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Ramanujan's Ranks,
Mock Theta Functions, and Beyond
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LECTURE 6 (under construction)
TRANSFORMATION PROPERTIES FOR DYSON'S RANK FUNCTION

(Includes joint work with Rishabh Sarma)

INTRO

Building Blocks of Generalized Eta-Products

Symmetry for $\mathcal{K}_{p,0}(\zeta_p, z)$

SKETCH OF THE PROOF OF MAIN SYMMETRY RESULT THEOREM

We begin with a quote from Dyson :

> *"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future."*

As noted in Lecture 5, all of Ramanujan's third order mock theta functions can be written in terms of the Dyson's rank function

$$R(z,q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq, z^{-1}q; q)_n}.$$

For example,

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = R(-1,q),$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots = R(\zeta_6, q).$$

In 2010, in addressing Dyson's challenge, Bringmann and Ono studied transformation properties for $R(\zeta, q)$ when ζ is a root of unity. In LECTURE 2, we discussed the identity for $R(\zeta_5, q)$ from Ramanujan Lost Notebook, when ζ_5 is a primitive 5-th root of unity. We rewrite the identity in terms of generalized eta-products :

$$q^{-\frac{1}{24}}\left(R(\zeta_5,q) - (\zeta_5+\zeta_5^4-2)\,\phi(q^5) + (1+2\zeta_5+2\zeta_5^4)\,q^{-2}\,\psi(q^5)\right)$$
$$= \frac{\eta(25z)\,\eta_{5,2}(5z)}{\eta_{5,1}(5z)^2} + \frac{\eta(25z)}{\eta_{5,1}(5z)} + (\zeta_5+\zeta_5^4)\,\frac{\eta(25z)}{\eta_{5,2}(5z)} - (\zeta_5+\zeta_5^4)\,\frac{\eta(25z)\,\eta_5}{\eta_{5,2}(5z}$$

where

The identity can be generalized to all primes $p > 3$.
Let $p > 3$ be prime and $1 \leq a \leq \frac{p-1}{2}$. Define

$$\Phi_{p,a}(q) := \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } 0 < 6a < p, \\ \displaystyle -1 + \sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } p < 6a < 3p, \end{cases}$$

and

$$\mathcal{R}_p\left(\zeta_p, z\right)$$

$$:= q^{-\frac{1}{24}} R(\zeta_p, q) - \left(\frac{12}{p}\right) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a-\frac{1}{2}(p-1)}\right) q^{\frac{a}{2}(p-3a)-\frac{p^2}{24}} \Phi_{p,a}(q^p).$$

Theorem (G., 2017)

Let $p > 3$ be prime. Then the function

$$\eta(p^2 z)\,\mathcal{R}_p(z)$$

is a weakly holomorphic modular form of weight 1 on the group $\Gamma_0(p^2) \cap \Gamma_1(p)$.

In 2010, Bringmann and Ono proved

Theorem (Bringmann and Ono)

If c is odd, $0 < a < c$ then

$$q^{-\frac{\ell}{24}} R\left(\zeta_c^a; q^\ell\right) + i \sin\left(\frac{\pi a}{c}\right) \sqrt{\frac{\ell}{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \ell\tau\right)\, d\tau}{\sqrt{-i(\tau+z)}}$$

is a harmonic Maass form of weight $1/2$ on $\Gamma_1(144f^2\ell')$, where $\Theta\left(\frac{a}{c}; \ell\tau\right)$ is a sum of certain weight $3/2$ theta functions, $\ell = lcm(2c^2, 24), \ell' = 1/24, f = \frac{2c}{gcd(c,6)}$.

For c odd, $0 < a < c$, define

$$\mathcal{G}_1\left(\frac{a}{c}; z\right) = \csc\left(\frac{\pi a}{c}\right) q^{-\frac{l}{24}} R\left(\zeta_c^a; q\right) + \frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \tau\right) \, d\tau}{\sqrt{-i(\tau + z)}}.$$

Then Bringmann and Ono proved that $\mathcal{G}_1\left(\frac{a}{c}; z\right)$ is one component of a vector-valued harmonic Maass form of weight $1/2$ on $SL_2(\mathbb{Z})$. In 2017, the speaker corrected one of the components $\mathcal{G}_2\left(\frac{a}{c}; z\right)$. This means that $q^{-\frac{1}{24}} R\left(\zeta_c^a; q\right)$ is a mock modular form of weight $1/2$. Zagier later showed that the result is true for all general ζ_c.

Theorem (G.)

Let $p > 3$ be prime and $s_p = \frac{1}{24}(p^2 - 1)$. Then the function

$$(q^p; q^p)_\infty \sum_{n=\left\lceil \frac{1}{p}(s_p) \right\rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn - s_p) \, \zeta_p^k \right) q^n$$

is a weakly holomorphic modular form of weight 1 on $\Gamma_1(p)$.

REMARK This improves a result of Ahlgren and Treneer (2008) who were confined to the group $\Gamma_1(576.p^4)$.

We consider all components of the p-dissection of the rank function $R(\zeta_p, q)$.

DEFINITION For $p > 3$ prime, $0 \leq m \leq p-1$ and $1 \leq d \leq p-1$, define $\mathcal{K}_{p,m}(\zeta_p^d; z)$ as follows :

(i) For $m = 0$ or $\left(\frac{-24m}{p}\right) = -1$ define

$$\mathcal{K}_{p,m}(\zeta_p^d; z)$$

$$:= q^{m/p} \prod_{n=1}^{\infty}(1-q^{pn}) \sum_{n=\left\lceil \frac{1}{p}(s_p - m)\right\rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn+m-s_p)\, \zeta_p^{kd}\right) q^n,$$

where $s_p = \frac{1}{24}(p^2-1)$, and $q = \exp(2\pi i z)$.

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where $s_p = \frac{1}{24}(p^2 - 1)$, and $q = \exp(2\pi i z)$.

(ii) For $\left(\frac{-24m}{p}\right) = 1$ define

$$\mathcal{K}_{p,m}(\zeta_p^d; z) := q^{m/p} \prod_{n=1}^{\infty}(1 - q^{pn})$$

$$\times \left(\sum_{n=\left\lceil \frac{1}{p}(s_p - m) \right\rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k, p, pn + m - s_p)\, \zeta_p^{kd} \right) q^n \right.$$

$$\left. - 4\chi_{12}(p)\,(-1)^{a+d+1} \sin\left(\frac{d\pi}{p}\right) \sin\left(\frac{6ad\pi}{p}\right) q^{\frac{1}{p}(\frac{a}{2}(p-3a)-m)}\, \Phi_{p,a}(q) \right),$$

where $1 \leq a \leq \frac{1}{2}(p-1)$ has been chosen so that

$$-24m \equiv (6a)^2 \pmod{p}.$$

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where $1 \leq a \leq \frac{1}{2}(p-1)$ has been chosen so that

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Theorem (G.)

Suppose $p > 3$ is prime, and $0 \leq m \leq p-1$. Then

(i) *$\mathcal{K}_{p,0}(\zeta_p, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_1(p)$.*

(ii) *For $1 \leq m \leq (p-1)$, $\mathcal{K}_{p,m}(\zeta_p, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(p)$, and*

$$\mathcal{K}_{p,m}(\zeta_p, z) \mid [A]_1 = \exp\left(\frac{2\pi i b m}{p}\right) \mathcal{K}_{p,m}(\zeta_p, z)$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$.

DEFINITION The weight k stroke operator is defined as

$$F(z)\,|\,[A]_k := (det(A))^{k/2}\,(cz+d)^{-k}\,F\,(A\,z)\,,$$

$$\text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Z}).$$

In joint work with Rishabh Sarma, we have extended this theorem to $\Gamma_0(p)$.

Theorem (G. and Sarma)

Suppose $p > 3$ is prime, $0 \le m \le p-1$, and $1 \le d \le p-1$. Then

$$\mathcal{K}_{p,m}(\zeta_p, z) \mid [A]_1 = \frac{\sin(\pi/p)}{\sin(d\pi/p)} (-1)^{d+1} \exp\left(\frac{2\pi i m a b}{p}\right) \mathcal{K}_{p,\overline{ma^2}}(\zeta_p^d, z),$$

if $A = \begin{pmatrix} a & b \\ p & d \end{pmatrix} \in \Gamma_0(p)$. *Here* $\overline{ma^2}$ *means reduction of* $ma^2 \pmod{p}$.

PROBLEM Find identities for $\mathcal{K}_{p,m}(\zeta_p, z)$ in terms of generalized eta-functions. This means we want weight one functions on $\Gamma(p)$ that satisfy the transformation property in Theorem (ii) (p.6) for $A \in \Gamma_1(p)$. As building blocks, we use the functions $\eta(pz)$ and Biagioli's

$$f_{p,r}(z) = \eta(pz)\eta_{p,r}(z) = (-1)^{\lfloor r/p \rfloor} q^{(p-2r)^2/8p}(q^r;q^p)_\infty(q^{p-r};q^p)_\infty(q^p;q^p$$

for $p, r \in \mathbb{Z}, p \geq 1$ and $p \nmid r$. These functions satisfy

$$f_{p,r}(z) = f_{p,p+r}(z) = f_{p,-r}(z)$$

and

$$f_{p,r}(Az) = (-1)^{rb+\lfloor ra/p \rfloor + \lfloor r/p \rfloor} \exp\left(\frac{\pi i ab}{p} r^2\right) \nu_{\theta_1}({}^pA) \sqrt{cz+d}\, f_{p,ra}(z)$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p), {}^pA = \begin{pmatrix} a & p\,b \\ c/p & d \end{pmatrix},$$

and $\nu_{\theta_1} = \nu_\eta^3$ where ν_η is the eta-multiplier.

Building Blocks of Generalized Eta-Products

For $\overrightarrow{n} = (n_0, n_1, n_2, \cdots, n_{\frac{p-1}{2}}) \in \mathbb{Z}^{\frac{1}{2}(p+1)}$, we define

$$j(p, \overrightarrow{n}, z) = \eta(pz)^{n_0} \prod_{k=1}^{\frac{p-1}{2}} f_{p,k}(z)^{n_k}.$$

Theorem

Let $p > 3$ be prime, $0 \leq m \leq p-1$. Then $j(p, \overrightarrow{n}, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(p)$ and satisfies

$$j(p, \overrightarrow{n}, z) \mid [A]_1 = \exp(\tfrac{2\pi ibm}{p}) j(p, \overrightarrow{n}, z)$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p)$ provided the following conditions are met :

$$(i) \qquad n_0 + \sum_{k=1}^{\frac{p-1}{2}} n_k = 2,$$

$$(ii) \qquad n_0 + 3\sum_{k=1}^{\frac{p-1}{2}} n_k \equiv 0 \pmod{24},$$

$$(iii) \qquad \sum_{k=1}^{\frac{p-1}{2}} k^2 n_k \equiv 2m \pmod{p}.$$

DEFINITION Let $\mathcal{F}(m,p)$ be the set of functions $j(p,\overrightarrow{n},z)$ satisfying the conditions in this theorem.

We need to know how $\Gamma_0(p)$ acts on the functions in $\mathcal{F}(m,p)$.

Theorem

Let $p>3$ be prime, $0\le m\le p-1$. Suppose $j(p,\overrightarrow{n},z)\in\mathcal{F}(m,p)$ and $1\le a\le \frac{p-1}{2}$. Choose $1\le d\le p-1$ so that $A=\begin{pmatrix} a & b\\ p & d\end{pmatrix}\in\Gamma_0(p)$. Then,

$$j(p,\overrightarrow{n},z)\mid[A]_1$$

$$=(-1)^{L(\overrightarrow{n},a,b,p)}\exp\left(\frac{2\pi iabm}{p}\right)\eta(pz)^{n_0}\prod_{k=1}^{\frac{p-1}{2}} f_{p,ka}(z)^{n_k},$$

where $L(\overrightarrow{n}, a, b, p) = a(b+1) \sum_{k=1}^{\frac{p-1}{2}} k n_k + \sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{ka}{p} \right\rfloor n_k.$

A PERMUTATION Let $p > 3$ be prime and $\overrightarrow{n} = (n_0, n_1, n_2, \cdots, n_{\frac{1}{2}(p-1)}) \in \mathbb{Z}^{\frac{p+1}{2}}$, and let $1 \leq a \leq \frac{1}{2}(p-1)$. Let π_a be the permutation $\pi_a : [\frac{1}{2}(p-1)] \to [\frac{1}{2}(p-1)]$, where $[\frac{1}{2}(p-1)] = \{1, 2, \cdots, \frac{1}{2}(p-1)\}$ defined by $\pi_a(i) = i'$ where $ai' \equiv \pm i \pmod{p}$. π_a induces a permutation on $\mathbb{Z}^{\frac{1}{2}(p+1)}$:

$$\pi_a(\overrightarrow{n}) = \left(n_0, n_{\pi_a(1)}, n_{\pi_a(2)}, \cdots, n_{\pi_a(\frac{1}{2}(p-1))}\right),$$

so that

$$j(p, \pi_a(\overrightarrow{n}), z) = \eta(pz)^{n_0} \prod_{k=1}^{\frac{p-1}{2}} f_{p,ak}(z)^{n_k}.$$

A previous theorem leads to symmetry for $\mathcal{K}_{p,m}(\zeta_p, z)$ functions.

Theorem

Let $p > 3$ be prime and $0 \le m \le p-1$. Suppose there is a set $\mathcal{B}$ of $\overrightarrow{n}$−vectors such that $\{j(p, \overrightarrow{n}, z) : \overrightarrow{n} \in \mathcal{B}\}$ is linearly independent (over $\mathbb{Q}$) and

$$\mathcal{K}_{p,m}(\zeta_p, z) = \sum_{\overrightarrow{n} \in \mathcal{B}} c(\overrightarrow{n}, \zeta_p)\, j(p, \overrightarrow{n}, z),$$

where the $c(\overrightarrow{n}, \zeta_p) \in \mathbb{Q}[\zeta_p]$. Then for $1 \le a \le \frac{p-1}{2}$, we have

$$\mathcal{K}_{p,\overline{ma^2}}(\zeta_p, z) = \sum_{\overrightarrow{n} \in \mathcal{B}} (-1)^{L(\overrightarrow{n},a,b,p)+a+d} \frac{\sin(\pi/p)}{\sin(a\pi/p)}\, c(\overrightarrow{n}, \zeta_p^a)\, j(p, \pi_a(\overrightarrow{n}), z)$$

where $1 \le d \le p-1$ and $ad \equiv 1 \pmod{p}$.

Examples (1) $p = 5$. From Ramanujan's 5-dissection of $R(\zeta_5, q)$ we have

$$\mathcal{K}_{5,0}(\zeta_5, z) = 0,$$

$$\mathcal{K}_{5,1}(\zeta_5, z) = \frac{\eta(5z)^2\,\eta_{5,2}(z)}{\eta_{5,1}(z)^2} = j(5, [3, -2, 1], z),$$

$$\mathcal{K}_{5,2}(\zeta_5, z) = \frac{\eta(5z)^2}{\eta_{5,1}(z)} = j(5, [3, -1, 0], z),$$

$$\mathcal{K}_{5,3}(\zeta_5, z) = (\zeta_5 + \zeta_5^4)\frac{\eta(5z)^2}{\eta_{5,2}(z)} = (\zeta_5 + \zeta_5^4)\,j(5, [3, 0, -1], z),$$

$$\mathcal{K}_{5,4}(\zeta_5, z) = -(\zeta_5 + \zeta_5^4)\,\frac{\eta(5z)^2\,\eta_{5,1}(z)}{\eta_{5,2}(z)^2} = -(\zeta_5 + \zeta_5^4)\,j(5, [3, 1, -2], z).$$

Let $\overrightarrow{n} = [3, -2, 1], p = 5, a = 2, b = 1, d = 3$. Now,

$$L(\overrightarrow{n}, a, b, p) = 4,$$
$$\frac{sin(\frac{\pi}{p})}{sin(\frac{a\pi}{p})} = \frac{sin(\frac{\pi}{5})}{sin(\frac{2\pi}{5})} = \zeta_5 + \zeta_5^4,$$
$$\pi_a(\overrightarrow{n}) = \pi_2[3, -2, 1] = [3, 1, -2].$$

Since

$$\mathcal{K}_{5,1}(\zeta_5, z) = j(5, [3, -2, 1], z),$$

we have

$$\mathcal{K}_{5,4}(\zeta_5, z) = (-1)^{L(\overrightarrow{n},a,b,p)+a+d} \frac{\sin(\pi/p)}{\sin(a\pi/p)} c(\overrightarrow{n}, \zeta_p^a) j(p, \pi_a(\overrightarrow{n}), z)$$
$$= -(\zeta_5 + \zeta_5^4) j(5, [3, 1, -2], z),$$

which confirms that $\mathcal{K}_{5,4}(\zeta_5, z) = -(\zeta_5 + \zeta_5^4) \dfrac{\eta(5z)^2 \, \eta_{5,1}(z)}{\eta_{5,2}(z)^2}$.

(2) $p = 7$. In the Lost Notebook, Ramanujan was close to finding the 7-dissection of $R(\zeta_7, q)$.

$$R_7(z)$$

$$= q^{-\frac{1}{24}}\Big(R(\zeta_7, q) + (\zeta_7 + \zeta_7^6 - 2)\,\phi_{7,1}(q^7) + (-\zeta_7^2 + \zeta_7^3 + \zeta_7^4 - \zeta_7^5)\,q^{-1}\,\phi_{7,2}($$

$$+ (1 + 2\zeta_7^2 + \zeta_7^3 + \zeta_7^4 + 2\zeta_7^5)\,q^{-5}\,\phi_{7,3}(q^7)\Big)$$

$$= (-1 + \zeta_7 + \zeta_7^6)\frac{\eta(49z)\,\eta_{7,3}(7z)}{\eta_{7,1}(7z)\,\eta_{7,2}(7z)} + \frac{\eta(49z)}{\eta_{7,1}(7z)} + (\zeta_7 + \zeta_7^6)\,\frac{\eta(49z)\,\eta_{7,2}(7z)}{\eta_{7,1}(7z)\,\eta_{7,3}(7z)}$$

$$+ (1 + \zeta_7^2 + \zeta_7^5)\,\frac{\eta(49z)}{\eta_{7,2}(7z)} - (\zeta_7^2 + \zeta_7^5)\,\frac{\eta(49z)}{\eta_{7,3}(7z)} - (1 + \zeta_7^3 + \zeta_7^4)\,\frac{\eta(49z)\,\eta_{7,1}(7}{\eta_{7,2}(7z)\,\eta_{7,3}(}$$

This implies that

$$\mathcal{K}_{7,0}(\zeta_7, z) = 0,$$

$$\mathcal{K}_{7,1}(\zeta_7, z) = -(1 + \zeta_7^3 + \zeta_7^4)\, \frac{\eta(7z)^2\, \eta_{7,1}(z)}{\eta_{7,2}(z)\, \eta_{7,3}(z)},$$

$$\mathcal{K}_{7,2}(\zeta_7, z) = (-1 + \zeta_7 + \zeta_7^6)\frac{\eta(7z)^2\, \eta_{7,3}(z)}{\eta_{7,1}(z)\, \eta_{7,2}(z)},$$

$$\mathcal{K}_{7,3}(\zeta_7, z) = \frac{\eta(7z)^2}{\eta_{7,1}(z)},$$

$$\mathcal{K}_{7,4}(\zeta_7, z) = (\zeta_7 + \zeta_7^6)\, \frac{\eta(7z)^2\, \eta_{7,2}(z)}{\eta_{7,1}(z)\, \eta_{7,3}(z)},$$

$$\mathcal{K}_{7,5}(\zeta_7, z) = (1 + \zeta_7^2 + \zeta_7^5)\, \frac{\eta(7z)^2}{\eta_{7,2}(z)},$$

$$\mathcal{K}_{7,6}(\zeta_7, z) = -(\zeta_7^2 + \zeta_7^5)\, \frac{\eta(7z)^2}{\eta_{7,3}(z)}.$$

Let $\overrightarrow{n} = [3, 1, -1, -1], p = 7, a = 2, b = 1, d = 4,\ c(\overrightarrow{n}, \zeta_p) = -(1 + \zeta_7^3 + \zeta_7^4)$. Now,

$$L(\overrightarrow{n}, a, b, p) = -7,$$
$$\frac{sin(\frac{\pi}{p})}{sin(\frac{a\pi}{p})} = \frac{sin(\frac{\pi}{7})}{sin(\frac{2\pi}{7})} = 1 + \zeta_7^2 + \zeta_7^5,$$
$$c(\overrightarrow{n}, \zeta_p^a) = -(1 + \zeta_7^6 + \zeta_7).$$

Since

$$\mathcal{K}_{7,1}(\zeta_7, z) = -(1 + \zeta_7^3 + \zeta_7^4)\, j(7, [3, 1, -1, -1], z),$$

we have

$$\begin{aligned}\mathcal{K}_{7,4}(\zeta_7, z) &= (-1)^{L(\overrightarrow{n}, a, b, p) + a + d}\, \frac{\sin(\pi/p)}{\sin(a\pi/p)}\, c(\overrightarrow{n}, \zeta_p^a)\, j(p, \pi_a(\overrightarrow{n}), z) \\ &= (-1)^{-7+2+4}(1 + \zeta_7^2 + \zeta_7^5)(-1)(1 + \zeta_7 + \zeta_7^6) j(7, [3, -1, 1, - \\ &= (\zeta_7 + \zeta_7^6)\, j(7, [3, -1, 1, -1], z)\end{aligned}$$

Symmetry for $\mathcal{K}_{p,0}(\zeta_p, z)$

Theorem

Let $p > 3$ be prime. Suppose there are t vectors $\overrightarrow{n_1}, \overrightarrow{n_2}, \cdots, \overrightarrow{n_t} \in \mathbb{Z}^{\frac{1}{2}(p+1)}$ such that the set of functions $j(p, \pi_r(\overrightarrow{n_k}), z), 1 \le k \le t, 1 \le r \le \frac{1}{2}(p-1)$ are linearly independent (over $\mathbb{Q}$) and

$$\mathcal{K}_{p,0}(\zeta_p, z) = \sum_{k=1}^{t} \sum_{r=1}^{\frac{1}{2}(p-1)} c_{p,r,k}(\zeta_p) j(p, \pi_r(\overrightarrow{n_k}), z),$$

where $c_{p,r,k}(\zeta_p) \in \mathbb{Q}[\zeta_p]$. Then for $1 \le d \le \frac{p-1}{2}$, and $\overrightarrow{n} = (n_0, n_1, n_2, \cdots, n_{\frac{p-1}{2}})$, we have

$$c_{p,d,k}(\zeta_p) = \frac{\sin(\pi/p)}{\sin(d\pi/p)} (-1)^{d+1+L(\overrightarrow{n},d)} c_{p,1,k}(\zeta_p^d)$$

Example (p=11) Atkin and Hussain (1958) derived identities for rank mod 11 by the method of the earlier paper by Atkin and Swinnerton-Dyer (1953). It turns out that

$$\mathcal{K}_{11,0}(\zeta_{11}, z) = (q^{11}; q^{11})_\infty \sum_{n=1}^{\infty} \left(\sum_{k=0}^{10} N(k, 11, 11n-5)\, \zeta_{11}^{k} \right) q^n$$

$$= \sum_{r=1}^{5} c_{11,r}(\zeta_{11})\, j(11, \pi_r(\overrightarrow{n}), z),$$

where $\overrightarrow{n} = [5, 0, 0, 0, -1, -2]$, and

$$c_{11,1} = 1 + 2\,({\zeta_{11}}^2 + {\zeta_{11}}^9) + 2\,({\zeta_{11}}^3 + {\zeta_{11}}^8) + ({\zeta_{11}}^4 + {\zeta_{11}}^7)\,,$$

$$c_{11,2} = -(1 + ({\zeta_{11}}^2 + {\zeta_{11}}^9) + ({\zeta_{11}}^3 + {\zeta_{11}}^8) + 2\,({\zeta_{11}}^4 + {\zeta_{11}}^7) + ({\zeta_{11}}^5 + \zeta_{1}$$

$$c_{11,3} = 3 + 2\,({\zeta_{11}}^3 + {\zeta_{11}}^8) + 2\,({\zeta_{11}}^4 + {\zeta_{11}}^7),$$

$$c_{11,4} = 4 + 2\,({\zeta_{11}}^2 + {\zeta_{11}}^9) + ({\zeta_{11}}^3 + {\zeta_{11}}^8) + 2\,({\zeta_{11}}^4 + {\zeta_{11}}^7) + 2\,({\zeta_{11}}^5 + \zeta$$

$$c_{11,5} = -(3 + ({\zeta_{11}}^2 + {\zeta_{11}}^9) + 2\,({\zeta_{11}}^3 + {\zeta_{11}}^8) - ({\zeta_{11}}^4 + {\zeta_{11}}^7) + 2\,({\zeta_{11}}^5 +$$

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$$= \sum_{r=1}^{5} c_{11,r}(\zeta_{11})\, j(11, \pi_r(\overrightarrow{n}), z),$$

where $\overrightarrow{n} = [5, 0, 0, 0, -1, -2]$, and

$$c_{11,1} = 1 + 2\,\left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + 2\left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + \left({\zeta_{11}}^4 + {\zeta_{11}}^7\right),$$

$$c_{11,2} = -(1 + \left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + \left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + 2\left({\zeta_{11}}^4 + {\zeta_{11}}^7\right) + \left({\zeta_{11}}^5 + \zeta_1$$

$$c_{11,3} = 3 + 2\left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + 2\left({\zeta_{11}}^4 + {\zeta_{11}}^7\right),$$

$$c_{11,4} = 4 + 2\left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + \left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + 2\left({\zeta_{11}}^4 + {\zeta_{11}}^7\right) + 2\left({\zeta_{11}}^5 + \zeta\right.$$

$$c_{11,5} = -(3 + \left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + 2\left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) - \left({\zeta_{11}}^4 + {\zeta_{11}}^7\right) + 2\left({\zeta_{11}}^5 +\right.$$

We find that

$$c_{11,1} = 1 + 2\left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + 2\left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + \left({\zeta_{11}}^4 + {\zeta_{11}}^7\right),$$

$$c_{11,2} = -\frac{\sin(\pi/11)}{\sin(2\pi/11)}(1 + 2\left({\zeta_{11}}^4 + {\zeta_{11}}^7\right) + 2\left({\zeta_{11}}^5 + {\zeta_{11}}^6\right) + \left({\zeta_{11}}^3 + {\zeta_{11}}^8\right)$$

$$c_{11,3} = -\frac{\sin(\pi/11)}{\sin(3\pi/11)}(1 + 2\left({\zeta_{11}}^5 + {\zeta_{11}}^6\right) + 2\left({\zeta_{11}}^2 + {\zeta_{11}}^9\right) + \left(\zeta_{11} + {\zeta_{11}}^{10}\right)$$

$$c_{11,4} = \frac{\sin(\pi/11)}{\sin(4\pi/11)}(1 + 2\left({\zeta_{11}}^3 + {\zeta_{11}}^8\right) + 2\left({\zeta_{11}}^{10} + \zeta_{11}\right) + \left({\zeta_{11}}^6 + {\zeta_{11}}^5\right)),$$

$$c_{11,5} = -\frac{\sin(\pi/11)}{\sin(5\pi/11)}(1 + 2\left(\zeta_{11} + {\zeta_{11}}^{10}\right) + 2\left({\zeta_{11}}^7 + {\zeta_{11}}^4\right) + \left({\zeta_{11}}^2 + {\zeta_{11}}^9\right)$$

SKETCH OF THE PROOF OF MAIN SYMMETRY RESULT THEOREM

We need to define the functions $\mathcal{F}_1\left(\frac{\ell}{p}; z\right)$ and $\mathcal{F}_2\left(\frac{\ell}{p}; z\right)$. The definitions are quite technical and rely on results of Bringmann and Ono (2010) and Garvan (2019). Assume $0 < a < c$ are integers, $q = \exp(2\pi i z)$.

$$M\left(\frac{a}{c}; z\right) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1 - q^{n+\frac{a}{c}}} q^{\frac{3}{2}n(n+1)},$$

$$N\left(\frac{a}{c}; z\right) := \frac{1}{(q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)\left(2 - 2\cos\left(\frac{2\pi a}{c}\right)\right)}{1 - 2\cos\left(\frac{2\pi a}{c}\right) q^n + q^{2n}} q^{\frac{1}{2}n(3n+1}$$

We note that

$$R(\zeta_c^a, q) = N\left(\frac{a}{c}; z\right)$$

and

$$\sum_{n=0}^{\infty} \frac{q^{cn^2}}{(q^a; a^c)_{n+1}(q^{c-a}; q^c)_n} = 1 + q^a\, M\left(\frac{a}{c}; cz\right).$$

We define

$$\mathcal{N}\left(\frac{a}{p};c\right)z := \csc\left(\frac{a\pi}{c}\right)\,q^{-\frac{1}{24}}\,N\left(\frac{a}{c};z\right),$$

$$\mathcal{M}\left(\frac{a}{p};c\right)z := 2q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}}\,M\left(\frac{a}{c};z\right).$$

We also need various weight 3/2 theta functions. For $0 \le k < N$ define

$$\widetilde{\theta}(k,N;z) := \sum_{m=-\infty}^{\infty}(Nm+k)\exp\left(\frac{\pi iz}{N}(Nm+k)^2\right),$$

$$\Theta_1(a,b,c;z) := \zeta_{c^2}^{3ab}\,\zeta_{2c}^{-a}\sum_{m=0}^{6c-1}(-1)^m \sin\left(\frac{\pi}{3}(2m+1)\right)\exp\left(\frac{-2\pi ima}{c}\right)\widetilde{\theta}($$

and

$$\Theta_2(a,b,c;z) := \sum_{\ell=0}^{2c-1}\left((-1)^\ell \exp\left(\frac{-\pi ib}{c}(6\ell+1)\right)\widetilde{\theta}(6c\ell+6a+c,12c^2;\right.$$

$$+(-1)^\ell \exp\left(\frac{-\pi ib}{c}(6\ell-1)\right)\widetilde{\theta}(6c\ell+6a-c,12c$$

for $0 \le a, b < c$. Also

$$\Theta_1\left(\frac{a}{c};z\right) = -\frac{i}{2c}\,\Theta_2(0,-a,c;z) = \sum_{n=-\infty}^{\infty}(-1)^n(6n+1)\sin\left(\frac{\pi a(6n+1)}{c}\right.$$

We also need some period integrals of theta functions :

$$T_1\left(\frac{a}{c};z\right) := -\frac{i}{\sqrt{3}}\int_{-\overline{z}}^{i\infty}\frac{\Theta_1\left(\frac{a}{c};\tau\right)}{\sqrt{-i(\tau+z)}}\,d\tau,$$

$$T_2\left(\frac{a}{c};z\right) := \frac{i}{3c}\int_{-\overline{z}}^{i\infty}\frac{\Theta_1(0,-a,c;\tau)}{\sqrt{-i(\tau+z)}}\,d\tau,$$

We define the following Harmonic Maass forms of weight $1/2$:

$$\mathcal{G}_1\left(\frac{a}{c};z\right) := \mathcal{N}\left(\frac{a}{p};c\right)z - T_1\left(\frac{a}{c};z\right),$$
$$\mathcal{G}_2\left(\frac{a}{c};z\right) := \mathcal{M}\left(\frac{a}{p};c\right)z + \varepsilon_2\left(\frac{a}{c};z\right) - T_2\left(\frac{a}{c};z\right)$$

where

$$\varepsilon_2\left(\frac{a}{c};z\right) := \begin{cases} 2\exp\left(-3\pi iz\left(\frac{a}{c}-\frac{1}{6}\right)^2\right) & \text{if } 0 < \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ 2\exp\left(-3\pi iz\left(\frac{a}{c}-\frac{5}{6}\right)^2\right) & \text{if } \frac{5}{6} < \frac{a}{c} < 1. \end{cases}$$

Finally, we define

$$\mathcal{F}_1\left(\frac{\ell}{p}; z\right) := \eta(z)\,\mathcal{G}_1\left(\frac{\ell}{p}; z\right),$$
$$\mathcal{F}_2\left(\frac{\ell}{p}; z\right) := \eta(z)\,\mathcal{G}_2\left(\frac{\ell}{p}; z\right)$$

where $p > 3$ is a prime and $1 \leq \ell < p$. Note : The term $\varepsilon_2\left(\frac{a}{c}; z\right)$ is missing from Bringmann and Ono's paper. Let

$$\mathcal{J}\left(\frac{d}{p}; z\right) = \eta(p^2 z)\,\left(\mathcal{N}\left(\frac{d}{p}; p\right) z - \right.$$
$$\left. 2\,\chi_{12}(p) \sum_{\ell=1}^{\frac{1}{2}(p-1)} (-1)^{\ell+d+1} \sin\left(\frac{6d\ell\pi}{p}\right)\,\left(\mathcal{M}\left(\frac{\ell}{p}; p\right) p^2 z + \varepsilon_2\left(\frac{\ell}{p}; p^2 z\right)\right)\right)$$

Then

$$\eta(p^2 z)\,\mathcal{R}_p\left(\zeta_p^d, z\right) = \sin\left(\frac{d\pi}{p}\right)\,\mathcal{J}\left(\frac{d}{p}; z\right),$$

and

Also defining

$$\mathcal{J}^*\left(\frac{d}{p};z\right) = \frac{\eta(p^2z)}{\eta(z)}\,\mathcal{F}_1\left(\frac{d}{p};z\right) - 2\left(\frac{12}{p}\right)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{k+d+1}\sin\left(\frac{6kd\pi}{p}\right.$$

we get

$$\mathcal{J}\left(\frac{d}{p};z\right) = \mathcal{J}^*\left(\frac{d}{p};z\right),$$

from the identity

$$T_1\left(\frac{d}{p};z\right) = 2\left(\frac{12}{p}\right)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{k+d+1}\sin\left(\frac{6kda\pi}{p}\right)\,T_2\left(\frac{k}{p};p^2z\right)$$

which comes from a corresponding identity of the theta functions in the integrals involved.

By studying the action of $U_{p,m}$ and $A = \begin{pmatrix} a & k \\ p & d \end{pmatrix} \in \Gamma_0(p)$ on $\mathcal{F}_1\left(\frac{d}{p}; z\right)$ and $\mathcal{F}_2\left(\frac{k}{p}; p^2 z\right)$, we find that

$$\mathcal{J}^*\left(\frac{d}{p}; z\right) \mid [U_{p,m}]_1 \mid [A]_1 = (-1)^{d+1} \zeta_p^{mak} \mathcal{J}^*\left(\frac{d}{p}; z\right) \mid \left[U_{p,ma^2}\right]_1$$

which completes the proof.

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