NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
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Frank Garvan
url: qseries.org/fgarvan

University of Florida

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LECTURE 5
RAMANUJAN'S MOCK THETA FUNCTIONS

RAMANUJAN'S LAST LETTER

- RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

WATSON

ZWEGERS

- ZWEGERS 2000 BREAKTHROUGH
- MODERN DEFINITION OF A MOCK THETA FUNCTION

RAMANUJAN'S LAST LETTER

S. Ramanujan to G. H. Hardy **12 January 1920**
University of Madras

I am extremely sorry for not writing you a single letter up to now . . . I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as ordinary ϑ-functions. I am sending you this letter with some examples . . .

RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

A *mock ϑ-function* is a function $M(q)$, holomorphic for $|q| < 1$, such that

(i) $M(q)$ has infinitely many exponential singularities at roots of unity,

(ii) under radial approach to every such singularity, $M(q)$ has an approximation consisting of a finite sum of terms with closed exponential factors, and an error term $O(1)$,

(iii) there is no ϑ-function $T(q)$ which differs from $M(q)$ by a "trivial function", i.e. a function bounded under radial approach to every root of unity.

It seems that by a ϑ-function Ramanujan means a quotient of series of the form

$$\sum_{n=-\infty}^{\infty} (-1)^{kn} q^{an^2+bn}$$

where $k = 0, 1,\ a, b$ are rational with $a > 0$.

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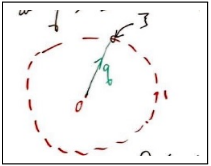
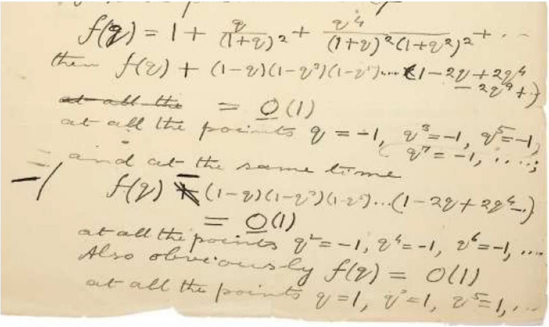
RADIAL LIMIT



EXAMPLE:



$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$

~~at all the~~ $= O(1)$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1, \ldots$;

and at the same time

$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$

$= O(1)$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1, \ldots$

Also obviously $f(q) = O(1)$

at all the points $q=1$, $q^3=1$, $q^5=1, \ldots$

EXAMPLE:

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2},$$

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if ξ is a primitive m-th root of unity then

$$f(q) = \begin{cases} O(1), & \text{if } m \text{ is odd}, \\ -T(q) + O(1), & \text{if } m \equiv 2 \pmod 4, \\ T(q) + O(1), & \text{if } m \equiv 0 \pmod 4, \end{cases}$$

radially as $q \to \xi$,

EXAMPLE:

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radially as $q \to \xi$, where $T(q)$ is ϑ-function

$$T(q) = (1-q)(1-q^3)(1-q^5)\cdots(1 - 2q + 2q^4 - 2q^9 + \cdots)$$

$$= \frac{\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\right)^2}{\sum_{n=-\infty}^{\infty} q^{2n^2+n}} = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{2n})^2}$$

PROOF SKETCH

$$\begin{aligned} 2\phi(-q) - f(q) &= f(q) + 4\psi(-q) \\ &= \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1+q)(1+q^2)(1+q^3)\cdots} = T(q) \end{aligned}$$

where

$$\phi(q) = 1 + \frac{q}{(1+q^2)} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}$$

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$$\psi(q) = \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^3)} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$

$$f(q) + T(q) = 2\phi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n} = O(1)$$

as $q \to \xi$ (primitive m-root of unity when $m \equiv 2 \pmod 4$)

$$f(q) + T(q) = 2\phi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q^2; q^2)_n} = O(1)$$

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$$f(q) - T(q) = -4\psi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q; q^2)_n} = O(1)$$

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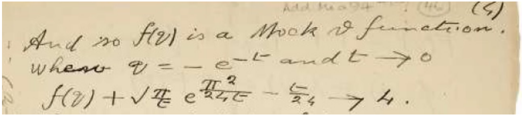
EXAMPLE:

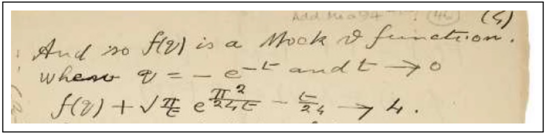

(5)

And so $f(q)$ is a Mock ϑ function.

When $q = -e^{-t}$ and $t \to 0$

$$f(q) + \sqrt{\frac{\pi}{t}}\, e^{\frac{\pi^2}{24t} - \frac{t}{24}} \to 4.$$

EXAMPLE:



When $q = -e^{-t}$ and $t \to 0$

$$f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4.$$

PROOF SKETCH

PROOF SKETCH

THE DEDEKIND ETA FUNCTION

$$\eta(\tau) := \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau)),$$

for $Im(\tau) > 0$. Then

$$\eta(24\tau) \in S_{1/2}(576, \chi_{12}),$$

where $\chi_{12}(n) = \left(\frac{12}{n}\right)$.

$$\eta(\tau+1) = \exp(\pi i\tau/12)\eta(\tau)$$
$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\nu_\eta(A)\sqrt{c\tau+d}\,\eta(\tau)$$

for $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\nu_\eta(A)\sqrt{c\tau+d}\,\eta(\tau)$$

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$$\eta\left(\frac{\tau}{2\tau+1}\right)=\exp\left(\frac{-\pi i}{3}\right)\sqrt{2\tau+1}\,\eta(\tau)$$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\nu_\eta(A)\sqrt{c\tau+d}\,\eta(\tau)$$

for $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$

$$\eta\left(\frac{\tau}{2\tau+1}\right)=\exp\left(\frac{-\pi i}{3}\right)\sqrt{2\tau+1}\,\eta(\tau)$$

When $q=-e^{-t}$ then

$$\prod_{n=1}^{\infty}(1-q^n)=\sqrt{\frac{\pi}{t}}\exp\left(\frac{t}{24}-\frac{\pi^2}{24t}\right)\prod_{n=1}^{\infty}(1-q_1^n),$$

where $q_1=-\exp\left(\frac{-\pi^2}{t}\right)$.

When $q = -e^{-t}$ then

$$\prod_{n=1}^{\infty}(1-q^{2n}) = \sqrt{\frac{\pi}{t}}\exp\left(-\frac{t}{12}-\frac{\pi^2}{12t}\right)\prod_{n=1}^{\infty}(1-q_1^{2n}),$$

where $q_1 = -\exp\left(\frac{-\pi^2}{t}\right)$.

When $q = -e^{-t}$ then

$$\prod_{n=1}^{\infty}(1-q^{2n}) = \sqrt{\frac{\pi}{t}}\,\exp\left(-\frac{t}{12}-\frac{\pi^2}{12t}\right)\prod_{n=1}^{\infty}(1-q_1^{2n}),$$

where $q_1 = -\exp\left(\frac{-\pi^2}{t}\right)$. When $q = -e^{-t}$

$$\begin{aligned} T(q) &= \prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{2n})^2} \\ &= \sqrt{\frac{\pi}{t}}\,\exp\left(3\left(\frac{t}{24}-\frac{\pi^2}{24t}\right)-2\left(\frac{t}{12}-\frac{\pi^2}{12t}\right)\right)\,T(q_1) \\ &= \sqrt{\frac{\pi}{t}}\,\exp\left(-\frac{t}{24}+\frac{\pi^2}{24t}\right)(1+O(q_1)) \\ &= \sqrt{\frac{\pi}{t}}\,\exp\left(-\frac{t}{24}+\frac{\pi^2}{24t}\right)+o(1) \end{aligned}$$

$$f(q) + T(q) = 2\phi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q^2; q^2)_n}$$

and

$$f(q) + T(q) = 2\phi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q^2; q^2)_n}$$

and

$$\lim_{q \to 1^-} 2\phi(q) = 4.$$

HENCE

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$$f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4.$$

GEORGE NEVILLE WATSON (1936)

The Final Problem: An Account of the Mock Theta Functions

THIRD ORDER MOCK THETA FUNCTIONS

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, + \cdots =$$

$$\phi(q) = 1 + \frac{q}{(1+q^2)} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}$$

$$\psi(q) = \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^3)} + \cdots = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$

$$\chi(q) = 1 + \frac{q}{(1-q+q^2)} + \cdots = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(-q^3;q^3)_n}$$

NOTATION

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$$[b_1, b_2, \ldots, b_r; q]_\infty$$
$$= (b_1; q)_\infty (b_1^{-1} q; q)_\infty (b_2; q)_\infty (b_2^{-1} q; q)_\infty \cdots (b_r; q)_\infty (b_r^{-1} q; q)_\infty$$

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$$S(a, b; q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2} a^n}{1 - bq^n}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n} = \frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right) \tag{A}$$

$$\frac{(q)_\infty^2}{[b_1,b_2,b_3;q]_\infty} = \frac{S(b_1^2/b_2b_3,b_1;q)}{[b_2/b_1,b_3/b_1;q]_\infty} + \frac{S(b_2^2/b_3b_1,b_2;q)}{[b_3/b_2,b_1/b_2;q]_\infty} + \frac{S(b_3^2/b_1b_2,b_3;q)}{[b_1/b_3,b_2/b_3;q]_\infty} \tag{B}$$

$$f(q)\prod_{r=1}^{\infty}(1-q^r)=1+4\sum_{n=1}^{\infty}\frac{(-)^n q^{\frac{1}{2}n(3n+1)}}{1+q^n},$$

$$\phi(q)\prod_{r=1}^{\infty}(1-q^r)=1+2\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)\,q^{\frac{1}{2}n(3n+1)}}{1+q^{2n}},$$

$$\chi(q)\prod_{r=1}^{\infty}(1-q^r)=1+\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)\,q^{\frac{1}{2}n(3n+1)}}{1-q^n+q^{2n}}.$$

THIRD ORDER FUNCTIONS IN TERMS OF THE RANK

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$(A) = R(z, q)$.

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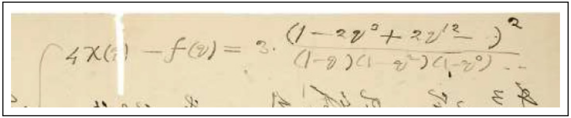
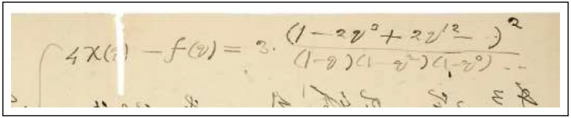
$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=R(-1,q)=\frac{2}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

$$\phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=R(i,q)=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{2n}}$$

$$\psi(q)=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}=-1+\frac{1}{1-q}R(q,q^4)=\frac{1}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n(}}{1-q^4}$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n^2}}{(-q^3;q^3)_n}=R(\zeta_6,q)=\frac{1}{2(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+}{1+q^{3n}}$$


$4\chi(q) - f(q) = 3 \cdot \frac{(1-2q^3+2q^{12}-\ )^2}{(1-q)(1-q^2)(1-q^3)\ldots}$



$$4\chi(q) - f(q) = 3\frac{(1 - 2q^3 + 2q^{12} - + \cdots)^2}{(1-q)(1-q^2)(1-q^3)\cdots}$$

PROOF

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CRANK GENERATING FUNCTION

$$C(z,q) = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} = \frac{(1-z)}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1-zq^n}$$

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$$C(-1,q) = \frac{(q;q)_\infty}{(-q;q)_\infty^2} = \frac{2}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}$$

$$4\chi(q) - f(q)$$
$$= \frac{2}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}((1+q^n)^2 - (1-q^n+q^{2n}))}{1+q^{3n}}$$
$$= \frac{6}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{3n}} = \frac{3}{(q)_\infty} C(-1, q^3)(q^3;q^3)_\infty$$
$$= \frac{3(q^3;q^3)_\infty^2}{(q)_\infty(-q^3;q^3)_\infty^2} = 3\frac{(1-2q^3+2q^{12} - + \cdots)^2}{(1-q)(1-q^2)(1-q^3)\cdots}$$

WATSON'S MODULAR TRANSFORMATIONS

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$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}$$

$$= \frac{1}{q}\left(-1 + \frac{1}{1-q} R(q,q^2)\right)$$

$$= \frac{1}{2(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1+q^{2n+1})}{(1-q^{2n+1})}$$

WATSON'S MODULAR TRANSFORMATIONS

$$
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&= \frac{1}{q}\left(-1+\frac{1}{1-q}R(q,q^2)\right) \\
&= \frac{1}{2(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1+q^{2n+1})}{(1-q^{2n+1})}
\end{aligned}
$$

$$
\begin{aligned}
q^{-1/24} f(q) = 2\sqrt{\frac{2\pi}{\alpha}}\, q_1^{4/3}\, \omega(q_1^2) \\
+ 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^{\infty} e^{-3\alpha x^2/2} \frac{\sinh(\alpha x)}{\sinh(3\alpha x/2)}\, dx
\end{aligned}
$$

where $q = e^{-\alpha}$, $q_1 = e^{-\beta}$, $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$.

This is the modular transformation $\tau \mapsto -1/\tau$ in disguise since if $\alpha = -\pi i \tau$ then $q = \exp(\pi i \tau)$

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THE STARTING POINT

$$(q)_\infty f(q) = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

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$$2\frac{(-1)^n \exp\left(-\frac{3\alpha n^2}{2} - \frac{\alpha n}{2}\right)}{1 + exp(-\alpha n)} = 2\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

$$(q)_\infty f(q) = \frac{1}{2\pi i}\left[\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} F(z)\,dz + \int_{\infty+i\varepsilon}^{-\infty+i\varepsilon} F(z)\,dz\right]$$

where $0 < \varepsilon < \pi/(3\alpha)$.

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THE TRICK

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where $0 < \varepsilon < \pi/(3\alpha)$.

THE TRICK is to use THE SADDLE POINT METHOD to move the lines of integration of certain integrals which pick up new residues corresponding to

$$q_1^{1/6}\,(q_1^4;q_1^4)_\infty q_1^{4/3}\,\omega(q_1^2)$$

$$(q)_\infty f(q) = \frac{1}{2\pi i}\left[\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} F(z)\,dz + \int_{\infty+i\varepsilon}^{-\infty+i\varepsilon} F(z)\,dz\right]$$

where $0 < \varepsilon < \pi/(3\alpha)$.
THE TRICK is to use THE SADDLE POINT METHOD to move the lines of integration of certain integrals which pick up new residues corresponding to

$$q_1^{1/6}\,(q_1^4;q_1^4)_\infty q_1^{4/3}\,\omega(q_1^2)$$

and a remainder term involving

$$q_1^{1/6}\,(q_1^4;q_1^4)_\infty \int_0^\infty e^{-3\alpha x^2/2}\frac{\sinh(\alpha x)}{\sinh(3\alpha x/2)}\,dx$$

The result then follows using the transformation

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule". I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{\frac{2}{3}\pi}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2 (1+e^{-3\pi})^2 \ldots (1+e^{-(2n+1)\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing "Day", "Night", "Evening", and "Dawn" which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

"Pale, beyond porch and portal,
  Crowned with calm leaves, she stands
Who gathers all things mortal
  With cold immortal hands".

ZWEGERS

Sander Zwegers

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$$F(\tau) = (f_0, f_1, f_2)^T$$

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$$f_0(\tau) = q^{-1/24} f(q), \quad f_1(\tau) = 2q^{1/3}\omega(q^{1/2}), \quad f_2(\tau) = 2q^{1/3}\omega(-q^{1/2}),$$

where $q = \exp(2\pi i\tau)$ and $\tau \in \mathfrak{h}$.

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$$\frac{1}{\sqrt{-i\tau}} F\left(\frac{-1}{\tau}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0-1 \end{bmatrix} F(\tau) + R(\tau), \quad \text{(WATSON)}$$

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where

$$R(\tau) = 4\sqrt{3}\sqrt{-i\tau}\,(j_2(\tau), -j_1(\tau), j_3(\tau))^T,$$

$$j_1(\tau) = \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin(2\pi\tau x)}{\sin(3\pi\tau x)}\, dx, \quad \ldots, j_3(\tau) = \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin(\pi\tau x)}{\sin(3\pi\tau x)}\, dx$$

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If we let $\tau = i$ in (WATSON) we have

$$F(i) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} F(i) + R(i),$$

$$j_1(\tau) = \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin(2\pi\tau x)}{\sin(3\pi\tau x)}\, dx, \quad \ldots, j_3(\tau) = \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin(\pi\tau x)}{\sin(3\pi\tau x)}\, dx$$

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$$R(i) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} F(i)$$

Reading off the THIRD component:

$$4\sqrt{3}j_3(i) = 4\sqrt{3}\int_0^\infty e^{-3\pi x^2}\frac{\sinh(\pi x)}{\sinh(3\pi)}\,dx = 4e^{-2\pi/3}\omega(-e^{-\pi}),$$

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ZWEGERS 2000 BREAKTHROUGH

Zwegers constructs three weight 3/2 theta functions

$$g_0(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n (n+1/3) \exp\left(3\pi i (n+1/3)^2 \tau\right),$$

$$g_1(\tau) = -\sum_{n=-\infty}^{\infty} (-1)^n (n+1/6) \exp\left(3\pi i (n+1/6)^2 \tau\right),$$

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so that $R(\tau) = -2i\sqrt{3} \int_0^{\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}}\, dz$ where $g(z) = (g_0(z), g_1(z), g_2(z))^T$.

On crucial ingredients is the result that

$$\int_{-\infty}^{\infty} \frac{e^{-\pi t y^2}}{y - ir}\, dy = \pi i r \int_0^{\infty} \frac{e^{-\pi r^2 u}}{\sqrt{u+t}}\, du$$

for $r, t > 0$.

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for $r, t > 0$. Finally ZWEGERS defines

$$G(\tau) = 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(z+\tau)}}\, dz,$$

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$$\frac{1}{\sqrt{-i\tau}} G\left(\frac{-1}{\tau}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0-1 \end{bmatrix} G(\tau) + R(\tau),$$

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which is the SAME TRANSFORMATION satisfied by $F(\tau)$.

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This means that $H(\tau)$ is a vector-valued real analytic modular form of weight $1/2$.

IN ADDITION Each component of F is holomorphic so that

$$\begin{aligned} \frac{\partial H}{\partial \overline{\tau}} &= -\frac{\partial G}{\partial \overline{\tau}} \\ &= \frac{(g_1(-\overline{\tau}), g_0(-\overline{\tau}), -g_2(-\overline{\tau}))^T}{\sqrt{2y}} \end{aligned}$$

so that

$$\frac{\partial}{\partial \tau}\sqrt{y}\frac{\partial H}{\partial \overline{\tau}} = 0$$

$$\frac{\partial}{\partial \tau}\sqrt{y}\frac{\partial}{\partial \overline{\tau}} = \sqrt{y}\frac{\partial^2}{\partial \tau \partial \overline{\tau}} - \frac{i}{4}\frac{1}{\sqrt{y}}\frac{\partial}{\partial \tau}$$

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THUS

$$\Delta_{1/2}H = 0$$

where

$$\Delta_{1/2} = -4y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\tau}$$

is the weight $1/2$ hyperbolic Laplacian.

Theorem (S. ZWEGERS (2000))

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is a vector-valued modular form of weigth $1/2$ *satisfying*

$$H(\tau+1) = \begin{bmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{bmatrix} H(\tau),$$

$$\frac{1}{\sqrt{-i\tau}} H\left(\frac{-1}{\tau}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1, \end{bmatrix} H(\tau)$$

and

$$\Delta_{1/2} H(\tau) = 0,$$

where $\Delta_{1/2} = -4y^2 \frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\tau}$ *is the weight* $1/2$ *hyperbolic Laplacian.*

In his last letter to Hardy, Ramanujan also described two sets of fifth order mock theta functions and list three seventh order functions.

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In his thesis (2002) Zweger derived similar results for Ramanujan's fifth and seventh order mock theta functions. In his second paper Watson proved the claims for the fifth order functions in the letter but was unable to find modular transformation properties because there were no known identities like

$$f(q) = \frac{2}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

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$$f(q) = \frac{2}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

Watson did not consider the seventh order functions. Selberg (1938) studied the seventh order functions near the unit circle.

In 1986, George Andrews found new identities for most of the fifth order functions and all of the seventh order functions in terms of indefinite theta series. For example Andrews found the following fifth order indentity using Bailey pair machinery:

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$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{(n(5n+1)/2-j^2} (1-q^{4n+2})$$

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Zwegers was able to use Andrews's identities a build a theory of non-holomorphic theta functions to find transformation formulas these fifth and seveth order functions, completing them to real analytic modular forms of weight 1/2 analogous to his third order result.

In addition Zwegers considered the Lerch series

$$\mu(u, v, \tau) := \frac{1}{\theta(\zeta, q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{n+1/2} q^{n(n+1)/2}}{1 - zq^n}$$

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where $\zeta = \exp(2\pi iu)$, $z = \exp(2\pi iv)$, $q = \exp(2\pi i\tau)$, and

$$\theta(z, q) = z^{1/2} q^{1/8} \sum_{m=-\infty}^{\infty} (-1)^m z^m q^{m(m+1)/2}$$

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where $\zeta = \exp(2\pi i u)$, $z = \exp(2\pi i v)$, $q = \exp(2\pi i \tau)$, and

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He was able this to a function that transform likes a Jacobi form:

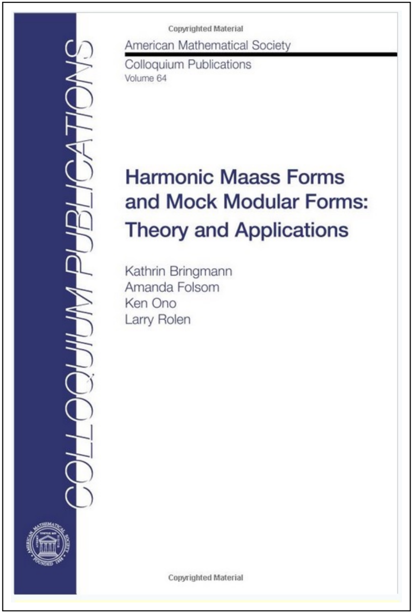
$$\widehat{\mu}(u, v, \tau) = \mu(u, v, \tau) + \frac{i}{2} R(u - v; \tau),$$

where $R(u; \tau) =$

$$\sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \left\{ \operatorname{sgn}(\nu) - E\left((\nu + a)\sqrt{2y} \right) \right\} (-1)^{\nu - 1/2} e^{-\pi i \nu^2 \tau - 2\pi i \nu u}$$

$y = \Im(\tau)$, $a = \frac{\Im(u)}{\Im(tau)}$,

$$E(z) = 2\int_0^z e^{-\pi u^2}\, du$$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Ken Ono
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In the Appendix of this book all of Ramanujan's mock theta functions are written in terms of $\mu(u, v, \tau)$ and thus allso fo Ramanujan's mock theta functions can be completed to transform like modular forms.

MODERN DEFINITION OF A MOCK THETA FUNCTION

Following BRUNIER AND FUNKE a weight k harmonic Maass form $f(\tau)$ on a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a smooth function $f\ :, \mathfrak{h} \longrightarrow \mathbb{C}$ satisying

(i)

$$f(A\tau) = \begin{cases} (c\tau + d)^k f(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\varepsilon_d^{-2k}(c\tau + d)^k f(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

(ii)

$$\Delta_k(f) = 0$$

where

$$\Delta_k = -4y^2 \frac{\partial^2}{\partial\tau\partial\overline{\tau}} + 2iky \frac{\partial}{\partial\tau}$$

$(\tau = x + iy)$

(iii) There is a polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$f(\tau) - P_f(\tau) = O(e^{-\varepsilon y}),$$

as $y \to \infty$ for some $\varepsilon > 0$, and analogous conditions at other cusps.

Let $\boxed{H_k(\Gamma)}$ denote the space of such functions.

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Let $\Gamma_1(N) \subset \Gamma$ for some positive integer N, and suppose $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ If $f \in H_k(\Gamma)$ then f has en expansion

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(1-k, -4\pi n y) q^n,$$

where $q = \exp(2\pi i \tau)$, $\tau = x + iy$,

$$\Gamma(s, z) = \int_z^\infty e^{-t} t^s \frac{dt}{t},$$

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where $q = \exp(2\pi i \tau)$, $\tau = x + iy$,

$$\Gamma(s, z) = \int_z^\infty e^{-t} t^s \frac{dt}{t},$$

$$f^+(tau) = \sum_{n \gg -\infty} c_f^+(n) q^n$$

is the $\boxed{\text{holomorphic part of } f}$, and

$$f^-(\tau) = \sum_{n<0} c_f^-(n) \Gamma(1-k, -4\pi n y) q^n$$

is the $\boxed{\text{non-holomorphic part of } f}$

$$f^{-}(\tau) = 2^{k-1} i \int_{-\overline{\tau}}^{i\infty} \frac{g^{c}(z)}{(-i(z+\tau))^{k}}\, dz$$

where $g(\tau) = \sum_{n=1}^{\infty} c_g(n) q^n \in S_{2-k}(\Gamma)$, and

$$g^{c}(\tau) = \sum_{n=1}^{\infty} \overline{c_g(n)} q^n = \overline{g(-\overline{\tau})}$$

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where $g(\tau) = \sum_{n=1}^{\infty} c_g(n) q^n \in S_{2-k}(\Gamma)$, and

$$g^c(\tau) = \sum_{n=1}^{\infty} \overline{c_g(n)} q^n = \overline{g(-\overline{\tau})}$$

The function $g(\tau)$ is called the shadow of f.

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The function $g(\tau)$ is called the shadow of f.
A mock modular form of weight k is the holomorphic part f^{+} of some harmonic Maass form of weight k for which f^{-} is non-trivial.

The **modern definition of a mock theta function** is a mock modular for of weight $1/2$ or $3/2$ whose shadow is a linear combination of unary theta functions.

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Theorem (ZWEGERS)

Ramanujan's mock theta functions are (up to multiplication by a power of q and addition of a constant) weight 1/2 mock modular forms. A Ramanujan mock theta function $F(\tau)$ has the form

$$F(\tau) = q^{\alpha} G^{+}(\tau) + c,$$

for some $\alpha \in \mathbb{Q}$, $c \in \mathbb{C}$, where $G^{+}(\tau)$ is the holomorphic part of a weight 1/2 harmonic Maass form whose shadow is a weight 3/2 unary theta function.