NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
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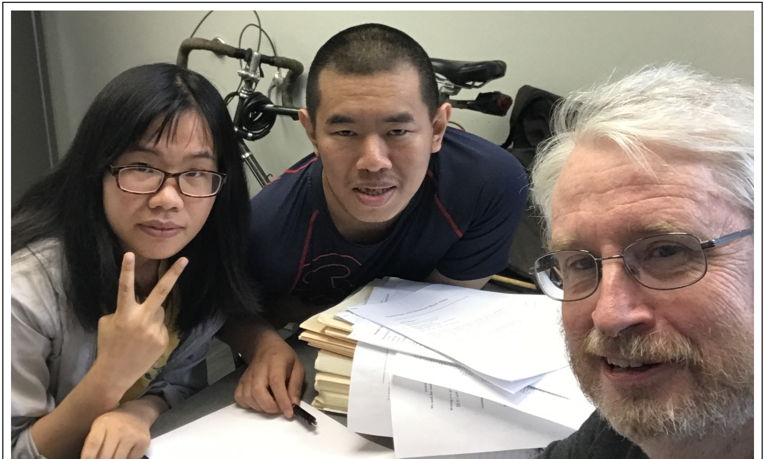
LECTURE 10 (under construction) CONGRUENCES FOR THE RANK AND CRANK PARITY FUNCTIONS

(Includes joint work with Dandan Chan and Rong Chen, Shanghai)



A *partition of n* is a representation of n as a sum of nonincreasing positive integers. Let $p(n)$ denote the number of partitions of n.

EULER: For $|q| < 1$

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}$$

$$= \frac{1}{\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}} = \frac{1}{1-q-q^2+q^5+q^7-\cdots}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^k)^2}$$

Let $q = e^{2\pi i\tau}$ with $\mathrm{Im}\tau > 0$, then

$$\eta(\tau) := e^{\pi i/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i\tau m}) = q^{1/24} \prod_{m=1}^{\infty} (1-q^m)$$

so that $\quad P(q) = \dfrac{q^{1/24}}{\eta(\tau)}.$

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Ramanujan's Partition Congruences (1919)

$$p(5n+4) \equiv 0 \pmod 5$$

$$p(7n+5) \equiv 0 \pmod 7$$

$$p(11n+6) \equiv 0 \pmod{11}$$

Ramanujan's conjecture: If $24\lambda \equiv 1 \pmod{5^\alpha 7^\beta 11^\gamma}$ then

$$p(5^\alpha 7^\beta 11^\gamma n + \lambda) \equiv 0 \pmod{5^\alpha 7^{[(\beta+2)/2]} 11^\gamma}.$$

Watson (1938) Atkin (1967)

THE RANK AND THE CRANK

Dyson's rank

Dyson (1944) defined the *RANK* of a partition as the largest part minus the number of parts.

Example. The rank of $14+8+2+2+1$ is $14-5=9$.

Let $N(m,n)$ denote the number of partitions of n with rank m.

Let $N(m,t,n)$ denote the number of partitions of n with rank $\equiv m \pmod{t}$.

Dyson's conjectures:

$$N(k,5,5n+4) = \frac{p(5n+4)}{5} \quad 0 \le k \le 4$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7} \quad 0 \le k \le 6$$

Atkin and Swinnerton-Dyer (1954)

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Dyson's (1944) CRANK CONJECTURE

G. (1986) A partial solution of Dyson's conjecture in terms of vector partitions.

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$$c(\pi) = \begin{cases} \ell(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0. \end{cases}$$

Then *The residue of the crank modulo t divides the partitions of* $p(tn+\delta)$ *into t equal classes for* $(t, \delta) = (5,4), (7,5), (11,6)$

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THE CRANK PARITY FUNCTION

Let $M(m,n)$ denote the number of partitions of n with crank m for $m > 1$. Then

$$\sum_{n=0}^{\infty}\sum_{m} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Let $M_e(n)$ (resp. $M_o(n)$) denote the number of partitions of n with even (resp. odd) crank. Then

$$\sum_{n=0}^{\infty}(M_e(n)-M_o(n))q^n = \frac{(q;q)_\infty}{(-q;q)_\infty^2} = \frac{(q;q)_\infty^3}{(q;q^2)_\infty^2} = q^{1/24}\frac{\eta(\tau)^3}{\eta(2\tau)^2}.$$

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Choi, Kang and Lovejoy (2009)

Theorem

thm:crankthm *For all* $\alpha \geq 0$ *we have*

$$M_e(n) - M_o(n) \equiv 0 \pmod{5^{\alpha+1}}, \qquad \text{if } 24n \equiv 1 \pmod{5^{2\alpha+1}}.$$

$$\sum_{n=0}^{\infty}(M_e(5n+4) - M_o(5n+4))q^n = 5\,\frac{(q;q^2)_\infty^2 (q^5;q^5)_\infty (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2}$$

$$= 5\,q^{-19/24}\,\frac{\eta(10\,\tau)^2\eta(5\,\tau)\eta(\tau)^2}{\eta(2\,\tau)^4}$$

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Chen, Chen and G. (2019)

Theorem

Let

$$\beta(n) = M_e(n) - M_o(n), \quad \boxed{eq:betadef}$$

for all $n \geq 0$. For each $\alpha \geq 1$ there is an integral constant K_α such that

$$\beta(49n-2) \equiv K_\alpha\, \beta(n) \pmod{7^\alpha}, \qquad \text{if } 24n \equiv 1 \pmod{7^\alpha}. \quad \boxed{eq:betan}$$

THE RANK PARITY FUNCTION

Recall $N(m,n)$ denotes the number of partitions of n with rank m. Then

$$\sum_{n=0}^{\infty}\sum_{m} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}.$$

Let $N_e(n)$ (resp. $N_o(n)$) denote the number of partitions of n with even (resp. odd) rank. Then

$$f(q) = \sum_{n=0}^{\infty}(N_e(n) - N_o(n))q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

Ramanujan mock theta function of third order

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Chen, Chen and G. (2019)

Theorem

Let

$$a_f(n) = N_e(n) - N_o(n), \quad \boxed{eq:afdef}$$

for all $n \geq 0$.

(i) *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_f(5^\alpha n+\delta_\alpha)+a_f(5^{\alpha-2}n+\delta_{\alpha-2}) \equiv 0 \pmod{5^{\left\lfloor \frac{1}{2}\alpha \right\rfloor}}, \quad \boxed{eq:rmod5}$$

where δ_α *satisfies* $0 < \delta_\alpha < 5^\alpha$ *and* $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$.

(ii) *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_f(7^\alpha n+\delta_\alpha)-a_f(7^{\alpha-2}n+\delta_{\alpha-2}) \equiv 0 \pmod{7^{\left\lfloor \frac{1}{2}(\alpha-1) \right\rfloor}}, \quad \boxed{eq:rmod7}$$

where δ_α *satisfies* $0 < \delta_\alpha < 7^\alpha$ *and* $24\delta_\alpha \equiv 1 \pmod{7^\alpha}$.

EXAMPLES

(i) $\alpha = 3$, $n = 0$.

$N_e(4) = 1$, $N_o(4) = 4$, $N_e(99) = 84623689$, $N_o(99) = 84606186$.

$$N_e(99) - N_o(99) + N_e(4) - N_o(4) = 17500 \equiv 0 \pmod 5.$$

(ii) $\alpha = 3$, $n = 0$.

$N_e(5) = 5$, $N_o(5) = 2$, $N_e(243) = 66989137386680$, $N_o(243) = 66989121958208$.

$$N_e(243) - N_o(243) - N_e(5) + N_o(5) = 15428469 = 3 \cdot 7 \cdot 17 \cdot 23 \cdot 1879 \equiv 0 \pmod 7.$$

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$N_o(99) = 84606186.$

$N_e(99) - N_o(99) + N_e(4) - N_o(4) = 17500 \equiv 0 \pmod 5.$

(ii) $\alpha = 3$, $n = 0$.

$N_e(5) = 5, \quad N_o(5) = 2, \quad N_e(243) = 66989137386680,$
$N_o(243) = 66989121958208.$

$N_e(243) - N_o(243) - N_e(5) + N_o(5) = 15428469 =$
$3 \cdot 7 \cdot 17 \cdot 23 \cdot 1879 \equiv 0 \pmod 7.$

$f(q)$ is a Mock Modular Form

Define

$$g_1(\tau) := -\sum_{n=-\infty}^{\infty} \left(n + \frac{1}{6}\right) e^{3\pi i (n+\frac{1}{6})^2},$$

and

$$M(z) := q^{-1/24} f(q) - 2i\sqrt{3} \int_{-\bar{z}}^{i\infty} \frac{g_1(\tau)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

where $q = \exp(2\pi i z)$, $z \in \mathfrak{H}$.

$$M(Az) = \psi(A)(cz + d)^{1/2} M(z)$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$. Zwegers (2001), Bringmann and Ono (2006), Ahlgren and Dunn (2019).

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The Atkin U_p Operator

For p prime define U_p by

$$U_p(f) = \frac{1}{p}\sum_{j=0}^{p} f\left(\frac{\tau+j}{p}\right). \quad \boxed{\text{eq:Updef}}$$

If $\quad f(\tau) = \sum_{m=m_0}^{\infty} a(m)q^m \quad$ then $\quad U_p(f) = \sum_{pm \geq m_0} a(pm)q^m.$

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where $q = \exp(2\pi i\tau)$.

$$g_1(24\tau) = -\frac{1}{6}\sum_{n\geq 1} n\chi(n)q^{n^2},$$

where

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 6 \\ -1 & \text{if } n \equiv 5 \pmod 6 \end{cases}$$

is a Dirichlet character mod 6.

So for any prime $p > 3$

$$U_p(g_1(24\tau)) = \chi(p)\, p\, g_1(24p\tau).$$

It follows that

$$U_p(M(24z)) - \chi(p)M(24pz)$$

is holomorphic.

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This means that it is probable that

$$\sum_{n=0}^{\infty} \left(a_f(\ell n - s_\ell) - \chi(\ell) a_f\left(\frac{n}{\ell}\right)\right) q^{n-\ell/24}$$

is a weakly holomorphic modular form of weight $1/2$ for any prime $\ell > 3$, where $s_\ell = \frac{1}{24}(\ell^2 - 1)$.

EXAMPLES

$\ell = 5$

$$\sum_{n=0}^{\infty}\left(a_f(5n-1)+a_f\left(\frac{n}{5}\right)\right)q^{n-5/24}$$
$$=\frac{\eta(2\tau)^4\eta(10\tau)^2}{\eta(\tau)\eta(4\tau)^3\eta(20\tau)}-4\,\frac{\eta(\tau)^2\eta(4\tau)^3\eta(5\tau)\eta(20\tau)}{\eta(2\tau)^5\eta(10\tau)}.$$

$\ell = 7$

$$\sum_{n=0}^{\infty}\left(a_f(7n-2)-a_f\left(\frac{n}{7}\right)\right)q^{n-7/24}$$
$$=-\frac{\eta(\tau)^3\eta(7\tau)^6}{\eta(2\tau)^5\eta(14\tau)^3}-6\,\frac{\eta(\tau)^4\eta(14\tau)^4}{\eta(2\tau)^6\eta(7\tau)}.$$

EXAMPLES

$\ell = 5$

$$\sum_{n=0}^{\infty} \left(a_f(5n-1) + a_f\left(\frac{n}{5}\right) \right) q^{n-5/24}$$
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IDEA OF PROOF

- Definition of a Modular Function
- Orders at cusp
- The Valence Formula
- Determining whether an eta-quotient is a modular function and finding order at cusps
- Construction a modular equation
- Determining $U_p(f)$ in terms of an infinite basis
- The rest ....

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Stroke Operator and Definition of Modular Function

Let $\mathscr{H} = \{\tau : \mathrm{Im}(\tau) > 0\}$ (the complex upper half-plane). For each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z})$, the set of integer 2×2 matrix with positive determinant, the bilinear transformation $M(\tau)$ is defined by

$$M\tau = M(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The stroke operator is defined by

$$\left(\,|\,[f]_M\,\right)(\tau) = f(M\tau),$$

and satisfies

$$|\,[f]_M\,S = \,\Big|\,[\,|\,[f]_M]_S\,.$$

A function $f : \mathscr{H} \longrightarrow \mathbb{C}$ is a *modular function* on Γ if the following conditions hold:

(i) f is holomorphic on $\mathscr{H}$.

(ii) $\mid [f]_V = f$ for all $V \in \Gamma$.

(iii) For every $A \in \Gamma(1)$ the function $\Big| [f]_A^{-1}$ has an expansion

$$(\Big| [f]_A^{-1})(\tau) = \sum_{m=m_0}^{\infty} b(m) \exp(2\pi i \tau m/\kappa)$$

on some half-plane $\{\tau : \operatorname{Im} \tau > h \geq 0\}$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$$\kappa = \min\left\{k > 0 : \pm A^{-1} T^k A \in \Gamma\right\}.$$

Fan Width, Orders at Cusps

The positive integer $\kappa = \kappa(\Gamma;\zeta)$ is called the *fan width* of Γ at the cusp $\zeta = A^{-1}\infty$. If $b(m_0) \neq 0$, then we write

$$\mathrm{ORD}(f;\zeta,\Gamma) = m_0$$

which is called the *order* of f at ζ with respect to Γ. We also write

$$\mathrm{ord}(f;\zeta) = \frac{m_0}{\kappa} = \frac{m_0}{\kappa(\Gamma,\zeta)},$$

which is called the *invariant order* of f at ζ. For each $z \in \mathscr{H}$, $\mathrm{ord}(f;z)$ denotes the order of f at z as an analytic function of z, and the order of f with respect to Γ is defined by

$$\mathrm{ORD}(f;z,\Gamma) = \frac{1}{\ell}\mathrm{ord}(f;z)$$

where ℓ is the order of z as a fixed point of Γ. We note $\ell = 1$, 2 or 3.

The Valence Formula If $f \neq 0$ is a modular function on Γ and $\mathscr{F}$ is any fundamental set for Γ then

$$\sum_{z \in \mathscr{F}} \mathrm{ORD}(f; z, \Gamma) = 0. \quad \boxed{\text{eq:valform}} \tag{1}$$

SKETCH OF PROOF OF RAMANUJAN's PARTITION CONGRUENCES

If $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$ then $p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}$.

Define

$$f(\tau) = \frac{\eta(25\tau)}{\eta(\tau)}, \qquad h(\tau) = \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6$$

Then $f(\tau)$ is a modular function $\Gamma_0(25)$ and $h(\tau)$ is a modular function on $\Gamma_0(5)$.

$$\mathrm{ORD}(h; 0, \Gamma_0(5)) = -1, \qquad \mathrm{ORD}(h; \infty, \Gamma_0(5)) = 1$$

$h(\tau)$ is a hauptmodul for $\Gamma_0(5)$.
$U_5(f)$ is a modular function on $\Gamma_0(5)$ and

$$\mathrm{ORD}(U_5(f); 0, \Gamma_0(5)) \geq -1, \qquad \mathrm{ORD}(U_5(f); \infty, \Gamma_0(5)) = 1$$

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$$U_5(f) = 5q + 30q^2 + \cdots$$
$$h = q + 6q^2 + \cdots$$

The Valence Formula implies

$$U_5(f) = 5\,h$$

$$\eta(5\tau) \sum_{n\geq 1} p(5n-1) q^{n-5/24} = 5 \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6$$

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}$$ THE MOST BEAUTIFUL

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A FIFTH ORDER MODULAR EQUATION

Define

$$
\begin{aligned}
\sigma_0(\tau) &= h(\tau) \\
\sigma_1(\tau) &= 6 \cdot 5\, h(\tau) + 5^3 h^2(\tau) \\
\sigma_2(\tau) &= 63 \cdot 5\, h(\tau) + 6 \cdot 5^4\, h(\tau)^2 + 5^6\, h(\tau)^3 \\
\sigma_3(\tau) &= 52 \cdot 5^2\, h(\tau) + 63 \cdot 5^4\, h(\tau)^2 + 6 \cdot 5^7\, h(\tau)^3 + 5^9\, h(\tau)^4 \\
\sigma_4(\tau) &= 63 \cdot 5^2\, h(\tau) + 52 \cdot 5^5\, h(\tau)^2 + 63 \cdot 5^7\, h(\tau)^3 + 6 \cdot 5^{10}\, h(\tau)^4 \\
&\qquad + 5^{12}\, h(\tau)^5
\end{aligned}
$$

Then

$$
h(\tau)^5 = \sum_{j=0}^{4} \sigma_j(5\tau)\, h(\tau)^j
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\end{aligned}$$

Then

$$h(\tau)^5 = \sum_{j=0}^{4} \sigma_j(5\tau)\, h(\tau)^j$$

A FUNDAMENTAL LEMMA

$$U_5(u\, h^k) = \sum_{j=0}^{4} \sigma_j(\tau)\, U_5\left(u\, h(\tau)^{j+k-5}\right)$$

INITIAL VALUES

$$U_5(1) = 1$$
$$U_5(h) = 63 \cdot 5\, h + 52 \cdot 5^4 h^2 + \cdots + 5^{11}\, h^5$$
$$U_5(h^2) = 104 \cdot 5\, h + 819 \cdot 5^4 h^2 + \cdots + 5^{23}\, h^{10}$$
$$U_5(h^3) = 189\, h + 9846 \cdot 5^3 h^2 + \cdots + 5^{35}\, h^{15}$$
$$U_5(h^4) = 24\, h + 8584 \cdot 5^3 h^2 + \cdots + 5^{47}\, h^{20}$$

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\end{aligned}$$

MORE INITIAL VALUES

$$
\begin{aligned}
U_5(f) &= 5\,h \\
U_5(f\,h) &= 28 \cdot 5\,h + 49 \cdot 5^4 h^2 + \cdots + 5^{13}\,h^6 \\
U_5(f\,h^2) &= 104\,h + 364 \cdot 5^4 h^2 + \cdots + 5^{25}\,h^{11} \\
U_5(f\,h^3) &= 19\,h + 13889 \cdot 5^2 h^2 + \cdots + 5^{37}\,h^{16} \\
U_5(f\,h^4) &= h + 67 \cdot 5^5 h^2 + \cdots + 5^{49}\,h^{21}
\end{aligned}
$$

INDUCTION

For p prime define the p-adic valuation on $\mathbb{Z}$ by

$$\nu_p(n) = \begin{cases} \max\{v \in \mathbb{N} : p^v \mid n\} & n \neq 0 \\ \infty & n = 0 \end{cases}$$

$$U_5(h^i) = \sum_{j \geq 1} a_{ij} h^j \quad \text{where } \nu_5(a_{ij}) \geq \lfloor \tfrac{1}{2}(5j - i - 1) \rfloor$$

$$U_5(f \cdot h^i) = \sum_{j \geq 1} b_{ij} h^j \quad \text{where } \nu_5(b_{ij}) \geq \lfloor \tfrac{1}{2}(5j - i - 2) \rfloor$$

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INDUCTION CONTINUED When α is odd then

$$\eta(5\tau)\sum_{n\geq 0} p(5^\alpha n+\delta_\alpha)q^{n+19/24} = \sum_{i\geq 1} x_{\alpha,i}h^i$$

where $\nu_5(x_{\alpha,i}) \geq \alpha + \lfloor \frac{1}{2}(5i-5)\rfloor$

When α is even

$$\eta(25\tau)\sum_{n\geq 0} p(5^\alpha n+\delta_\alpha)q^{n+23/24} = \sum_{i\geq 1} x_{\alpha,i}f\cdot h^i$$

where $\nu_5(x_{\alpha,i}) \geq \alpha + \lfloor \frac{1}{2}(5i-4)\rfloor$

RESULT

$$p(5^\alpha n+\delta_\alpha) \equiv 0 \pmod{5^\alpha}.$$

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RESULT

$$p(5^\alpha n+\delta_\alpha)\equiv 0 \pmod{5^\alpha}.$$

PROOF OF RANK PARITY CONGRUENCE

J-NOTATION

$$J_b = (q^b; q^b)_\infty,$$

Theorem

$$\sum_{n=0}^{\infty} (a_f(5n-1) + a_f(n/5))q^n = \frac{J_2^4 J_{10}^2}{J_1 J_4^3 J_{20}} - 4q \frac{J_1^2 J_4^3 J_5 J_{20}}{J_2^5 J_{10}}.$$

Define

$$t := t(\tau) := \frac{\eta(\tau)^2\eta(10\tau)^4}{\eta(2\tau)^4\eta(5\tau)^2}$$
$$= q - 2\,q^2 + 3\,q^3 - 6\,q^4 + 11\,q^5 - 16\,q^6 + 24\,q^7 - 38\,q^8 + 57\,q^9 - 82\,q$$

We note that $t(\tau)$ is a Hauptmodul for $\Gamma_0(10)$ As an application of our algorithm we prove the following theorem which will be needed later.

Theorem

Let

$$\sigma_0(\tau) = -t, \tag{2}$$

$$\sigma_1(\tau) = -5t^2 + 2\cdot 5t, \tag{3}$$

$$\sigma_2(\tau) = -5^2t^3 + 2\cdot 5^2t^2 - 7\cdot 5t, \tag{4}$$

$$\sigma_3(\tau) = -5^3t^4 + 2\cdot 5^3t^3 - 7\cdot 5^2t^2 + 12\cdot 5t, \tag{5}$$

$$\sigma_4(\tau) = -5^4t^5 + 2\cdot 5^4t^4 - 7\cdot 5^3t^3 + 12\cdot 5^2t^2 - 11\cdot 5t, \tag{6}$$

where $t = t(\tau)$ is defined above. Then

$$t(\tau)^5 + \sum_{j=0}^{4} \sigma_j(5\tau)\, t(\tau)^j = 0.$$

Lemma (A Fundamental Lemma)

Suppose $u = u(\tau)$, and j is any integer. Then

$$U_5(u\,t^j) = -\sum_{l=0}^{4} \sigma_l(\tau)\,U_5(u\,t^{j+l-5}),$$

where $t = t(\tau)$ is defined in above and the $\sigma_j(\tau)$ are given above.

Lemma

Let $u = u(\tau)$, and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l + 4$ there exist Laurent polynomials $p_{u,k}(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$U_5(u\,t^k) = v\,p_{u,k}(t), \tag{7}$$

and

$$ord_t(p_{u,k}(t)) \geq \left\lceil \frac{k+s}{5} \right\rceil, \tag{8}$$

for a fixed integer s, where $t = t(\tau)$ is defined in above and where $v = v(\tau)$. Then there exists a sequence of Laurent polynomials $p_{u,k}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that above and above hold for all $k \in \mathbb{Z}$.

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Let $u = u(\tau)$, and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l+4$ there exist Laurent polynomials $p_{u,k}(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$U_5(u\, t^k) = v\, p_{u,k}(t), \tag{9}$$

where

$$p_{u,k}(t) = \sum_n c_u(k,n)\, t^n, \quad \nu_5(c_u(k,n)) \geq \left\lfloor \frac{3n-k+r}{4} \right\rfloor$$

for a fixed integer r, where $t = t(\tau)$ is defined in above and where $v = v(\tau)$. Then there exists a sequence of Laurent polynomials $p_{u,k}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that above holds for $k > l+4$, where

$$p_{u,k}(t) = \sum_n c_u(k,n)\, t^n, \quad \text{and} \quad \nu_5(c_u(k,n)) \geq \left\lfloor \frac{3n-k+r+2}{4} \right\rfloor.$$

We define the following functions which will be needed in the proof of the main Theorem

$$P_A := \frac{J_{10}^2 J_5 J_2^6}{J_{20} J_4^3 J_1^5} - 4\frac{qJ_{20} J_5^2 J_4^3}{J_{10} J_2^3 J_1^2}, P_B := \frac{J_{10}^6 J_2^2 J_1}{qJ_{20}^3 J_5^5 J_4} + 4\frac{qJ_{20}^3 J_4 J_1^2}{J_{10}^3 J_5^2 J_2}$$

$$A := \frac{J_{50}^2 J_1^4}{J_{25}^4 J_2^2}, \quad B := \frac{qJ_{25}}{J_1}.$$

For $f = f(\tau)$ we define

$$U_A(f) := U_5(A\, f), \qquad U_B(f) := U_5(B\, f). \tag{10}$$

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First we need some initial values of $U_A(P_A\, t^k)$ and $U_B(P_B\, t^k)$.

Lemma

Group I

$$U_A(P_A) = P_B(5^4 t^5 - 7\cdot 5^3 t^4 + 14\cdot 5^2 t^3 - 2\cdot 5^2 t^2 + t),$$
$$U_A(P_A t^{-1}) = -P_B t,$$
$$U_A(P_A t^{-2}) = -5 P_B t^2,$$
$$U_A(P_A t^{-3}) = -5^2 P_B t^3,$$
$$U_A(P_A t^{-4}) = -5^3 P_B t^4.$$

Group II

$$U_B(P_B) = P_A,$$
$$U_B(P_B t^{-1}) = P_A(-5t + 2),$$
$$U_B(P_B t^{-2}) = P_A(5^2 t^2 - 8 \cdot 5t + 8),$$
$$U_B(P_B t^{-3}) = P_A(5^3 t^3 - 34 \cdot 5t + 34),$$
$$U_B(P_B t^{-4}) = P_A(-5^4 t^4 + 16 \cdot 5^3 t^3 - 36 \cdot 5^2 t^2 - 128 \cdot 5t + 6 \cdot 5$$

Following PAULE-RADU a map $a : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is called a *discrete array* if for each i the map $a(i,-) : \mathbb{Z} \longrightarrow \mathbb{Z}$, by $j \mapsto a(i,j)$ has finite support.

Lemma

There exist discrete arrays a and b such that for $k \geq 1$

$$U_A(P_A\, t^k) = P_B \sum_{n \geq \lceil (k+5)/5 \rceil} a(k,n)\, t^n, \text{ where } \nu_5(a(k,n)) \geq \left\lfloor \frac{3n-k}{4} \right\rfloor,$$

$$U_B(P_B\, t^k) = P_A \sum_{n \geq \lceil k/5 \rceil} b(k,n)\, t^n, \text{ where } \nu_5(b(k,n)) \geq \left\lfloor \frac{3n-k+2}{4} \right\rfloor.$$

PROOF OF MAIN THEOREM For $\alpha \geq 1$ define an integer δ_α by $0 < \delta_\alpha < 5^\alpha$ and $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$. Then

$$\delta_{2\alpha} = \frac{23 \times 5^{2\alpha} + 1}{24}, \qquad \delta_{2\alpha+1} = \frac{19 \times 5^{2\alpha+1} + 1}{24}.$$

We let

$$\lambda_{2\alpha} = \lambda_{2\alpha+1} = \frac{5}{24}(1 - 5^{2\alpha}).$$

For $n \geq 0$ we define

$$c_f(n) := a_f(5n-1) + a_f(n/5). \tag{11}$$

We find that for $\alpha \geq 3$

$$\sum_{n=0}^{\infty} \left(a_f(5^\alpha n + \delta_\alpha) + a_f(5^{\alpha-2} n + \delta_{\alpha-2})\right) q^{n+1} = \sum_{n=1}^{\infty} c_f(5^{\alpha-1} n + \lambda_{\alpha-1}) q^n. \tag{12}$$

We define the sequence of functions $(L_\alpha)_{\alpha=0}^{\infty}$ by $L_0 := P_A$ and for $\alpha \geq 0$

$$L_{2\alpha+1} := U_A(L_{2\alpha}), \qquad \text{and} \qquad L_{2\alpha+2} := U_B(L_{2\alpha+1}).$$

Lemma

For $\alpha \geq 0$,

$$L_{2\alpha} = \frac{J_5 J_2^2}{J_1^4} \sum_{n=0}^{\infty} c_f(5^{2\alpha} n + \lambda_{2\alpha}) q^n,$$

and

$$L_{2\alpha+1} = \frac{J_{10}^2 J_1}{J_5^4} \sum_{n=0}^{\infty} c_f(5^{2\alpha+1} n + \lambda_{2\alpha+1}) q^n.$$

Our main result for the rank parity function modulo powers of 5 is the following theorem.

Theorem

There exists a discrete array ℓ such that for $\alpha \geq 1$

$$L_{2\alpha} = P_A \sum_{n\geq 1} \ell(2\alpha, n)\, t^n, \quad \textit{where}$$

$$\nu_5(\ell(2\alpha, n)) \geq \alpha + \left\lfloor \frac{3n-3}{4} \right\rfloor,$$

$$L_{2\alpha+1} = P_B \sum_{n\geq 2} \ell(2\alpha+1, n)\, t^n, \quad \textit{where}$$

$$\nu_5(\ell(2\alpha+1, n)) \geq \alpha + 1 + \left\lfloor \frac{3n-6}{4} \right\rfloor.$$

Corollary

For $\alpha \geq 1$ and all $n \geq 0$ we have

$$c_f(5^{2\alpha}n + \lambda_{2\alpha}) \equiv 0 \pmod{5^{\alpha}},$$
$$c_f(5^{2\alpha+1}n + \lambda_{2\alpha+1}) \equiv 0 \pmod{5^{\alpha+1}}.$$

PROOF OF THEOREM We define the discrete array ℓ recursively. Define

$\ell(1,1) = 1$, $\ell(1,2) = -2 \cdot 5^2$, $\ell(1,3) = 14 \cdot 5^2$, $\ell(1,4) = -7 \cdot 5^3$, $\ell(1,5) =$
and $\quad \ell(1,k) = 0, \quad$ for $k \geq 6$.

For $\alpha \geq 1$ define

$$\ell(2\alpha, n) = \sum_{k \geq 1} \ell(2\alpha - 1, k)\, b(k,n) \qquad (\text{for } n \geq 1),$$

and

$$\ell(2\alpha + 1, n) = \sum_{k \geq 1} \ell(2\alpha, k)\, a(k,n) \qquad (\text{for } n \geq 2),$$

where a and b are the discrete arrays given a previous Lemma

From a Lemma (Group I) and similarly

$$L_1 = U_A(L_0) = U_A(P_A) = P_B \sum_{n=1}^{5} \ell(1,n)\, t^n,$$

$$\text{where} \quad \nu_5(\ell(1,n)) \geq \left\lfloor \frac{3n-2}{4} \right\rfloor.$$

$$L_2 = U_B(L_1) = \sum_{n=1}^{5} \ell(1,n) U_B(P_B t^n),$$

$$= \sum_{n=1}^{5} \ell(1,n) P_A \sum_{k\geq 1} b(n,k) t^k$$

$$= P_A \sum_{n\geq 1} \sum_{k=1}^{5} \ell(1,k) b(k,n) t^n$$

$$= P_A \sum_{n\geq 1} \ell(2,n) t^n,$$

where

$$\nu_5(\ell(2,n)) \geq \min_{1\leq k\leq 5}\left(\nu_5(\ell(1,k)) + \nu_5(b(k,n)\right)$$
$$\geq \min_{1\leq k\leq 5}\left(\left\lfloor \frac{3k-2}{4} \right\rfloor + \left\lfloor \frac{3n-k+2}{4} \right\rfloor \right) = \left\lfloor \frac{3n+1}{4} \right\rfloor,$$

since when $k = 1$, $\lfloor \frac{3k-2}{4} \rfloor + \lfloor \frac{3n-k+2}{4} \rfloor = \lfloor \frac{3n+1}{4} \rfloor$, and for $k \geq 2$,

$$\left\lfloor \frac{3k-2}{4} \right\rfloor + \left\lfloor \frac{3n-k+2}{4} \right\rfloor \geq \left\lfloor \frac{3n+2k-3}{4} \right\rfloor \geq \left\lfloor \frac{3n+1}{4} \right\rfloor.$$

Thus the result holds for $L_{2\alpha}$ when $\alpha = 1$. We proceed by induction. Suppose the result holds for $L_{2\alpha}$ for a given $\alpha \geq 1$. Then by Lemma ? and equation ?

we have

$$
\begin{aligned}
L_{2\alpha+1} = U_A(L_{2\alpha}) &= \sum_{n\geq 1} \ell(2\alpha, n) U_A(P_A t^n), \\
&= \sum_{n\geq 1} \ell(2\alpha, n) P_B \sum_{k\geq 2} a(n,k) t^k \\
&= P_B \sum_{n\geq 2} \sum_{k\geq 1} \ell(2\alpha, k) a(k,n) t^n \\
&= P_B \sum_{n\geq 2} \ell(2\alpha+1, n) t^n,
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_5(\ell(2\alpha+1, n)) &\geq \min_{1\leq k}\Big(\nu_5(\ell(2\alpha, k)) + \nu_5(a(k,n))\Big) \geq \min_{1\leq k}\Big(\alpha + \Big\lfloor \frac{3k-3}{4}
\\
&\geq \alpha + 1 + \Big\lfloor \frac{3n-6}{4} \Big\rfloor,
\end{aligned}
$$

since when $k = 1$, $\lfloor \frac{3k-3}{4} \rfloor + \lfloor \frac{3n-k}{4} \rfloor = 1 + \lfloor \frac{3n-5}{4} \rfloor$, and for $k \geq 2$,

$$\left\lfloor \frac{3k-3}{4} \right\rfloor + \left\lfloor \frac{3n-k}{4} \right\rfloor \geq \left\lfloor \frac{3n+2k-6}{4} \right\rfloor \geq 1 + \left\lfloor \frac{3n-6}{4} \right\rfloor .$$

Thus the result holds for $L_{2\alpha+1}$. Suppose the result holds for $L_{2\alpha+1}$ for a given $\alpha \geq 1$. Then again by Lemma ?and equation ? we have

$$\begin{aligned} L_{2\alpha+2} = U_B(L_{2\alpha+1}) &= \sum_{n\geq 2} \ell(2\alpha+1,n) U_B(P_B t^n), \\ &= \sum_{n\geq 2} \ell(2\alpha+1,n) P_A \sum_{k\geq 1} b(n,k) t^k \\ &= P_A \sum_{n\geq 1} \sum_{k\geq 2} \ell(2\alpha+1,k) b(k,n) t^n \\ &= P_A \sum_{n\geq 1} \ell(2\alpha+2,n) t^n, \end{aligned}$$

where $\ell(2\alpha+1,1)=0$. Here

$$\begin{aligned} \nu_5(\ell(2\alpha+2,n)) &\geq \min_{2\leq k} \Big(\nu_5(\ell(2\alpha+1,k)) + \nu_5(b(k,n) \Big) \\ &\geq \min_{2\leq k} \left(\alpha + 1 + \left\lfloor \frac{3k-6}{4} \right\rfloor + \left\lfloor \frac{3n-k+2}{4} \right\rfloor \right) \end{aligned}$$

$$\geq \min_{2\leq k}\left(\alpha+1+\left\lfloor\frac{3n+2k-7}{4}\right\rfloor\right)=\alpha+1+\left\lfloor\frac{3n-3}{4}\right\rfloor.$$

Thus the result holds for $L_{2\alpha+2}$, and the result holds in general by induction.