NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
May 16-20, 2022
The University of Texas Rio Grande Valley

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May 19, 2022

LECTURE 8
ZAGIER'S HIGHER ORDER MOCK THETA FUNCTIONS
(Includes notes of Jonathan Bradley-Thrush)

THE MOCK THETA CONJECTURES

- SEVENTH ORDER MOCK THETA FUNCTIONS
- SEVENTH ORDER MOCK THETA “CONJECTURES”
- RANK MOD 7

ZWEGERS

ZAGIER'S 11TH ORDER MOCK THETA FUNCTIONS

- AN 11TH ORDER MOCK THETA CONJECTURE

HIGHER ORDER MOCK THETA FUNCTIONS AND HOLOMORPHIC PROJECTION

- HECKE-ROGERS SERIES OF WEIGHT 2

THE MOCK THETA CONJECTURES

RAMANUJAN, ANDREWS and G. (1989)

THE FIRST MOCK THETA CONJECTURE

$$N(1,5,5n) = N(0,5,5n) + \rho_0(n),$$

where $\rho_0(n)$ is the number of partitions of n with unique smallest part and other parts $\leq$ double the smallest part.

EXAMPLE $N(1,5,25) = 393$, $N(0,5,25) = 390$, $\rho_0(5) = 3$ since the relevant partitions are 5, $3+2$, $2+2+1$.

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THE SECOND MOCK THETA CONJECTURE

$$2N(2,5,5n+3) = N(1,5,5n+3) + N(0,5,5n+3) + \rho_1(n),$$

where $\rho_1(n)$ is the number of partitions of n with unique smallest part and other parts $\leq$ one plus double the smallest part.

$$\sum_{n=1}^{\infty} \rho(n) q^n = \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n}$$

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These identities and conjectures are related to a page from Ramanujan's Lost Notebook. We find that

$$\sum_{n=1}^{\infty} \rho(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}$$
$$= 3\Phi(q) + 1 - A(q),$$

where the functions $\Phi(q)$, $A(q)$ occur in Ramanujan's 5-dissection of $R(\zeta_5, q)$.

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THE FIRST CLUE = ANDREWS FIFTH ORDER HECKE-ROGERS IDENTITY

$$f_0(q) = \sum_{n=0} \frac{q^{n^2}}{(-q;q)_n}$$
$$= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{n(5n+1)/2-j^2} (1 - q^{4n+2}).$$

HICKERSON's (1988) incredible proof

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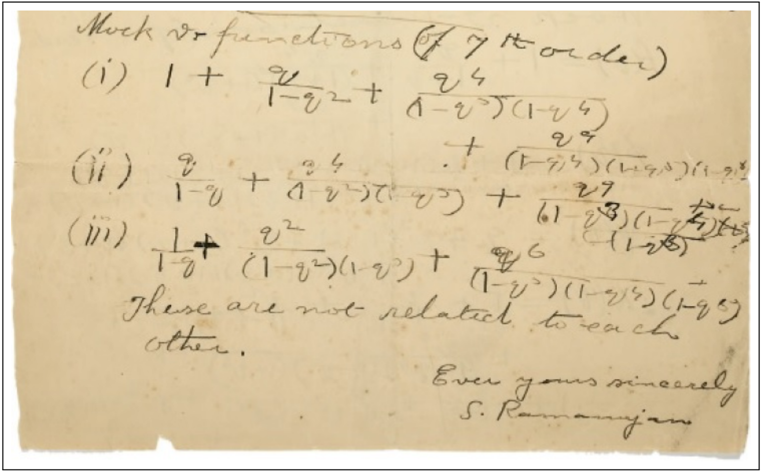
SEVENTH ORDER FUNCTIONS

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}$$
$$= 1 + q + q^3 + q^4 + q^5 + 2q^7 + q^8 + 2q^9 + \cdots ,$$
$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n}$$
$$= q + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + 2q^7 + 2q^8 + 3q^9 + \cdots ,$$
$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}$$
$$= 1 + q + 2q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 2q^7 + 3q^8 + 3q^9 + \cdots .$$



Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

SEVENTH ORDER MOCK THETA "CONJECTURES"

$$\mathcal{F}_0(q) - 2 = 2\,q\,g(q,q^7) - \frac{j(q^3,q^7)^2}{(q;q)_\infty}$$

$$\mathcal{F}_1(q) = 2\,q^2\,g(q^2,q^7) + q\,\frac{j(q,q^7)^2}{(q;q)_\infty}$$

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where

$$z\,g(z,q) = -1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(z;q)_{n+1}(z^{-1}q;q)_n} = -1 + \frac{1}{(1-z)}\,R(z,q),$$

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HICKERSON (1988)

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FIRST 7TH MOCK THETA CONJECTURE REPHRASED

Let $N(r,t,n)$ denote the number of partitions of n with rank congruent to r mod t. Let $\gamma_0(n)$ denote the number of partitions of n where the number of smallest parts equals the size of the smallest part and all other parts $\leq$ the double of the smallest part. Then

$$N(0,7,7n) = N(2,7,7n) + \gamma_0(n)$$

EXAMPLE

$N(0,7,49) = 24791$, $N(2,7,49) = 24789$, and $\gamma_0(7) = 2$ with the relevant partitions being $2+2+2+1$, $3+2+2$.

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Alternative proofs of the Seventh Order Mock Theta Conjectures have been given by ANDERSEN (2016) using Zweger's results, the theory of harmonic Maass forms and some results from LECTURE 6. The following theorem of Zwegers was crucial:

Theorem (ZWEGERS (2002))

Define

$$M_7(\tau) = \begin{pmatrix} q^{-1/168}\,\mathcal{F}_0(q) \\ -q^{-25/168}\,\mathcal{F}_1(q) \\ q^{47/168}\,\mathcal{F}_2(q) \end{pmatrix}, \quad R_7(\tau) = \begin{pmatrix} R_{7,1}(\tau) \\ R_{7,2}(\tau) \\ R_{7,3}(\tau) \end{pmatrix}$$

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where

$$R_{p,j}(\tau) = \sum_{n \equiv j \pmod p} \left(\frac{12}{n}\right) \mathrm{sgn}(n) \beta\left(\frac{n^2 y}{6p}\right) q^{-n^2/(24p)},$$

$$\beta(x) = \int_x^\infty u^{-1/2} e^{-\pi u}\, du, \quad q = \exp(2\pi i \tau), \quad \tau = x + iy$$

The function

$$\widehat{M_7}(\tau) = M_7(\tau) + R_7(\tau)$$

satisfies

$$\widehat{M_7}(\tau+1) = \begin{pmatrix} \zeta_{168}^{-1} & 0 & 0 \\ 0 & \zeta_{168}^{-25} & 0 \\ 0 & 0 & \zeta_{168}^{-121} \end{pmatrix} \widehat{M_7}(\tau),$$

and

$$\widehat{M_7}(-1/\tau) = \sqrt{\tau/7i} \begin{pmatrix} 2\sin\frac{\pi}{7} & -2\sin\frac{2\pi}{7} & 2\sin\frac{3\pi}{7} \\ -2\sin\frac{2\pi}{7} & -2\sin\frac{3\pi}{7} & -2\sin\frac{\pi}{7} \\ 2\sin\frac{3\pi}{7} & -2\sin\frac{\pi}{7} & -2\sin\frac{2\pi}{7} \end{pmatrix} \widehat{M_7}(\tau)$$

NOTE: This reformulation of Zweger's result is due to Zagier.

and

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Séminaire BOURBAKI Novembre 2007
60ème année, 2006-2007, n° 986

RAMANUJAN'S MOCK THETA FUNCTIONS
AND THEIR APPLICATIONS
[d'après ZWEGERS and BRINGMANN–ONO]

by **Don ZAGIER**

INTRODUCTION

One of the most romantic stories in the history of mathematics is that of the friendship between Hardy and Ramanujan. It began and ended with two famous letters. The first, sent by Ramanujan to Hardy in 1913, presents its author as a penniless clerk in a Madras shipping office who has made some discoveries that "are termed by the local mathematicians as 'startling'." Hardy spent the night with Littlewood convincing himself that the letter was the work of a genius and not of a fraud and promptly invited Ramanujan to come to England for what was to become one of the most famous mathematical collaborations in history. The other letter was sent in 1920, also by Ramanujan to Hardy, just three months before his death at the age of 32 in India, to which he had returned after five years in England. Here he recovers briefly from his illness and depression to tell Hardy excitedly about a new class of functions that he has discovered and that he calls "mock theta functions."

Zagier also stated an analog of $\widehat{M}_7(\tau)$ for the fifth order functions $\chi_0(q)$, $\chi_1(q)$.

Theorem (ZAGIER (2007))

Define

$$M_5(\tau) = -\frac{2}{3}\begin{pmatrix} q^{-1/120}(2-\chi_0(q)) \\ q^{71/120}\chi_1(q) \end{pmatrix}, \quad R_5(\tau) = \begin{pmatrix} R_{5,1}(\tau) \\ R_{5,2}(\tau) \end{pmatrix}$$

where

$$R_{p,j}(\tau) = \sum_{n\equiv j \pmod p} \left(\frac{12}{n}\right)\operatorname{sgn}(n)\beta(\frac{n^2 y}{6p})q^{-n^2/(24p)},$$

$$\beta(x) = \int_x^\infty u^{-1/2}e^{-\pi u}\,du,$$

$$q = \exp(2\pi i\tau), \quad \tau = x + iy$$

The function

$$\widehat{M_5}(\tau) = M_5(\tau) + R_5(\tau)$$

satisfies

$$\widehat{M_5}(\tau+1) = \begin{pmatrix} \zeta_{120}^{-1} & 0 \\ 0 & \zeta_{120}^{-49} \end{pmatrix} \widehat{M_5}(\tau),$$

and

$$\widehat{M_5}(-1/\tau) = \sqrt{\tau/5i} \begin{pmatrix} -2\sin\frac{\pi}{5} & 2\sin\frac{2\pi}{5} \\ 2\sin\frac{2\pi}{5} & 2\sin\frac{\pi}{5} \end{pmatrix} \widehat{M_5}(\tau)$$

Similarly, for the mock theta functions of order 7, as well as Hickerson's identity for $\eta(\tau)M_{7,j}(\tau)$ as an indefinite binary theta series, we find the representation

$$\eta(7\tau)\,M_{7,j}(\tau) \;=\; \sum_{\substack{|r|>|s|,\; rs>0\\ 2r\equiv -2s\equiv j \pmod 7}} \operatorname{sgn}(r)\,\bigl(2\varepsilon_6(s)-\varepsilon_2(r)\varepsilon_3(s)-\varepsilon_3(r)\varepsilon_2(s)\bigr)\,q^{rs/42}$$

(where $\varepsilon_N(s)=1$ if $s\equiv 0 \pmod N$ and 0 otherwise) of the product of $M_{7,j}(\tau)$ with $\eta(7\tau)$ as a "mock Eisenstein series" of weight 1 (explaining the smallness of the Fourier coefficients of this product that was mentioned in §1), and also the representation

$$\eta(\tau)^3\,M_{7,j}(\tau) \;=\; \sum_{\substack{m>2|n|/9\\ n\equiv j \pmod 7}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\Bigl(m\operatorname{sgn}(n)\,-\,\frac{3n}{14}\Bigr)\,q^{m^2/8-n^2/168}$$

of the product of $M_{7,j}(\tau)$ with $\eta(\tau)^3$ as an indefinite theta series of weight 2. What's

EXAMPLE The first component gives

$$\mathcal{F}_0(q) = \frac{1}{(q)_\infty^3} \sum_{m=1}^{\infty} \sum_{\substack{n=-\lfloor 9m/2 \rfloor \\ n \equiv 1 \pmod 7}}^{\lfloor 9m/2 \rfloor} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{3n}{14}\right) \times q^{m^2/8 - n^2/168 - 5/42}$$

ZAGIER indicated there should be analogs of these two theorem for all primes $p > 3$.

of the product of $M_{7,j}(\tau)$ with $\eta(\tau)^3$ as an indefinite theta series of weight 2. What's more, by methods obtained in a reasonably straightforward way by generalizing methods from standard modular form theory (holomorphic projection, Rankin–Cohen brackets, etc.), one can produce infinitely many new examples of mock theta functions or of more general types of mock modular forms. In particular, we can construct vector-valued mock modular forms $M_p(\tau) = \big(M_{p,j}(\tau) = -M_{p,-j}(\tau)\big)_{j \pmod p}$ of length $(p-1)/2$ of order $p > 3$ for any prime p by a formula like the one just given for M_7, e.g.,

$$M_{11,j}(\tau) = \frac{1}{\eta(\tau)^3} \sum_{\substack{m>2|n|/11 \\ n\equiv j \pmod{11}}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{n}{6}\right) q^{m^2/8 - n^2/264}$$

for $p = 11$, in such a way that the completed function $\widehat{M}_p(\tau) = \big(M_{p,j}(\tau)\big)_{j \pmod p}$ with $\widehat{M}_{p,j}(\tau) = M_{p,j}(\tau) + R_{p,j}(\tau)$ transforms like a vector-valued modular form of weight $\frac{1}{2}$ on $\mathrm{SL}(2,\mathbb{Z})$, thus directly generalizing the previous two cases $p = 5$ and $p = 7$.

Conjecture (AN 11TH ORDER MOCK THETA CONJECTURE)

Define $\mu_{11,1}$ *and* $\phi_{11,2}(n)$ *by*

$$\sum_{n=0}^{\infty} \mu_{11,1}(n) q^n = \frac{1}{(q)_\infty^3} \sum_{\substack{2|n|/11<m \\ n\equiv 1 \pmod{11}}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{n}{6}\right) \times q^{\frac{1}{8}m^2 - \frac{1}{264}n^2 - 4/33}$$

and

$$\sum_{n=0}^{\infty} \phi_{11,2}(n) q^n = \sum_{n=0}^{\infty} \frac{q^{11n^2}}{(q^2;q^{11})_{n+1}(q^9;q^{11})_n}$$

Then

$$\mu_{11,1}(n) = \frac{1}{12}\left(11\,N(0,11,11n) - p(11n) - 2\,\phi_{11,2}(n)\right)$$

EXAMPLE $11\,N(0,11,22) = 1034$, $p(22) = 1002$, $2\phi_{11,2}(0) = 2$, and $\mu_{11,1}(2) = \frac{5}{2}$.

$$\sum_{n=0}^{\infty} \mu_{11,1}(n) q^n = \frac{5}{6} + \frac{5}{6}q + \frac{5}{2}q^2 + 10q^3 + \frac{125}{6}q^4 + \frac{313}{6}q^5 + \frac{310}{3}q^6 + \cdots$$

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HIGHER ORDER MOCK THETA FUNCTIONS AND HOLOMORPHIC PROJECTION

We explain how Zagier's higher order mock theta functions are constructed using holomorphic projection. In LECTURE 6 we defined the harmonic Maass forms $\mathcal{G}_1$, $\mathcal{G}_2$ related to the Dyson rank function $R(\zeta_p, q)$.

For $p \geq 5$ prime and $1 \leq a \leq \frac{1}{2}(p-1)$ define

$$\mathcal{G}_1\left(\frac{a}{p}; z\right) = \csc\left(\frac{a\pi}{p}\right) q^{-\frac{1}{24}} R(\zeta_p^a, q) + \frac{i}{\sqrt{3}} \int_{-\overline{z}}^{i\infty} \frac{\Theta_1\left(\frac{a}{p}; \tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau,$$

$$\mathcal{G}_2\left(\frac{a}{p}; z\right) = 2\,\Phi_{p,a}(z/p) - \frac{i}{3p} \int_{-\overline{z}}^{i\infty} \frac{\Theta_1(0, -a, p; \tau)}{\sqrt{-i(\tau+z)}}\, d\tau,$$

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$$- \frac{i}{3p} \int_{-\overline{z}}^{i\infty} \frac{\Theta_1(0, -a, p; \tau)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

where

$$\Theta_1(0,-a,p;pz) = (-1)^a \left(\frac{12}{p}\right) \frac{\sqrt{3}}{2} \sum_{\substack{j=-\infty \\ j\equiv 6a \pmod p}}^{\infty} \left(\frac{12}{j}\right) j\, q^{j^2/(24p)}$$

$$\Phi_{p,a}(z)$$

$$= q^{\frac{1}{2p}a(p-3a)-\frac{p}{24}} \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } 0 < 6a < p, \\ \displaystyle -1 + \sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } p < 6a < 3p \end{cases}$$

Theorem (G. (2016-2019))
The functions $\mathcal{G}_1\left(\frac{a}{p}; z\right)$ and $\mathcal{G}_2\left(\frac{a}{p}; p^2 z\right)$ are harmonic Maass forms of weight $1/2$ on $\Gamma_0(p^2) \cap \Gamma_1(p)$.

Proposition
Let $p > 3$ be prime. Then

$$\Theta_1\left(\frac{d}{p}; z\right) = (-1)^d \frac{2}{\sqrt{3}} \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \sin\left(\frac{6ad\pi}{p}\right) \times \Theta_1(0, -a, p; p^2 z).$$

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Proposition

Let $p > 3$ be prime. Then

$$\mathcal{G}_1\left(\frac{a}{p}; \frac{z}{p}\right) = H_1(a,p,z) - \left(\frac{12}{p}\right) \sum_{k=1}^{\frac{1}{2}(p-1)} (-1)^{a+k} \sin\left(\frac{6ak\pi}{p}\right) \times \mathcal{G}_2\left(\frac{a}{p}; pz\right)$$

for some holomorphic function $H_1(a,p,z)$.

Proposition

Let $p > 3$ be prime. Then

$$\pi_{hol}^{(2)}\left(\eta^3(\tau)\mathcal{G}_2^-\left(\frac{a}{p}; pz\right)\right)$$

$$= \frac{(-1)^a}{\sqrt{3p}}\left(\frac{12}{p}\right) \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ 3mp^2-n^2>0 \\ n\equiv 6a \pmod p}}^{\infty} \left(\frac{-4}{m}\right)\left(\frac{12}{2}\right)\left(\sqrt{3p}\, m \operatorname{sgn}(n) - n\right)$$

$$\times\, q^{m^2/8-n^2/(24p)}$$

Proof

$$\eta^3(\tau) = \sum_{m=1}^{\infty} m\left(\frac{-4}{m}\right) \exp\left(\frac{2\pi i m^2 \tau}{8}\right)$$

and

$$\mathcal{G}_2^-\left(\frac{a}{p}; pz\right) = -\frac{i}{3\sqrt{p}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(0, -a, p; p\tau)}{\sqrt{-i(\tau+z)}}\, d\tau$$

$$= -\frac{(-1)^a i}{2\sqrt{3p}} \left(\frac{12}{p}\right) \sum_{n=-\infty}^{\infty} n\left(\frac{12}{n}\right) \int_{-\bar{z}}^{i\infty} \frac{\exp\left(\frac{2\pi i n^2 \tau}{24p}\right)}{\sqrt{-i(\tau+z)}}\, d\tau$$

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Let $a = 1/8$, $b = 1/(24p)$ in LECTURE 7 result.

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Let $a = 1/8$, $b = 1/(24p)$ in LECTURE 7 result.

DESCRIPTION: **Holomorphic Projection of eta(tau)^3*G^(-1)(a/p,p*z)**
EXAMPLE: **a=1, p=5**

```
> with(qseries):
> with(ramamocktheta):
> currentdir("H:\\math\\research\\higher-order-mock-theta"):
> kron:=(m,n)->NumberTheory[KroneckerSymbol](m,n):
> sgnm:=m->if m>=0 then 1 else -1 fi:
> roundQS:=QS->local j: add(round(coeff(QS,q,j))*q^j,j=0..qdegree(QS)):
> ZAGHPROJSUM:=proc(a,p,T,adjx)
    local x,m,n,QF,lastn,j:
    x:=0: j:=modp(6*a,p):
    QF:=(m,n)->m^2/8 - n^2/24/p-adjx:
    for m from 1 to T do
      lastn:=floor(sqrt(3*p)*m):
      for n from j by p to lastn do
        x:= x + kron(-4,m)*kron(12,n)*(sqrt(3*p)*m*sgnm(n)-n)*q^QF(m,n):
      od:
      for n from j-p by -p to -lastn do
        x:= x + kron(-4,m)*kron(12,n)*(m*sqrt(3*p)*sgnm(n)-n)*q^QF(m,n):
      od:
    od:
  RETURN(x*(-1)^a/sqrt(3*p)):
  end:
```

```
> s1:=evalf(series(ZAGHPROJSUM(1,5,200,7/60),q,20));
```

$$s1 := -0.6653005370 + 2.667878828\,q - 1.311822372\,q^2 - 4.011967652\,q^3 + 3.279555904\,q^5 + 6.676209994\,q^6 - 3.279555904\,q^7 + 1.311822372\,q^8 - 5.247289436\,q^9 - 8.676209994\,q^{10} + 4.591378276\,q^{12} + 2.623644718\,q^{13} + 2.846931658\,q^{14} + 10.67620999\,q^{15} + 0.6559111808\,q^{16} - 6.847989840\,q^{18} + O(q^{20}) \quad (1)$$

```
> s2:=series(3/2*s1,q,20);
```

$$s2 := -0.9979508055 + 4.001818242\,q - 1.967733558\,q^2 - 6.017951478\,q^3 + 4.919333856\,q^5 + 10.01431499\,q^6 - 4.919333856\,q^7 + 1.967733558\,q^8 - 7.870934154\,q^9 - 13.01431499\,q^{10} + 6.887067414\,q^{12} + 3.935467077\,q^{13} + 4.270397487\,q^{14} + 16.01431498\,q^{15} + 0.9838667712\,q^{16} - 10.27198476\,q^{18} + O(q^{20}) \quad (2)$$

```
> s3:=roundQS(s2);
```

$$s3 := -10\,q^{18} + q^{16} + 16\,q^{15} + 4\,q^{14} + 4\,q^{13} + 7\,q^{12} - 13\,q^{10} - 8\,q^9 + 2\,q^8 - 5\,q^7 + 10\,q^6 + 5\,q^5 - 6\,q^3 - 2\,q^2 + 4\,q - 1 \quad (3)$$

```
> s4:=series(s3/etaq(q,1,100)^3,q,20);
```

$$s4 := -1 + q + q^2 + 2\,q^3 + q^4 + 3\,q^5 + 2\,q^6 + 3\,q^7 + 3\,q^8 + 5\,q^9 + 3\,q^{10} + 6\,q^{11} + 5\,q^{12} + 7\,q^{13} + 7\,q^{14} + 9\,q^{15} + 7\,q^{16} + 12\,q^{17} + 11\,q^{18} + 13\,q^{19} + O(q^{20}) \quad (4)$$

```
> with(qsOEIS):
> matchOEIS(qs2L(s4,1,17));
There were 1 matches  (returning the first 1)

        53262, "Coefficients of the 5th-order mock theta function chi_0(q)."
                              pp1                                   (5)
> series(add(q^n/aqprod(q^(n+1),q,n),n=0..20)-2-s4,q,20);
                             O(q^20)                                (6)
```

It seems that

$$\pi_{\text{hol}}^{(2)}\left(\eta^3(\tau)\mathcal{G}_2^-\left(\frac{1}{5};5z\right)\right)$$
$$=\eta^3(\tau)^3\left(q^{-1/120}\,\frac{2}{3}\left(\chi_0(q)-2\right)\right)$$

where

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_n}$$

Corollary

Let $p > 3$ be prime and let $1 \leq a \leq \frac{1}{2}(p-1)$. Then

$$M_{p,a}(\tau) = \frac{1}{\eta^3(\tau)} \frac{(-1)^a}{\sqrt{3p}} \left(\frac{12}{p}\right) \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ 3mp^2-n^2>0 \\ n\equiv 6a \pmod{p}}}^{\infty} \left(\frac{-4}{m}\right)\left(\frac{12}{2}\right)\left(\sqrt{3p}\, m \operatorname{sgn}(n) - n\right) \times q^{m^2/8-n^2/(24p)}$$

is a mock modular form (holomorphic part of a harmonic Maass form of weight 1/2)

$$M_{p,a}(\tau) = \frac{1}{\eta^3(\tau)} \pi_{\text{hol}}\left(\eta^3(\tau)\mathcal{G}_2^-(a/p; p\tau)\right).$$

We know that

$$\mathcal{G}_2\left(\frac{a}{p};\frac{-1}{z}\right)=\sqrt{-iz}\,\mathcal{G}_1\left(\frac{a}{p};z\right)$$

so that

$$\mathcal{G}_2\left(\frac{a}{p};p\frac{-1}{z}\right)=\sqrt{-iz/p}\,\mathcal{G}_1\left(\frac{a}{p};\frac{z}{p}\right)$$

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DEFINE

$$\widehat{M}^*_{p,a}(\tau)=\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p};p\tau\right)-\pi^{(2)}_{\text{hol}}\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p};p\tau\right)\right)$$

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$$\mathcal{G}_2\left(\frac{a}{p}; p\frac{-1}{z}\right) = \sqrt{-iz/p}\,\mathcal{G}_1\left(\frac{a}{p}; \frac{z}{p}\right)$$

DEFINE

$$\widehat{M}^*_{p,a}(\tau) = \eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right) - \pi^{(2)}_{\text{hol}}\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)\right)$$

THEN $\widehat{M}^*_{p,a}(\tau)$ transforms like a modular form of weight 2.

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DEFINE

$$\widehat{M}^*_{p,a}(\tau)=\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p};p\tau\right)-\pi^{(2)}_{\text{hol}}\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p};p\tau\right)\right)$$

THEN $\widehat{M}^*_{p,a}(\tau)$ transforms like a modular form of weight 2.

$$\left(\eta^3(z)\,\mathcal{G}_2\left(\frac{a}{p};pz\right)\right)\Big|_{(2)}S = -\frac{1}{\sqrt{p}}\,\eta^3(z)\,\mathcal{G}_1\left(\frac{a}{p};\frac{z}{p}\right)$$

$$= \frac{1}{\sqrt{p}}\left(\frac{12}{p}\right)\eta^3(z)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{a+k}2\sin\left(\frac{6ak\pi}{p}\right)\mathcal{G}_2\left(\frac{k}{p};pz\right)$$
$$+ H(a,p,z)$$

where $H(a,p,z)$ is some holomorphic function of z.

Now we ASSUME that holomorphic projection and stroke operator commute in this case. That is

$$\pi_{\mathrm{hol}}^{(2)}\left(F(z)\Big|_{(2)} S\right) = \left(\pi_{\mathrm{hol}}^{(2)}\left(F(z)\right)\right)\Big|_{(2)} S$$

for

$$F(z) = \eta^3(z)\,\mathcal{G}_2\left(\frac{a}{p}; pz\right)$$

Therefore

$$\widehat{M}^{*}_{p,a}(\tau)\Big|_{(2)} S = \left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)\right)\Big|_{(2)} S - \pi^{(2)}_{\mathrm{hol}}\left(\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)\right)\Big|_{(2)} S\right)$$

Therefore

$$\widehat{M}^*_{p,a}(\tau)\bigg|_{(2)} S = \left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)\right)\bigg|_{(2)} S \\ - \pi^{(2)}_{\mathrm{hol}}\left(\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)\right)\bigg|_{(2)} S\right)$$

Therefore

$$\widehat{M}^*_{p,a}(\tau)\Big|_{(2)} S$$

$$= \frac{1}{\sqrt{p}}\left(\frac{12}{p}\right)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{a+k}2\sin\left(\frac{6ak\pi}{p}\right)\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{k}{p};p\tau\right)\right) + H(a,p,\tau)$$

$$- \frac{1}{\sqrt{p}}\left(\frac{12}{p}\right)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{a+k}2\sin\left(\frac{6ak\pi}{p}\right)\pi^{(2)}_{\text{hol}}\left(\eta^3(\tau)\,\mathcal{G}_2\left(\frac{k}{p};p\tau\right)\right) - H(a,p,\tau)$$

(since $H(a,p,\tau)$ is holomorphic)

$$= \frac{1}{\sqrt{p}} \left(\frac{12}{p}\right) \sum_{k=1}^{\frac{1}{2}(p-1)} (-1)^{a+k} 2 \sin\left(\frac{6ak\pi}{p}\right) \left\{\eta^3(\tau)\, \mathcal{G}_2\left(\frac{k}{p}; p\tau\right) - \pi_{\mathrm{hol}}^{(2)}\left(\eta^3(\tau)\, \mathcal{G}_2\left(\frac{k}{p}; p\tau\right)\right)\right\}$$

$$= \frac{1}{\sqrt{p}} \left(\frac{12}{p}\right) \sum_{k=1}^{\frac{1}{2}(p-1)} (-1)^{a+k} 2 \sin\left(\frac{6ak\pi}{p}\right) \widehat{M}^{*}_{p,k}(\tau)$$

DEFINE

$$\widehat{M}_{p,a}(\tau) = \frac{1}{\eta^3(\tau)}\widehat{M}^*_{p,a}(\tau).$$

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THEN

$$\widehat{M}^*_{p,a}\left(\frac{-1}{\tau}\right)$$

$$= \sqrt{-i\tau/p}\left(\frac{12}{p}\right)\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^{a+k}2\sin\left(\frac{6ak\pi}{p}\right)\widehat{M}_{p,k}(\tau)$$

DEFINE

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Theorem

Let $p > 3$ be prime, and let

$$\widehat{M}_p(\tau) = \left(\widehat{M}_{p,a}(\tau)\right)^T_{1\le a\le \frac{1}{2}(p-1)}.$$

THEN

$$\widehat{M}_p\left(\frac{-1}{\tau}\right)$$
$$= \sqrt{-i\tau/p}\left(\frac{12}{p}\right)\left((-1)^{a+k} 2\sin\left(\frac{6ak\pi}{p}\right)\right)_{\substack{1\le k\le \frac{1}{2}(p-1)\\ 1\le a\le \frac{1}{2}(p-1)}} \widehat{M}_p(\tau)$$

where each $\widehat{M}_{p,a}(\tau)$ is a harmonic Maass form of weight $1/2$ with shadow proportional to

$$\sum_{\substack{n=-\infty \\ n\equiv 6a \pmod p}}^{\infty} n\left(\frac{12}{n}\right) q^{n^2/(24p)}$$

HECKE-ROGERS SERIES OF WEIGHT 2

Using the method of LECTURE 7 we can rewrite the mock modular form $M_{p,a}(\tau)$ as a Hecke-Rogers type series (JONATHAN BRADLEY-THRUSH)

Theorem (JONATHAN BRADLEY-THRUSH)

$$M_{p,a}(\tau) = \frac{1}{\eta^3(\tau)}(-1)^a \left(\frac{12}{p}\right) \sum_{m=1}^{\infty} \sum_{\substack{n=-\lfloor \frac{12\beta pm}{\alpha+1} \rfloor \\ n \equiv 6a \pmod p}}^{\lfloor \frac{12\beta pm}{\alpha+1} \rfloor} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{n}{12\beta p}(\alpha+\varepsilon_p)\right) \times q^{m^2/8-n^2/(24p)}$$

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where $(x, y) = (\alpha, \beta)$ is the fundamental solution to

$$x^2 - 48py^2 = 1,$$

and

$$\varepsilon_p = \left(\frac{-4}{\alpha}\right)\left(\frac{12}{\alpha}\right)\begin{cases}1 & \text{if } \alpha \equiv 1 \pmod{p}\\ -1 & \text{if } \alpha \equiv -1 \pmod{p}\end{cases}$$

We note that the functions $\widehat{M}_{p,a}(\tau)$ and $\mathcal{G}_2\left(\frac{a}{p}; p\tau\right)$ have the same non-holomorphic part. It follows that

$$\begin{aligned}\mathcal{G}_2\left(\frac{a}{p}; p\tau\right) - \widehat{M}_{p,a}(\tau) \\ = \mathcal{G}_2^+\left(\frac{a}{p}; p\tau\right) + M_{p,a}(\tau) \\ = M_{p,a}(\tau) + 2\Phi_{p,a}(\tau)\end{aligned}$$

is a weakly holomorphic modular form.

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is a weakly holomorphic modular form. This means that there are analogs of the Mock Theta Conjectures to all primes $p > 3$.

Theorem

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Theorem

Let $p > 3$ be prime and let $1 \leq a \leq \frac{1}{2}(p-1)$. Then

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EXAMPLES

(1) $\mathbf{p = 5}$, $\mathbf{a = 1}$ The fundamental solution to

$$x^2 - 48 \cdot 5 \cdot y^2 = 1$$

is $(x, y) = (\alpha, \beta) = (31, 2)$, $\varepsilon_5 = 1$, $\frac{12\beta p}{\alpha + 1} = \frac{15}{4}$

$M_{5,1}(\tau)$

$$= \frac{1}{\eta^3(\tau)} \sum_{m=1}^{\infty} \sum_{\substack{n=-\lfloor \frac{15m}{4} \rfloor \\ n \equiv 1 \pmod 5}}^{\lfloor \frac{15m}{4} \rfloor} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{4n}{15}\right) q^{m^2/8 - n^2/120}$$

EXAMPLES

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$$M_{5,1}(\tau) + 2\Phi_{5,1}(\tau)$$

is a weakly holomorphic modular form of weight 1/2. It turns out that

$$M_{5,1}(\tau) + 2\Phi_{5,1}(\tau) = \frac{2}{3}\eta(5\tau)\frac{\eta_{5,2}(\tau)}{\eta_{5,1}^2(\tau)}$$

$$M_{5,1}(\tau) + 2\Phi_{5,1}(\tau)$$

is a weakly holomorphic modular form of weight $1/2$. It turns out that

$$M_{5,1}(\tau) + 2\Phi_{5,1}(\tau) = \frac{2}{3}\eta(5\tau)\frac{\eta_{5,2}(\tau)}{\eta_{5,1}^2(\tau)}$$

EQUIVALENTLY

$$\frac{1}{(q)_\infty^3}\sum_{m=1}^{\infty}\sum_{\substack{n=-\lfloor\frac{15m}{4}\rfloor \\ n\equiv 1 \pmod 5}}^{\lfloor\frac{15m}{4}\rfloor}\left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n)-\frac{4n}{15}\right)q^{m^2/8-n^2/120-7/60}$$

$$=-2\left(-1+\sum_{n=1}^{\infty}\frac{5n^2}{(q;q^5)_{n+1}(q^4,q^5)_n}\right)+\frac{2}{3}\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q^5)_\infty^2(q^4;q^5)_\infty^2}$$

$$=\frac{2}{3}\left(-3\Phi(q)+A(q)\right)$$

$$=\frac{2}{3}\left(2-\chi_0(q)\right)$$

EXAMPLE

$$M_{5,2}(\tau) + 2\Phi_{5,2}(\tau) = -\frac{2q^{\frac{71}{120}}(q, q^4, q^5; q^5)_\infty}{3(q^2, q^3; q^5)_\infty^2},$$

$$M_{7,1}(\tau) + 2\Phi_{7,1}(\tau) = \frac{q^{-\frac{1}{168}}(q^3, q^4, q^7; q^7)_\infty}{(q, q^2, q^5, q^6; q^7)_\infty},$$

$$M_{7,2}(\tau) + 2\Phi_{7,2}(\tau) = -\frac{q^{\frac{143}{168}}(q, q^6, q^7; q^7)_\infty}{(q^2, q^3, q^4, q^5; q^7)_\infty},$$

$$M_{7,3}(\tau) + 2\Phi_{7,3}(\tau) = -\frac{q^{\frac{47}{168}}(q^2, q^5, q^7; q^7)_\infty}{(q, q^3, q^4, q^6; q^7)_\infty}.$$

EXAMPLE

(2) $\mathbf{p = 11}$, $\mathbf{a = 1}$

$$M_{11,1}(\tau) = \frac{-1}{\eta^3(\tau)} \sum_{m=1}^{\infty} \sum_{\substack{n=-\lfloor \frac{11m}{2} \rfloor \\ n \equiv 6 \pmod{11}}}^{\lfloor \frac{11m}{2} \rfloor} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m \operatorname{sgn}(n) - \frac{n}{6}\right) q^{m^2/8 - n^2/264}$$

THEN

$$M_{11,1}(\tau) + 2\Phi_{11,1}(\tau)$$

is a weakly holomorphic modular form of weight $1/2$.

EXAMPLE

(2) $\mathbf{p} = \mathbf{11}$, $\mathbf{a} = \mathbf{1}$

$$M_{11,1}(\tau)$$

$$= \frac{-1}{\eta^3(\tau)} \sum_{m=1}^{\infty} \sum_{\substack{n=-\lfloor \frac{11m}{2} \rfloor \\ n \equiv 6 \pmod{11}}}^{\lfloor \frac{11m}{2} \rfloor} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{n}{6}\right) q^{m^2/8 - n^2/264}$$

THEN

$$M_{11,1}(\tau) + 2\Phi_{11,1}(\tau)$$

is a weakly holomorphic modular form of weight $1/2$.

We find that

$$M_{11,1}(\tau) + 2\Phi_{11,1}(\tau)$$
$$= \frac{1}{6}\,\frac{\eta^4(11\tau)}{\eta^3(\tau)}\left(\frac{11\eta_{11,3}(\tau)\,\eta_{11,4}(\tau)^3}{\eta_{11,1}(\tau)\,\eta_{11,2}(\tau)} - \frac{5\eta_{11,1}(\tau)\,\eta_{11,4}(\tau)^2}{\eta_{11,5}(\tau)} - \frac{5\eta_{11,2}(\tau)\,\eta_{11,3}(\tau)^2}{\eta_{11,5}(\tau)} - 17\eta_{11,4}(\tau)\,\eta_{11,5}(\tau) - \frac{5\eta_{11,1}(\tau)^2\,\eta_{11,3}(\tau)}{\eta_{11,5}(\tau)} - 16\eta_{11,2}(\tau)^2\right)$$

Ramanujan an Hardy: Vom ersten bis zum letzten Brief

Don Zagier

„Mock-Thetafunktionen" von beliebiger Primzahlordnung

Für eine Primzahl $p \geq 5$ und $j = 1, 2, \ldots, \frac{p-1}{2}$ sei

$$M_{p,j}(q) =$$

$$\frac{1}{\eta(z)^3} \sum_{\substack{m,n \in \mathbb{Z} \\ |n| < \frac{t-1}{u} m \\ n \equiv j \pmod{p}}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right) \operatorname{sgn}(n) \left(m - \frac{u|n|}{t-1}\right) q^{\frac{3pm^2 - n^2}{24p}}$$

wobei $t^2 - 3pu^2 = 1$, $t, u > 0$. Dann sind die $M_{p,j}(q)$ Mock-Modulformen und sind für $p = 5$ oder 7 genau Ramanujans „Mock-Thetafunktionen" der „Ordnung" 5 und 7.