):
> RELS11 := [Q^3 - 3*R^2 + 2*P, Q*R - 1]:
> GB := Basis(RELS11, tdeg(P,Q,R),
characteristic=11):
> NormalForm( (Q^3 - R^2)^5, GB, tdeg(P,Q,R),
characteristic=11);
```

$$P^5 + 8P^3Q + 7P^2R + 6$$

$$(Q^3-R^2)^5 \equiv q^5 \prod_{n=1}^{\infty}(1-q^n)^{120} \equiv P^5+8P^3Q+7P^2R+6QR \pmod{11}$$

a quasi-modular form of weight 10

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a quasi-modular form of weight 10

$$q^5 \prod_{n=1}^{\infty}(1-q^n)^{120}$$
$$\equiv \sum_{n=1}^{\infty} \left(10n^4\sigma_1(n)+3n^3\sigma_3(n)+3n^2\sigma_5(n)+6n\sigma_7(n)\right) q^n \pmod{11}.$$

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$$\sum_{n=0}^{\infty} p(n)q^{n+5} = q^5 \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = q^5 \prod_{n=1}^{\infty} \frac{(1-q^n)^{120}}{(1-q^n)^{121}}$$

$$\equiv \left(\sum_{n=1}^{\infty} \left(10n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) + 6n\sigma_7(n)\right) q^n \right) \sum_{k=0}^{\infty} p(k)q^{121}$$

Picking out the coeff of q^{11m} we have

$$p(11m-5) \equiv 0 \pmod{11}.$$

$$\sum_{n=0}^{\infty} p(n)q^{n+5} = q^5 \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = q^5 \prod_{n=1}^{\infty} \frac{(1-q^n)^{120}}{(1-q^n)^{121}}$$

$$\equiv \left(\sum_{n=1}^{\infty} \left(10n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) + 6n\sigma_7(n) \right) q^n \right) \sum_{k=0}^{\infty} p(k)q^{121}$$

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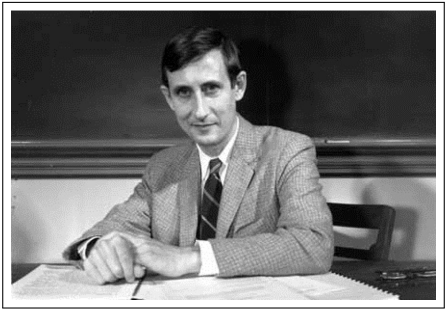

DYSON'S RANK CONJECTURE (1944, 1947)

PROBLEMS FOR SOLUTION

4259. *Proposed by Richard Bellman, Princeton University*

If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1 + x^{n_k}} = x \prod_{k=1}^{\infty} (1 + x^{n_k}), \qquad |x| < 1,$$

show that, except perhaps for order,

$$n_k = 2^k.$$

4260. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC inscribe two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose sides are parallel to the medians. Show that (1) the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$, have the same centroid and the same Brocard angle; (2) the triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in an ellipse concentric and homothetic to the inscribed Steiner ellipse, the ratio of homothety being $1/\sqrt{3}$.

4261. *Proposed by F. J. Dyson, Trinity College, Cambridge, England*

The number of partitions of an integer n into a sum of positive integral parts is denoted by $p(n)$. The result of subtracting the number of parts in a partition from the largest part is a positive or negative integer called the rank of the partition. Ramanujan proved that $p(5n+4)$ is always divisible by 5, and $p(7n+5)$ by 7. Show that the number of partitions of $5n+4$ whose ranks are congruent modulo 5 to a given residue is the same whichever of the five residues is chosen, and the number of partitions of $7n+5$ whose ranks are congruent modulo 7 to a given residue is the same whichever of the seven residues is chosen.

Notation

$$(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a)_\infty = (a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1})$$

Notation

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The Dyson rank of a partition is the largest part minus the number of parts.

Example

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

Let $N(m,n)$ denote the number of partitions of n with rank m. Then

$$R(z,q) = \sum_{n\geq 0}\sum_{m} N(m,n) z^m q^n$$

$$= 1 + \sum_{k\geq 1} \frac{q^{k^2}}{(zq;q)_k (z^{-1}q;q)_k}$$

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The Dyson rank of a partition is the largest part minus the number of parts.

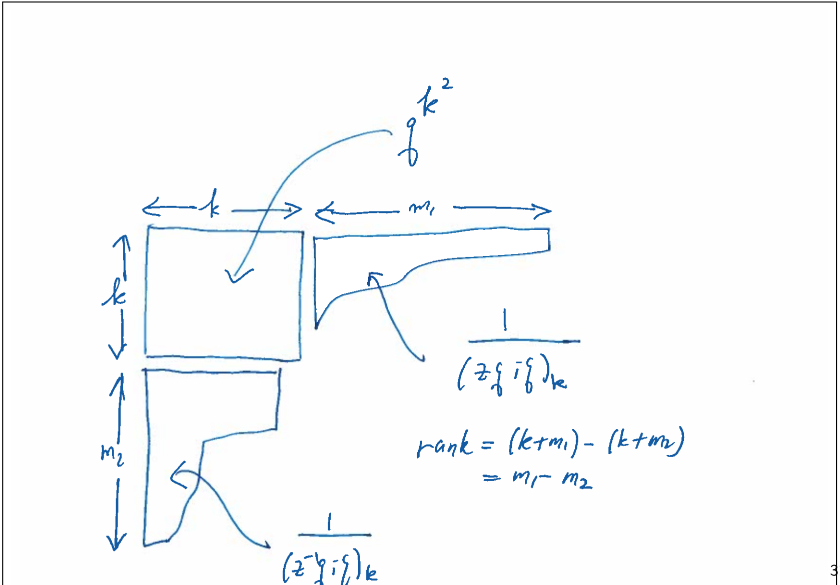
Example

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

Let $N(m,n)$ denote the number of partitions of n with rank m. Then

$$R(z,q) = \sum_{n\geq 0}\sum_{m} N(m,n) z^m q^n$$

$$= 1 + \sum_{k\geq 1} \frac{q^{k^2}}{(zq;q)_k (z^{-1}q;q)_k}$$


q^{k^2}
k
m_1
k
m_2
$\frac{1}{(zq;q)_k}$
rank = $(k+m_1)-(k+m_2)$
$= m_1 - m_2$
$\frac{1}{(z^{-1}q;q)_k}$

Let $N(r,t,n)$ denote the number of partitions of n with rank $\equiv r \pmod{t}$.

Example

$$N(0,5,4) = N(1,5,4) = N(2,5,4) = N(3,5,4) = N(4,5,4) = 1$$

Dyson's Rank Conjectures (1944)

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{1}{5}p(5n+4)$$

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{1}{7}p(7n+5)$$

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Atkin and Swinnerton-Dyer (1953)

Let $\zeta_p = \exp(2\pi i/p)$. Then

$$R(\zeta_p, q) = \sum_{n\geq 0}\sum_{m} N(m,n)\zeta_p^m q^n$$
$$= \sum_{n\geq 0}\left(\sum_{k=0}^{p-1} N(k,p,n)\zeta_p^k\right) q^n.$$

$DRC.1 \iff$ Coeff of q^{5n+4} in $R(\zeta_5, q) = 0$

$DRC.2 \iff$ Coeff of q^{7n+5} in $R(\zeta_7, q) = 0$

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$$R(\zeta_p, q) = \sum_{n\geq 0}\sum_{m} N(m,n)\zeta_p^m q^n = \sum_{n\geq 0}\left(\sum_{k=0}^{p-1} N(k,p,n)\zeta_p^k\right) q^n.$$

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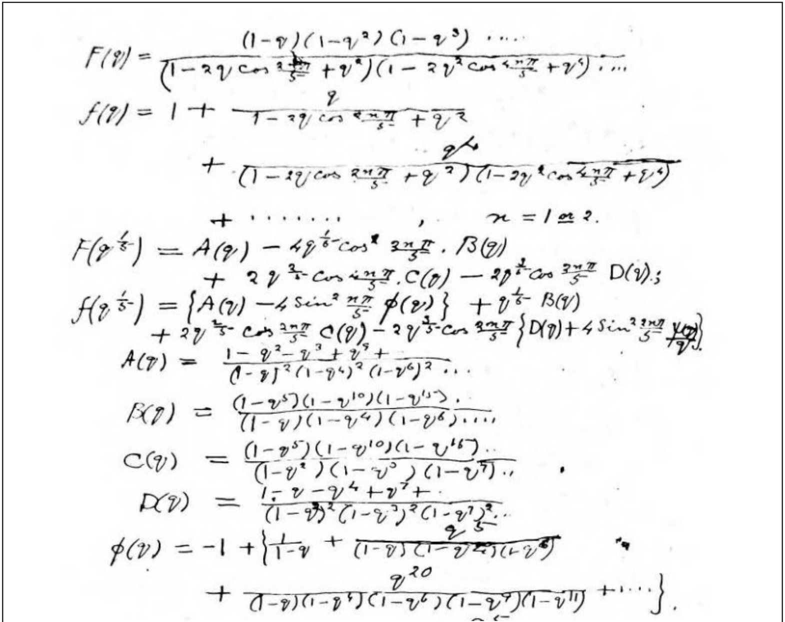
GEORGE ANDREWS



$$F(q)=\frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{4n\pi}{5}+q^4)\cdots}$$

$$f(q)=1+\frac{q}{1-2q\cos\frac{2n\pi}{5}+q^2}+\frac{q^4}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{4n\pi}{5}+q^4)}+\cdots\cdots, \quad n=1 \text{ or } 2.$$

$$F(q^{\frac{1}{5}})=A(q)-4q^{\frac{1}{5}}\cos^2\frac{2n\pi}{5}.B(q)+2q^{\frac{2}{5}}\cos\frac{4n\pi}{5}.C(q)-2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}D(q);$$

$$f(q^{\frac{1}{5}})=\left\{A(q)-4\sin^2\frac{n\pi}{5}\phi(q)\right\}+q^{\frac{1}{5}}B(q)+2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}C(q)-2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(q)+4\sin^2\frac{2n\pi}{5}\frac{\psi(q)}{q}\right\}.$$

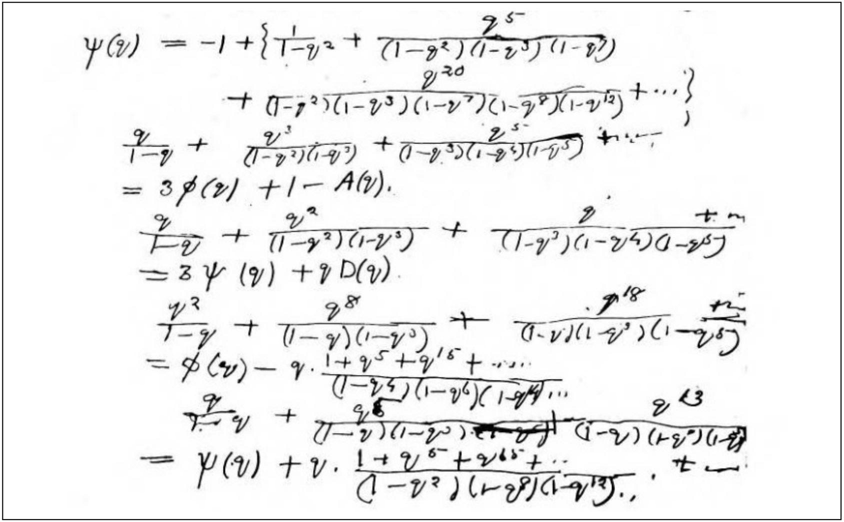
$$A(q)=\frac{1-q^2-q^3+q^9+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$B(q)=\frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots}$$

$$C(q)=\frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots}.$$

$$D(q)=\frac{1-q-q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots}$$

$$\phi(q)=-1+\left\{\frac{1}{1-q}+\frac{q^5}{(1-q)(1-q^4)(1-q^6)}+\frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})}+\cdots\right\}.$$



$$\psi(q) = -1 + \left\{ \frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})} + \cdots \right\}$$

$$\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots = 3\phi(q) + 1 - A(q).$$

$$\frac{q}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q}{(1-q^3)(1-q^4)(1-q^5)} + \cdots = 3\psi(q) + q D(q).$$

$$\frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \frac{q^{18}}{(1-q)(1-q^3)(1-q^5)} + \cdots = \phi(q) - q \cdot \frac{1+q^5+q^{16}+\cdots}{(1-q^5)(1-q^6)(1-q^{14})\cdots}$$

$$\frac{q}{1-q} + \frac{q^6}{(1-q)(1-q^3)} + \frac{q^{13}}{(1-q)(1-q^3)(1-q^5)} + \cdots = \psi(q) + q \cdot \frac{1+q^6+q^{16}+\cdots}{(1-q^2)(1-q^8)(1-q^{12})\cdots}$$

$$
\begin{aligned}
R(\zeta_5, q) = {} & A(q^5) + (3 - \zeta_5^2 - \zeta_5^3)\phi(q^5) \\
& + qB(q^5) \\
& + q^2(\zeta_5 + \zeta_5^4)C(q^5) \\
& + q^3\left((1 + \zeta_5^2 + \zeta_5^3)D(q^5) + (1 + 2\zeta_5^2 + 2\zeta_5^3)\frac{\psi(q^5)}{q^5}\right)
\end{aligned}
$$

where

$$A(q) = \frac{1 - q - q^3 + q^9 + \cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$\phi(q) = -1 + \frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \cdots$$

$RLN.1 \quad \Longleftrightarrow \quad ASD.Thm$

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$RLN.1 \iff ASD.Thm$

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$$RLN.1 \iff ASD.Thm$$

$$\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\cdots$$
$$=3\phi(q)+1-A(q)$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{4}N(r,5,5n)\zeta_5^r q^n$$
$$=\sum_{n=0}^{\infty}\left((N(0,5,5n)-N(1,5,5n)+(\zeta_5^2+\zeta_5^3)(N(2,5,5n)-N(1,5,5n)\right) q$$
$$=A(q)-(3+\zeta_5^2+\zeta_5^3)\phi(q).$$

$$\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\cdots$$
$$=3\phi(q)+1-A(q)$$

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$$=A(q)-(3+\zeta_5^2+\zeta_5^3)\phi(q).$$

$$-3\phi(q) + A(q) = \sum_{n=0}^{\infty} \left((N(0,5,5n) - N(1,5,5n) \right) q^n$$

$$\phi(q) = \sum_{n=0}^{\infty} \left((N(1,5,5n) - N(2,5,5n) \right) q^n$$

FIFTH ORDER MOCK THETA CONJECTURE

$$\chi_0(q) - 2 = \sum_{n=0}^{\infty} \left((N(1,5,5n) - N(0,5,5n) \right) q^n$$

$$-3\phi(q) + A(q) = \sum_{n=0}^{\infty} ((N(0,5,5n) - N(1,5,5n))\, q^n$$

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$$\chi_0(q) - 2 = \sum_{n=0}^{\infty} ((N(1,5,5n) - N(0,5,5n))\, q^n$$

```
> with(qseries):
> with(modforms):
> series(modp(sift(P,q,5,1,50),5),q,6);
```

$$1 + q + q^2 + q^3 + 2q^4 + q^5 + 0(q^6)$$

```
> series(modp(sift(P,q,7,1,50),7),q,8);
```

$$1 + q + q^2 + q^3 + q^4 + q^5 + 2q^6 + q^7 + O(q^8)$$

```
> series(modp(sift(P,q,11,1,130),11),q,14);
```

$$1 + q^2 + q^3 + q^4 + q^6 + q^8 + q^9 + q^{10} + q^{11} + O(q^{12})$$

```
> series(modp(sift(P,q,13,1,130),13),q,12);
```

$$1 + 5q + 7q^2 + 2q^3 + 4q^4 + 4q^5 + 10q^6 + 8q^7 + 11q^8 + 4q^9 + O(q^{10})$$

$$\sum_{n=0}^{\infty} p(5n+1)q^n \equiv 1 + q + q^2 + q^3 + 2q^4 + q^5 + \cdots \pmod 5$$

$$\sum_{n=0}^{\infty} p(7n+1)q^n \equiv 1 + q + q^2 + q^3 + q^4 + q^5 + \cdots \pmod 7$$

$$\sum_{n=0}^{\infty} p(11n+1)q^n \equiv 1 + q^2 + q^3 + q^4 + q^6 + \cdots \pmod{11}$$

$$p(5n+1) \equiv N(0,5,5n+1) - N(1,5,5n+1) \pmod 5$$

$$p(7n+1) \equiv N(0,5,7n+1) - N(3,7,7n+1) \pmod 7$$

$$\sum_{n=0}^{\infty} p(5n+1)q^n \equiv 1 + q + q^2 + q^3 + 2q^4 + q^5 + \cdots \pmod{5}$$

$$\sum_{n=0}^{\infty} p(7n+1)q^n \equiv 1 + q + q^2 + q^3 + q^4 + q^5 + \cdots \pmod{7}$$

$$\sum_{n=0}^{\infty} p(11n+1)q^n \equiv 1 + q^2 + q^3 + q^4 + q^6 + \cdots \pmod{11}$$

$$p(5n+1) \equiv N(0,5,5n+1) - N(1,5,5n+1) \pmod{5}$$

$$p(7n+1) \equiv N(0,5,7n+1) - N(3,7,7n+1) \pmod{7}$$

I hold in fact:
That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $M(m, q, n)$ *the number of partitions of* n *whose crank is congruent to* m *modulo* q*;*
that $M(m, q, n) = M(q - m, q, n)$*;*
that

$$\begin{aligned} M(0, 11, 11n+6) &= M(1, 11, 11n+6) = M(2, 11, 11n+6) \\ &= M(3, 11, 11n+6) = M(4, 11, 11n+6); \end{aligned}$$

that numerous other relations exist analogous to (12)–(19), and in particular

$$\begin{aligned} M(1, 11, 11n+1) &= M(2, 11, 11n+1) \\ &= M(3, 11, 11n+1) = M(4, 11, 11n+1); \end{aligned}$$

that $M(m, 11, n)$ *has a generating function not completely different in form from (24);*
that the values of the differences such as $M(0, 11, n) - M(4, 11, n)$ *are always extremely small compared with* $p(n)$*.*
Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

THE ANDREWS-G. CRANK

For a partition π let $\ell(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. Then the *crank*, $c(\pi)$ is given by

$$c(\pi) = \begin{cases} \ell(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0, \end{cases}$$

For example, the partition $6 + 5 + 4 + 4 + 2$ has crank $= 6$ (the largest part since there are no ones). For the partition $6 + 5 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1$, $\omega = 4$, and $\mu = 2$ so that the crank $= 2 - 4 = -2$.

Theorem

The residue of the crank mod 11 *divides the partitions of* $11n+6$ *into* 11 *equal classes. Also, analogous results hold mod* 5*, mod* 7 *for the partitions of* $5n+4$ *and* $7n+5$ *respectively.*

We illustrate with an example.

Partitions of 6	Crank (mod 11)
6	$6 \equiv 6$
$5+1$	$1-1 \equiv 0$
$4+2$	$4 \equiv 4$
$4+1+1$	$1-2 \equiv 10$
$3+3$	$3 \equiv 3$
$3+2+1$	$2-1 \equiv 1$
$3+1+1+1$	$0-3 \equiv 8$
$2+2+2$	$2 \equiv 2$
$2+2+1+1$	$0-2 \equiv 9$
$2+1+1+1+1$	$0-4 \equiv 7$
$1+1+1+1+1+1$	$0-6 \equiv 5$

We see that the crank mod 11 divides the partitions of 6 into 11 equal classes — one partition in each class.

WHERE DID THE CRANK COME FROM?

Define

$$C(z,q) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

RAMANUJAN

Then

$$\begin{aligned} C(\zeta_5, q) = {} & A(q^5) \\ & - q(2+\zeta_5^2+\zeta_5^3)B(q^5) \\ & + q^2(\zeta_5^2+\zeta_5^3)C(q^5) \\ & + q^3(1+\zeta_5^2+\zeta_5^3)D(q^5) \end{aligned}$$

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Then

$$\begin{aligned} C(\zeta_5, q) &= A(q^5) \\ &- q(2 + \zeta_5^2 + \zeta_5^3)B(q^5) \\ &+ q^2(\zeta_5^2 + \zeta_5^3)C(q^5) \\ &+ q^3(1 + \zeta_5^2 + \zeta_5^3)D(q^5) \end{aligned}$$

$$\text{Coeff of } q^{5n+4} \text{ in } C(\zeta_5, q) = 0$$
$$\text{Coeff of } q^{7n+5} \text{ in } C(\zeta_7, q) = 0$$
$$\text{Coeff of } q^{11n+6} \text{ in } C(\zeta_{11}, q) = 0$$

The CRANK was found by interpreting the generating function $C(z, q)$ in terms of partitions.