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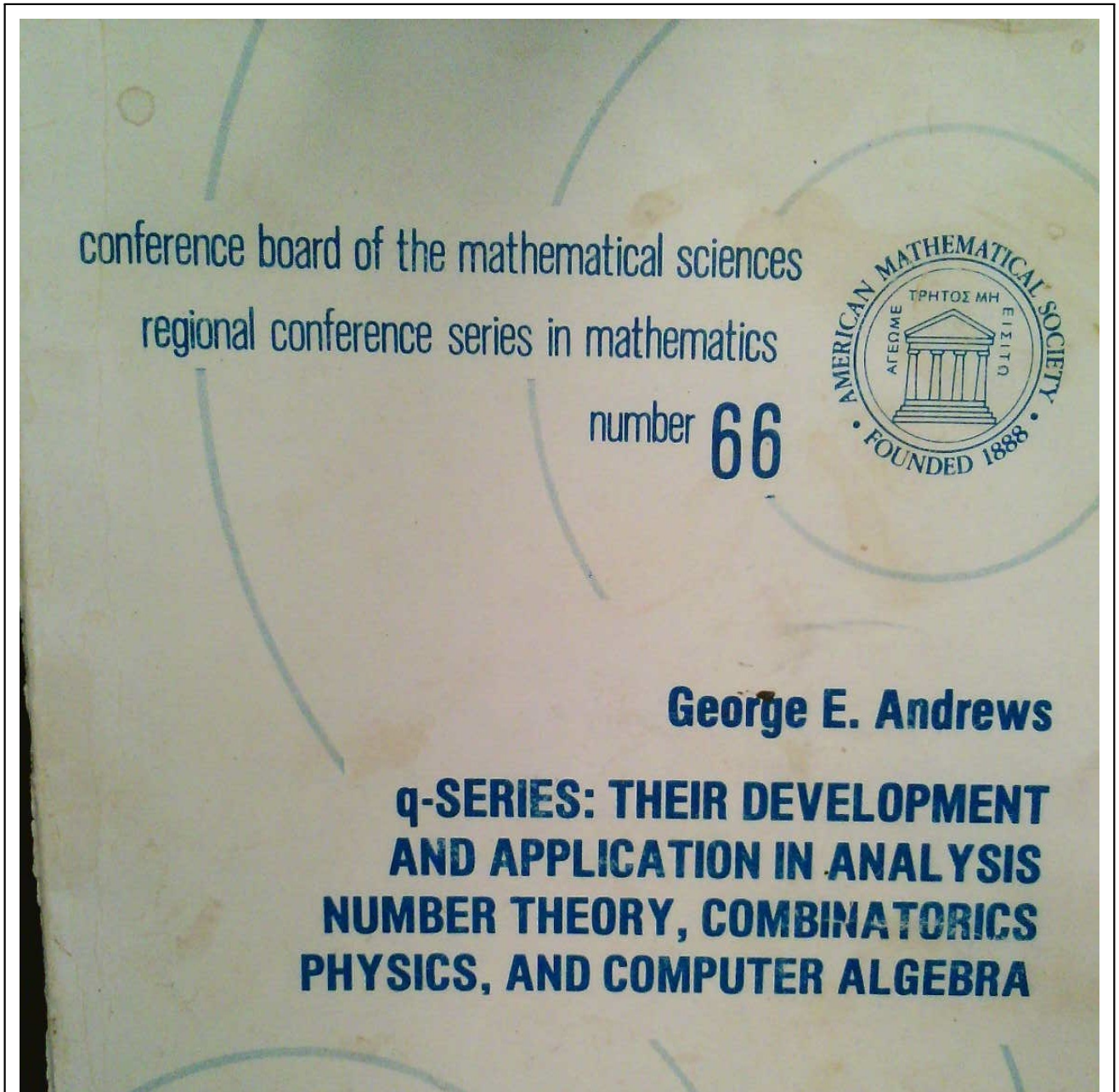
LECTURE 3
CRANK AND RANK CONGRUENCES - PART 1

BACKGROUND ON MODULAR FORMS AND HECKE OPERATORS

PARTITION CONGRUENCES FOR PRIMES $> \ell$

CRANK AND RANK CONGRUENCES

- THE CRANK-RANK PDE AND RANK AND CRANK MOMENTS
- ℓ-INTEGRAL QUASI-MODULAR FORMS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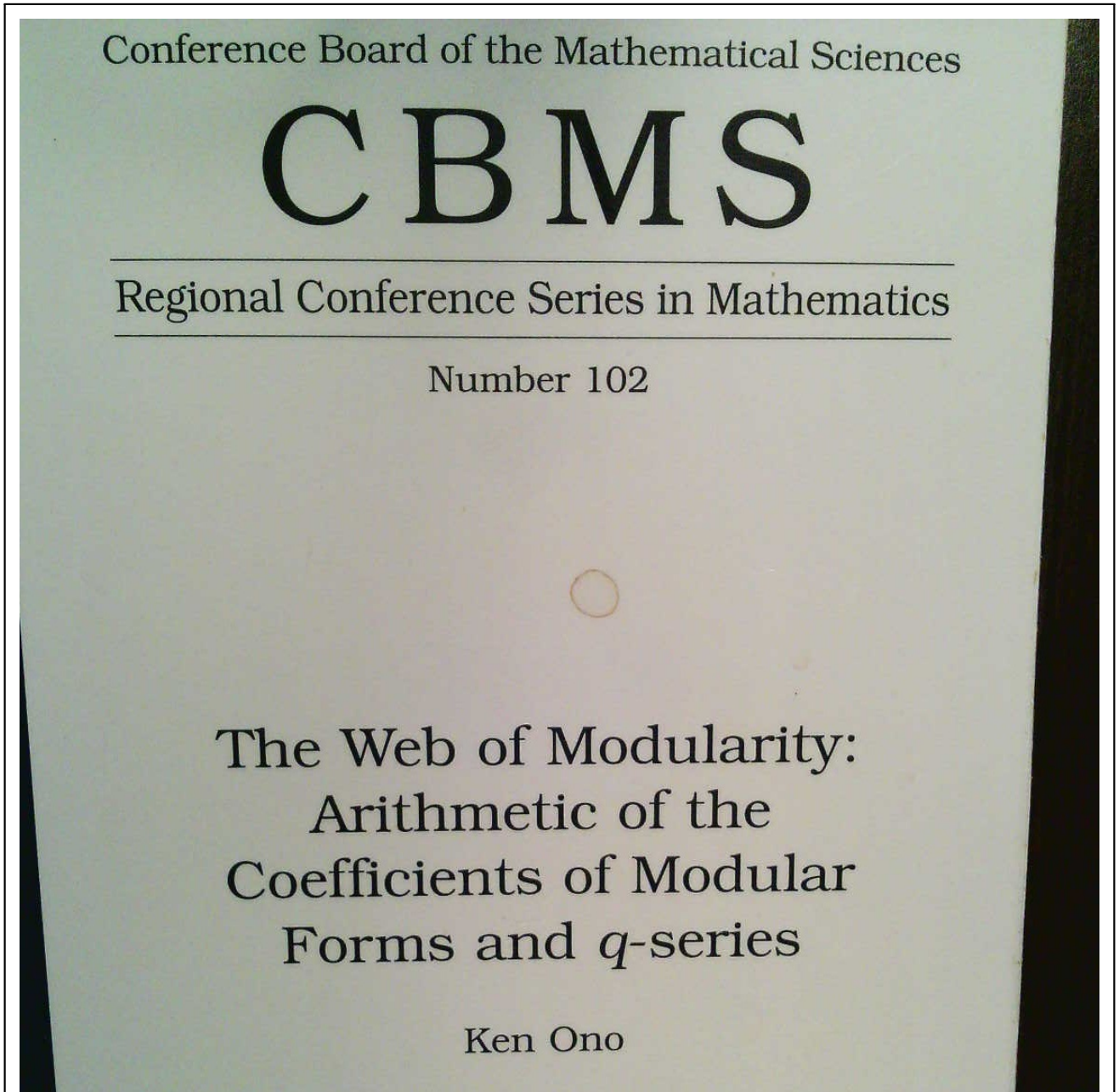
AND APPLICATION IN ANALYSIS
NUMBER THEORY, COMBINATORICS
PHYSICS, AND COMPUTER ALGEBRA


Conference Board of the Mathematical Sciences
CBMS
Regional Conference Series in Mathematics
Number 102
The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series
Ken Ono

BACKGROUND ON MODULAR FORMS AND HECKE OPERATORS

1.2. Integer weight modular forms

The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on functions $f(z) : \mathcal{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathcal{H}$ and k is an integer, then define the "slash" operator $|_k$ by

$$(f|_k\gamma)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z), \tag{1.3}$$

where

$$\gamma z := \frac{az+b}{cz+d}.$$

Definition 1.8. Suppose that $f(z)$ is a meromorphic function on $\mathcal{H}$, that $k \in \mathbb{Z}$, and that Γ is a congruence subgroup of level N. Then $f(z)$ is called a *meromorphic modular form with integer weight k on Γ* if the following hold:

(1) We have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathcal{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(2) If $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_k\gamma_0)(z)$ has a Fourier expansion of the form

$$(f|_k\gamma_0)(z) = \sum a_{\gamma_0}(n) q_N^n,$$

Let N be a positive integer and k a nonnegative integer, and let χ be a Dirichlet character mod N.

$M_k(N)$	denotes the space of entire modular forms of weight k on $\Gamma_0(N)$
$S_k(N)$	denotes the space of entire cusp forms of weight k on $\Gamma_0(N)$
$M_k(N,\chi)$	denotes the space of entire modular forms of weight k on $\Gamma_0(N)$ with character χ
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For the following spaces we assume $4 \mid N$.

$M_{k+\frac{1}{2},\chi}(N)$ denotes the space of entire modular forms of half integral weight $k+\frac{1}{2}$ on $\Gamma_0(N)$ with character χ

$S_{k+\frac{1}{2},\chi}(N)$ denotes the space of entire cusp forms of half integral weight $k+\frac{1}{2}$ on $\Gamma_0(N)$ with character χ

For ℓ prime the half integral weight Hecke operator

$$T_{k,N,\chi}(\ell^2) = T(\ell^2)$$

is given by

$$f \mid T(\ell^2) = \sum_{n=0}^{\infty} c(n)q^n,$$

where

$$c(n) = a(\ell^2 n) + \chi(\ell)\left(\frac{(-1)^k n}{\ell}\right)\ell^{k-1}a(n) + \chi(\ell^2)\ell^{2k-1}a(n/\ell^2),$$

$$f = \sum_{n=0}^{\infty} a(n)q^n$$

and $a(k) = 0$ if k is not a nonnegative integer.

We note that $T_{k,N,\chi}(\ell^2)$ preserve the spaces $M_{k+\frac{1}{2},\chi}(N)$ and $S_{k+\frac{1}{2},\chi}(N)$.

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THE DEDEKIND ETA FUNCTION

$$\eta(\tau) := \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi in\tau),$$

for $Im(\tau) > 0$. Then

$$\eta(24\tau) \in S_{1/2}(576, \chi_{12}),$$

where $\chi_{12}(n) = \left(\frac{12}{n}\right)$.

Proposition

Let $1 \leq r \leq 23$ *with* $(r, 24) = 1$, *suppose* m *is a nonnegative even integer, and* $\ell > 3$ *is prime. Then the Hecke operator* $T(\ell^2)$ *preserves the following subspace of* $S_{m+\frac{1}{2}}(576, \chi_{12})$*:*

$$\mathcal{C}_{r,m} = \{\eta^r(24\tau)F(24\tau) \,:\, F \in M_m(1)\}$$

Proposition

Let $1 \leq r \leq 23$ with $(r, 24) = 1$, suppose m is a nonnegative even integer, and $\ell > 3$ is prime. Then the Hecke operator $T(\ell^2)$ preserves the following subspace of $S_{m+\frac{1}{2}}(576, \chi_{12})$:

$$\mathcal{C}_{r,m} = \{\eta^r(24\tau)F(24\tau) \,:\, F \in M_m(1)\}$$

RETURN TO MOD 11 EXAMPLE

Corollary

Let r, m be as above. If $F \neq 0$, $F \in M_m(1)$ and $\dim M_m(1) = 1$ then $g(\tau) = \eta^r(24\tau)F(24\tau)$ is a Hecke eigenform on $S_{m+\frac{1}{2}}(576, \chi_{12})$.

Suppose k is a nonnegative even integer. For ℓ prime define the WEIGHT k HECKE OPERATOR by

$$f \mid T(\ell) = \sum_{n=0}^{\infty} c(n) q^n,$$

where

$$c(n) = a(\ell n) + \ell^{k-1} a(n/\ell),$$

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$$f = \sum_{n=0} a(n) q^n \in M_k(1).$$

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For $\ell > 3$ define

$$h_\ell(\tau) = (\eta(\tau)\eta(\ell\tau))^{\ell-1}$$

Proposition

Let $\ell > 3$ be prime and k a nonnegative even integer. Suppose $F(\tau) \in M_k(1)$. Then

$$(h_\ell(\tau)F(\tau)) \mid U(\ell) + (-1)^{(\ell-1)/2}\ell^{k+(\ell-1)/2-1}h_\ell(\tau)F(\ell\tau) \in S_{k+\ell-1}(1)$$

Here

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The $U(\ell)$ operator acts by

$$\sum_{n=0}^{\infty} a(n)q^n \mid U(\ell) = \sum_{n=0}^{\infty} a(\ell n)q^n$$

For $F \in M_k(1)$ DEFINE $p(F, n)$ by

$$\sum_{n=0}^{\infty} p(F, n) q^n = \frac{F(\tau)}{\prod_{n=1}^{\infty}(1 - q^n)}$$

where $q = \exp(2\pi i \tau)$

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SOME CONSTANTS. Define $1 \leq \beta_\ell \leq \ell - 1$, by $24\beta_\ell \equiv 1 \pmod{\ell}$. Let

$$r_\ell = \frac{1}{\ell}(24\beta_\ell - 1), \quad \lambda_\ell = \frac{1}{24\ell}(\ell^2 + 24\beta_\ell - 1)$$

RETURN TO BIG THEOREM

Theorem (G.)

Suppose $\ell > 3$ is prime, k is a nonnegative even integer, and $F(\tau) \in M_k(1) \cap \mathbb{Z}[[q]]$. Then

$$\sum_{n=0}^{\infty} p(F, \ell n + \beta_\ell) q^n \equiv E(q)^{r_\ell} g(\tau) \pmod{\ell}$$

for some $g \in M_{k+\ell-1-12\lambda_\ell}(1) \cap \mathbb{Z}[[q]]$,

$$E(q) = \prod_{n=1}^{\infty} (1 - q^n)$$

PARTITION CONGRUENCES FOR PRIMES $> \ell$

<u>EXAMPLE</u> $\ell = 13$, $\beta_{13} = 6$, $r_{13} = 11$, $\lambda_{13} = 1$.

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$$\sum_{n=0}^{\infty} p(13n+6)q^n \equiv 11E(q)^{11} \pmod{13}$$

so that

$$\sum_{n=0}^{\infty} p(13n+6)q^{24n+11} \equiv 11\eta(24\tau)^{11} \pmod{13}$$

PARTITION CONGRUENCES FOR PRIMES $> \ell$

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Since $\dim M_0(1) = 1$ we have for each prime $\ell > 3$

$$\eta(24\tau)^{11} \mid T(\ell^2) = \gamma_\ell \eta(24\tau)^{11}$$

for some constant γ_ℓ.

We wish to find an eigenvalue $\gamma_\ell \equiv 0 \pmod{13}$

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DEFINE $Q(n)$ by

$$\sum_n Q(n) q^n = \sum_{n=0}^{\infty} p(13n+6)\, q^{24n+11}$$

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so that $Q(m) = p(\frac{1}{24}(13m+1))$ if $m \equiv 11 \pmod{24}$.

Since $\eta(24\tau)^{11} \in S_{5+\frac{1}{2}}(576, \chi_{12})$

$$Q(59^2 m) + \left(\frac{-m}{59}\right) 59^4 Q(m) + 59^9 Q(m/59^2) \equiv 0 \pmod{13}$$

We want $m \equiv 0 \pmod{59}$
$m \not\equiv 0 \pmod{59^2}$ and $m \equiv 11 \pmod{24}$

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OLIVER ATKIN

> An obvious superficial use of computers is to find numerical accidents which are needed to apply known results. However, it is often more difficult to discover results in this subject than to prove them, and an informed search on the machine may enable one to find out precisely what happens. A striking example of this is Theorem 2 below, which was

OLIVER ATKIN [3 SEP 1996 on NMBRTHRY]

$$p(1140773130436436432134058026060201612619574856085125n+ \\ 1278827052061576887278324769721420299) \equiv 0 \pmod{113}$$

OLIVER ATKIN [3 SEP 1996 on NMBRTHRY]

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$p(n) \equiv 0 \pmod{73}$ if $\left(\frac{24n-1}{73}\right) = -1$ and $24n-1$ is divisible by 5^{10655} and not by 5^{10656}.

$$1140773130436436432134058026060201612619574856085125$$
$$= (5)^3\,(7)^4\,(13)^3\,(17)^3\,(19)^4\,(37)^4\,(113)\,(337)^3\,(661)^3\,(1049)^3$$

$$24\beta - 1 = (5)^2\,(7)^3\,(13)^2\,(17)^2\,(19)^3\,(37)^3\,(337)^2\,(661)^2\,(1049)^2\,(3863)$$

where

$$\beta = 1278827052061576887278324769721420299$$

Theorem (K. ONO (2000))

Let $m > 3$ be prime and $k \geq 1$. There is a positive proportion of primes ℓ such that

$$p\left(\frac{m^k \ell^3 n + 1}{24}\right) \equiv 0 \pmod{m}$$

for $(n, \ell) = 1$.

Theorem (S. AHLGREN (2000))

Let M be a positive integer coprime to 6. Let G_M be the product of all the prime factors of M. Then a positive proportion of the primes $\ell \equiv -1 \pmod{576M}$ have the property that

$$p\left(\frac{G_M \ell^3 n + 1}{24}\right) \equiv 0 \pmod{M}$$

for $(n, \ell) = 1$.

CRANK AND RANK CONGRUENCES

Conjecture (K. ONO)

For every prime $\ell \geq 5$, and integers $j \geq 1$ there are infinitely many arithmetic progressions $An + B$ for which

$$M(r, \ell, An + B) \equiv 0 \pmod{\ell^j}$$

for all $n \geq 0$ and all $0 \leq r \leq \ell - 1$.

CRANK AND RANK CONGRUENCES

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K. MAHLBURG (2005)

Theorem (K. BRINGMANN (2009))

For every prime $\ell \geq 5$, and integers $m, u \geq 1$ there is a positive proportion of primes $p \equiv -1 \pmod{24\ell}$ such that for every $0 \leq r \leq \ell^m - 1$

$$N\left(r, \ell^m, \frac{p^3 n + 1}{24}\right) \equiv 0 \pmod{\ell^u}$$

where n is a quadratic residue mod ℓ and not divisiple by p.

K. BRINGMANN AND K. ONO (2010)

THE CRANK-RANK PDE AND RANK AND CRANK MOMENTS

Theorem (A.O.L. ATKIN and G. (2003))

$$z(q;q)_\infty^2 \left(C^*(z,q)\right)^3 = \left(3\delta_q + \tfrac{1}{2}\delta_z + \tfrac{1}{2}\delta_z^2\right) R^*(z,q),$$

where

$$C^*(z,q) = \frac{1}{1-z}C(z,q), \quad R^*(z,q) = \frac{1}{1-z}R(z,q),$$

$$\delta_q = q\frac{d}{dq}, \quad \delta_z = z\frac{d}{dz}$$

CRANK AND RANK MOMENTS

$$M_k(n) = \sum_m m^k M(m,n), \qquad C_k(q) = \sum_{n=0}^{\infty} M_k(n) q^n,$$

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EISENSTEIN SERIES REVISITED

$$\Phi_j(q) = \sum_{n=1}^{\infty} \sigma_j(n) q^n, \qquad \sigma_j(n) = \sum_{d|n} d^j,$$

$$E_n(\tau) = 1 - \frac{2n}{B_n} \Phi_{n-1}(q)$$

CRANK MOMENTS AS QUASIMODULAR FORMS

Theorem (ATKIN and G.)

For $n \geq 1$ there are integers $\alpha_{a_1,a_2,\ldots,a_n}$ such that

$$C_{2n}(q) = P \sum_{a_1+2a_2+\cdots+na_n=n} \alpha_{a_1,a_2,\ldots,a_n} \Phi_1^{a_1} \Phi_3^{a_2} \cdots \Phi_{2n-1}^{a_n}$$

where

$$P = \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty}$$

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where

$$P = \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty}$$

$$C_k(q) = (\delta_z)^k C(z,q)\Big|_{z=1}, \qquad R_k(q) = (\delta_z)^k R(z,q)\Big|_{z=1},$$

CRANK-RANK MOMENT EQUATION

$$\sum_{i=0}^{a/2-1} \binom{a}{2i} \sum_{\substack{\alpha+\beta+\gamma=a-2i \\ \alpha,\ \beta,\ \gamma \text{ even} \geq 0}} \binom{a-2i}{\alpha,\beta,\gamma} C_\alpha\, C_\beta\, C_\gamma\, P^{-2} - 3\left(2^{a-1}-1\right) C_2$$

$$= \frac{1}{2}(a-1)(a-2)R_a + 6\sum_{i=1}^{a/2-1} \binom{a}{2i}\left(2^{2i-1}-1\right)\delta_q(R_{a-2i})$$

$$+ \sum_{i=1}^{a/2-1}\left[\binom{a}{2i+2}\left(2^{2i+1}-1\right) - 2^{2i}\binom{a}{2i+1} + \binom{a}{2i}\right] R_{a-2i}.$$

THE SPACE $\mathcal{W}_{2n}$

Let n be a positive integer. The set

$$\Big\{\Phi_1^a\Phi_3^b\Phi_5^c \,:\, 1 \le a+2b+3c \le n \\ \text{with } a,\ b,\ c \text{ nonnegative integers}\Big\}$$

is linearly independent (over $\mathbb{C}$) and spans a vector space we denote by $\mathcal{W}_{2n}$. Let

$$P\mathcal{W}_k = \{G\,P \,:\, G \in \mathcal{W}_k\}.$$

PROPERTIES OF CRANK MOMENTS

(i) For $n > 1$ $\Phi_{2n-1} \in \mathrm{Span}\{\Phi_3^b \Phi_5^c \,:\, 2 \leq 2b + 3c \leq n\} \subset \mathcal{W}_{2n}$.

(ii) For $n \geq 1$, $C_{2n} \in P\mathcal{W}_{2n}$.

(iii) For $m \geq 0$ and $n \geq 1$, $\delta_q^m(\mathcal{W}_{2n}) \subset \mathcal{W}_{2n+2m}$.

(iv) For $n \geq 1$ $\dim \mathcal{W}_{2n} = 2n + \sum_{k=2}^{n} (2n - 2k + 1) \dim M_{2k}(1)$

(v) For $m \geq 1$ $\delta_q^m(P) \in P\mathcal{W}_{2m}$.

(vi) For $m \geq 0$ and $n \geq 1$, $\delta_q^m(C_{2n}) \in P\mathcal{W}_{2n+2m}$.

DEFINE

$$Y_{2k} := \sum_{i=0}^{k-1} \binom{2k}{2i} \sum_{\substack{\alpha+\beta+\gamma=2k-2i \\ \alpha,\beta,\gamma \text{ even } \geq 0}} \binom{2k-2i}{\alpha,\beta,\gamma} C_\alpha\, C_\beta\, C_\gamma\, P^{-2} - 3\left(2^{2k-1}-1\right) C_2.$$

Corollary

For $n \geq 2$, $Y_{2n} \in P\mathcal{W}_{2n}$

Then

$$R_{2k} = \frac{1}{(2k-1)(k-1)} Y_{2k} - \frac{1}{(2k-1)(k-1)} \left(\sum_{i=1}^{k-1} 6 \binom{2k}{2i} \left(2^{2i-1} - 1\right) \delta_q(R_{2k-2i}) + \sum_{i=1}^{k-1} \left[\binom{2k}{2i+2} \left(2^{2i+1} - 1\right) - 2^{2i} \binom{2k}{2i+1} + \binom{2k}{2i} \right] R_{2k-2i} \right).$$

Theorem (ATKIN and G.)

For $k \geq 2$

$$R_{2k} = P_k(\delta_q)\, R_2 + \sum_{j=2}^{k} Q_{k,j}(\delta_q)\, Y_{2j},$$

where $P_k(x)$, $Q_{k,j} \in \mathbb{Q}[x]$ *have degrees* $k-1$ *and* $k-j$ *resp. Further the polynomials* $P_k(x)$ *satisfy* $P_0(x) := 0$, $P_1(x) = 1$ *and for* $k \geq 2$ *we have the recurrence relation*

$$P_k(x) = (1 - 12x)\, P_{k-1}(x) - 36x^2\, P_{k-2}(x)$$

Corollary

For $k \geq 2$ the polynomial $P_k(x)$ has integer coefficients. If $\ell > 3$ is prime then

$$P_{\frac{\ell+1}{2}}(x) \equiv \frac{\ell+1}{2}\left(1 + (1-24x)^{\frac{\ell-1}{2}}\right) \pmod{\ell}. \tag{1}$$

RETURN TO 2ND RANK COROLLARY

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RETURN TO 2ND RANK COROLLARY

DEFINE for $k \geq 1$

$$\mathcal{C}_{2k} = \left\{\delta_q^m\left(C_{2j}\right) : 1 \leq j \leq k, j+m \leq k\right\} \subset P\mathcal{W}_{2k}$$

Theorem (CRANK AND RANK MOMENT RELATIONS)

(i) *For $k = 2, 3, 4, 5$ there are polynomials $\widetilde{Q}_{k,j}(x) \in \mathbb{Q}[x]$ of degree $k - j$ $(1 \leq j \leq k)$ such that*

$$N_{2k}(n) = P_k(n)N_2(n) + \sum_{j=1}^{k} \widetilde{Q}_{k,j}(n)M_{2j}(n)$$

(ii)

$$N_{12}(n) = -(6n-1)(18n-1)(12n-1)(36n^2-24n+1)N_2(n) + \sum_{j=1}^{6} \widetilde{Q}_{6,j}(n)M_{2j}(n) - \frac{3316336128}{24599722121}p_{23}(n-1)$$

RETURN TO MOD 11 EXAMPLE

(iii)

$$N_{14}(n) = \frac{91}{138}(13 - 36n)N_{12}(n) + \left(P_7(n) - \frac{91}{138}(13 - 36n)\right)N_2(n) + \sum_{j=1}^{7} \widetilde{Q}_{6,7}(n)M_{2j}(n)$$

Here

$$\sum_{n=0}^{\infty} p_{23}(n)q^n = (q)_\infty^{23}$$

Proof Since $Y_{2k} \in P\mathcal{W}_{2k}$, $\mathcal{C}_{2k} \subset P\mathcal{W}_{2k}$, $|\mathcal{C}_{2k}| = \frac{1}{2}k(k+1) = \dim P\mathcal{W}_{2k}$ for $1 \leq k \leq 5$ there must be a non-trivial relation between Y_{2k} and the elements of $\mathcal{C}_{2k}$. Part (i) follows.

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Since $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in M_{12}(1)$

$$P\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{23} \ = \sum_{n=0}^{\infty} p_{23}(n)q^n \in P\mathcal{W}_{12}.$$

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Since Y_{12}, $P\Delta \in P\mathcal{W}_{12}$, $\mathcal{C}_{12} \subset P\mathcal{W}_{12}$, $|\mathcal{C}_{12}| = 21$ and $\dim P\mathcal{W}_{12} = 22$ there is a non-trivial relation which gives (ii).

Proof Since $Y_{2k} \in P\mathcal{W}_{2k}$, $\mathcal{C}_{2k} \subset P\mathcal{W}_{2k}$, $|\mathcal{C}_{2k}| = \frac{1}{2}k(k+1) = \dim P\mathcal{W}_{2k}$ for $1 \leq k \leq 5$ there must be a non-trivial relation between Y_{2k} and the elements of $\mathcal{C}_{2k}$. Part (i) follows.

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EXAMPLES For $n \geq 0$ we have

$$N_4(n) = \frac{2}{3}\,(-3\,n-1)\;M_2(n) + \frac{8}{3}\,M_4(n) + (-12\,n+1)\;N_2(n)$$

$$N_6(n) = \frac{2}{33}\,\left(324\,n^2+69\,n-10\right)\;M_2(n) + \frac{20}{33}\,(-45\,n+4)\;M_4(n) + \frac{18}{11}\,M_6(n) + \left(108\,n^2-24\,n+1\right)\;N_2(n)$$

ℓ-INTEGRAL QUASI-MODULAR FORMS

Let $\ell > 3$ be prime. A rational $\frac{m}{n}$ is $\boxed{\ell\text{-integral}}$ if $\ell \nmid n$. A function is an $\boxed{\ell\text{-integral quasi-modular form of weight } k}$ if it can be written as a sum of functions

$$E_2^a(\tau)F_b(\tau)$$

where $F_b(\tau) \in M_b(1)$ has ℓ-integral coefficients , $a \geq 0$ and $2a + b = k$. For k a nonnegative even integer we let $\mathcal{X}_k$ denote the set of functions that are sums of ℓ-integral quasi-modular form of weight $\leq k$. Let

$$P\mathcal{X}_k = \{G\,P \,:\, G \in \mathcal{X}_k\}.$$

Lemma (VON STAUDT-KUMMER)

(i) *If* $(\ell - 1) \mid 2n$ *then* $\ell B_{2n} \equiv -1 \pmod{l}$.

(ii) *If* $(\ell - 1) \nmid 2n$ *then* $\frac{B_{2n}}{2n}$ *is* ℓ*-integral and its residue class mod* ℓ *only depends on* $2n \pmod{\ell - 1}$.

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Corollary

Let $\ell > 3$ *be prime. Then the Eisenstein series* $E_2, \ldots, E_{\ell-1}$ *and* $E_{\ell+1}$ *are* ℓ*-integral and*

$$E_{\ell-1} \equiv 1 \pmod{\ell}, \qquad E_{\ell+1} \equiv E_2 \pmod{\ell}.$$

Proposition

Let $\ell > 3$ be prime.

(i) *For $k \geq 0$ even we have $\delta_q(\mathcal{X}_k) \subset \mathcal{X}_{k+2}$.*

(ii) *For $k \geq 0$ even and $m \geq 0$ we have $\delta_q^m(P\mathcal{X}_k) \subset P\mathcal{X}_{k+2m}$.*

(iii) *For $1 \leq j \leq \frac{\ell+1}{2}$ except $j = \frac{\ell-1}{2}$ we have $C_{2j} \in P\mathcal{X}_{2j}$.*

(iv) *$C_{\ell-1} = 2P\Phi_{\ell-2} + PG$ for some $G \in \mathcal{X}_{\ell-1}$.*

(v) *For $1 \leq j \leq \frac{\ell-3}{2}$ we have $Y_{2j} \in P\mathcal{X}_{2j}$.*

(vi) *$Y_{\ell-1} = 6P\Phi_{\ell-2} + PG$ for some $G \in \mathcal{X}_{\ell-1}$.*

Theorem

Let $\ell > 3$ be prime. Then for $2 \leq k \leq \frac{\ell-3}{2}$ we have

$$R_{2k} - P_k(\delta_q)R_2 \in P\mathcal{X}_{2k}$$

Also,

$$R_{\ell-1} - P_{\frac{\ell-1}{2}}(\delta_q)R_2 = 2P\Phi_{\ell-2} + PG_R,$$

for some $G_R \in \mathcal{X}_{\ell-1}$.

Theorem

Let $\ell > 3$ be prime. Then

$$R_{\ell+1} - P_{\frac{\ell+1}{2}}(\delta_q)R_2 \in P\mathcal{X}_{\ell+1}.$$

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For $\ell > 3$ prime and $\epsilon \in \{-1, 0, 1\}$ we define the operator $\boxed{U^*_{\epsilon,\ell}}$ which acts on q-series by

$$U^*_{\epsilon,\ell}\left(\sum a(n)q^n\right) = \sum_{\left(\frac{1-24n}{\ell}\right)=\epsilon} a(n)q^n.$$

The following corollary follows from POLYNOMIAL CONGRUENCE.

Corollary

Let $\ell > 3$ be prime and suppose $\epsilon = -1$ or 0. Then

$$U^*_{\epsilon,\ell}(R_2) \equiv U^*_{\epsilon,\ell}(G_\ell P) \pmod{\ell}$$

where $G_\ell \in \mathcal{X}_{\ell+1}$.

Theorem (BRINGMANN, G. and MAHLBURG)

Let $\ell > 3$ be prime and suppose β_ℓ, r_ℓ, and λ_ℓ are defined EARLIER. *Then*

(i)

$$\sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n+r_\ell} \equiv \eta^{r_\ell}(24\tau) G_{\ell,R,2}(24\tau) \pmod{\ell},$$

where $G_{\ell,R,2}(\tau)$ is a sum of integral modular forms of weight $\leq \frac{\ell(\ell+3)-r_\ell-1}{2}$.

(ii) *For $2 \leq k \leq \frac{\ell-3}{2}$,*

$$\sum_{n=0}^{\infty} N_{2k}(\ell n + \beta_\ell) q^{24n+r_\ell} \equiv c_k \sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n+r_\ell} + \eta^{r_\ell}(24\tau) G_{\ell,R,2k}(24\tau) \pmod{\ell}.$$

where $G_{\ell,R,2k}(\tau)$ is a sum of integral modular forms of weight $\leq k(\ell+1)-1+\frac{1}{2}(\ell-r)$, and c_k is some integer.

(iii) $$\sum_{n=0}^{\infty} N_{\ell-1}(\ell n+\beta_\ell)q^{24n+r_\ell}$$
$$\equiv c_{\ell-1}\sum_{n=0}^{\infty} N_2(\ell n+\beta_\ell)q^{24n+r_\ell}+\eta^{r_\ell}(24\tau)G_{\ell,R,\ell-1}(24\tau)$$
$$+\frac{1}{\ell}\eta^{r_\ell}(24\tau)\left(H^{(1)}_{\ell,R,\ell-1}(24\tau)-H^{(2)}_{\ell,R,\ell-1}(24\tau)\right) \pmod{\ell}.$$

where $G_{\ell,R,\ell-1}(\tau)$ is a sum of integral modular forms of weight $\leq \frac{\ell(\ell+1)-r_\ell-3}{2}$ $c_{\ell-1}$ is some integer, and $H^{(1)}_{\ell,R,\ell-1}(\tau)$, $H^{(2)}_{\ell,R,\ell-1}(\tau)$ are integral modular forms of weight $\frac{\ell(\ell-1)-r_\ell-1}{2}$ and $\frac{\ell(\ell+1)-r_\ell-3}{2}$ respectively such that

$$H^{(1)}_{\ell,R,\ell-1}(\tau)\equiv H^{(2)}_{\ell,R,\ell-1}(\tau) \pmod{\ell}.$$

EXAMPLE $\ell = 11$, $r_{11} = 13$, $\lambda_{11} = 1$ and $\beta_{11} = 6$. According to Theorem ON SECOND RANK

$$R_{12} - P_6(\delta_q)R_2 \in P\mathcal{X}_{12},$$

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$$\sum_{n=0}^{\infty} N_2(11n+6)q^n \equiv E(q)^{13}G(\tau) \pmod{13}$$

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for some sum of 13-integral modular forms of weight ≤ 70. The result may be greatly simplified. INSTEAD we use part (ii) CRANK RANK MOMENT RELATIONS THEOREM.

Reducing this mod 11 we find that

$$\begin{aligned} N_{12}(11n+6) &\equiv 6N_2(11n+6) + 5M_2(11n_6) + 4M_4(11n+6) \\ &+ 4M_6(11n+6) + 7M_8(11n+6) + 3M_{12}(11n+6) \\ &+ 7p_{23}(11n+5) \pmod{11} \end{aligned}$$

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and

$$-5N_2(11n+6) \equiv 7p_{23}(11n+5) \pmod{11}$$

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since

$$M(0,11,11n+6) = \cdots = M(10,11,11n+6)$$

$$\sum_{n=0}^{\infty} p_{23}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{23} \equiv \prod_{n=1}^{\infty}(1-q^{11n})^2 \prod_{n=1}^{\infty}(1-q^n) \pmod{11}$$

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$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$$

$$\sum_{n=0}^{\infty} p_{23}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{23} \equiv \prod_{n=1}^{\infty}(1-q^{11n})^2 \prod_{n=1}^{\infty}(1-q^n) \pmod{11}$$

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$$

$n(3n-1)/2 \equiv 5 \pmod{11}$ if and only if $n \equiv 2 \pmod{11}$

$$\sum_{n=0}^{\infty} p_{23}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{23} \equiv \prod_{n=1}^{\infty}(1-q^{11n})^2 \prod_{n=1}^{\infty}(1-q^n) \pmod{11}$$

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$$n(3n-1)/2 \equiv 5 \pmod{11} \qquad \text{if and only if} \qquad n \equiv 2 \pmod{11}$$

$$\sum_{n=0}^{\infty} p_{23}(11n+5)q^{11n+5} \equiv q^5 \prod_{n=1}^{\infty}(1-q^{11n})^2 \prod_{n=1}^{\infty}(1-q^{121n}) \pmod{11}$$

$$\sum_{n=0}^{\infty} p_{23}(11n+5)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^2 \prod_{n=1}^{\infty}(1-q^{11n}) \pmod{11}$$
$$\equiv \prod_{n=1}^{\infty}(1-q^n)^{13}$$

$$\sum_{n=0}^{\infty} p_{23}(11n+5)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^2 \prod_{n=1}^{\infty}(1-q^{11n}) \pmod{11}$$
$$\equiv \prod_{n=1}^{\infty}(1-q^n)^{13}$$

and therefore

$$\sum_{n=0}^{\infty} N_2(11n+6)q^n \equiv 3E(q)^{13} \pmod{11}$$

Theorem

$$\sum_{n=0}^{\infty} N_2(11n+6)\,q^n \equiv 3E^{13}(q) \pmod{11},$$

$$\sum_{n=0}^{\infty} N_4(11n+6)\,q^n \equiv 7E^{13}(q) \pmod{11},$$

$$\sum_{n=0}^{\infty} N_6(11n+6)\,q^n \equiv E^{13}(q)\,(4+E_4(\tau)) \pmod{11},$$

$$\sum_{n=0}^{\infty} N_8(11n+6)\,q^n \equiv E^{13}(q)\,(5+6E_4(\tau)+6E_6(\tau)) \pmod{11},$$

$$\sum_{n=0}^{\infty} N_{10}(11n+6)\,q^n \equiv E^{13}(q)\,(5+4E_4(\tau)+6E_6(\tau) + 6{E_4}^2(\tau)) \pmod{11}.$$

Corollary

$$N(r, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11},$$
$$N(r, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11},$$

for all $0 \leq r \leq 10$.

Corollary

$$N(r, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11},$$
$$N(r, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11},$$

for all $0 \leq r \leq 10$.

Proof Sketch For each r there exist integral constants a_r, b_r, c_r, d_r such that

$$\sum_{n=0}^{\infty} N(r, 11, \tfrac{1}{24}(11n+1))q^n \equiv \eta^{13}(24\tau)(a_r + b_r E_4(24\tau) + c_r E_6(24\tau) + d_r E_4^2(24\tau)) \pmod{11}.$$

Since

$$\dim M_0(1) = \dim M_4(1) = \dim M_6(1) = \dim M_8(1) = 1$$

By EIGENFORM COROLLARY each of the functions

$$\eta^{13}(24\tau), \quad \eta^{13}(24\tau)E_4(24\tau), \quad \eta^{13}(24\tau)E_6(24\tau), \quad \eta^{13}(24\tau)E_4^2(24\tau),$$

is a Hecke eigenform in its corresponding space.

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is a Hecke eigenform in its corresponding space. The first result follows from finding the relevant $0 \pmod{11}$ common eigenvalues.