# NSF/CBMS Research Conference
# Ramanujan's Ranks,
# Mock Theta Functions, and Beyond
# May 16-20, 2022
# The University of Texas Rio Grande Valley

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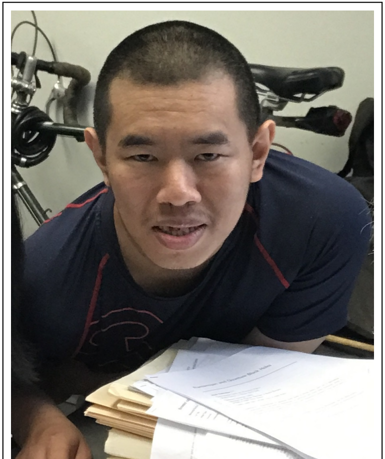
University of Florida

May 20, 2022

# LECTURE 9 (under construction)
# THE HURWITZ CLASS NUMBER, MOCK THETA FUNCTIONS AND THE UNIMODAL SEQUENCE CONJECTURES
(Includes joint work with Rong Chen, Shanghai)



## THE SMALLEST PARTS PARTITION FUNCTION

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# GEORGE ANDREWS

# spt-function

- Andrews (2008) defined the function $\mathrm{spt}(n)$ as the total number of appearances of the smallest parts in the partitions of $n$. For example,

$$\dot{4}, \quad 3+\dot{1}, \quad \dot{2}+\dot{2}, \quad 2+\dot{1}+\dot{1}, \; \dot{1}+\dot{1}+\dot{1}+\dot{1}.$$

Hence, $\mathrm{spt}(4) = 10$.

| $n$ | $\mathrm{spt}(n)$ |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 10 |
| 5 | 14 |
| 6 | 26 |
| $\vdots$ | |
| 10 | 119 |
| $\vdots$ | |
| 100 | 1545832615 |
| $\vdots$ | |
| 1000 | 600656570957882248155746472836274 |
| $\vdots$ | |

- BRINGMANN-MAHLBURG (2009)

$$\mathrm{spt}(n) \sim \frac{2\sqrt{6n}}{\pi} p(n) \sim \frac{1}{\sqrt{8n}} e^{\pi\sqrt{\frac{2n}{3}}}$$

- ANDREWS (2008)

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n$$
$$= \sum_{n=1}^{\infty} (q^n + 2q^{2n} + 3q^{3n} + \cdots) \cdot \frac{1}{(1-q^{n+1})(1-q^{n+2})\cdots}$$
$$= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty}$$
$$= q + 3q^2 + 5q^3 + 10q^4 + 14q^5 + 26q^6 + 35q^7 + \cdots,$$

where $(a)_\infty = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.

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where $(a)_\infty = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.

# NOTATION

$$(a)_0 := (a;q)_0 := 1,$$
$$(a)_n := (a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

when $n$ is a nonnegative integer.

$$(a)_\infty := (a;q)_\infty := \prod_{m=1}^{\infty}(1-aq^{m-1})$$

if $|q| < 1$.

# SPT and Maass Forms

BRINGMANN (2008)

- $$\mathcal{M}(z) := \sum_{n=0}^{\infty} (12\mathrm{spt}(n) + (24n-1)p(n))\, q^{n-1/24} - \frac{3\sqrt{3}i}{\pi} \int_{-\bar{z}}^{i\infty} \frac{\eta(\tau)\, d\tau}{(-i(z+\tau))^{3/2}}$$

  Then

  $$\mathcal{M}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^{3/2}}{\nu_\eta(A)} \mathcal{M}(z).$$

- $\mathcal{M}(24z)$ is a weight $\frac{3}{2}$ weak Maass form $\mathcal{M}(z)$ on $\Gamma_0(576)$ with Nebentypus $\chi_{12}$.

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# SPT-Congruences

- Andrews (2008) proved that

$$spt(5n+4) \equiv 0 \pmod{5}, \tag{1}$$
$$spt(7n+5) \equiv 0 \pmod{7}, \tag{2}$$
$$spt(13n+6) \equiv 0 \pmod{13}. \tag{3}$$

- G. (unpublished)

$$\sum_{n=1}^{\infty} \mathrm{spt}(5n-1)q^n + 5\sum_{n=1}^{\infty} \mathrm{spt}(n)q^{5n}$$
$$= \frac{5}{2}\sum_{n=1}^{\infty}(\sigma(5n)-\sigma(n))q^n \times \prod_{n=1}^{\infty}\frac{1}{(1-q^{5n})}$$
$$+ \frac{25q}{2}\left(1+\sum_{n=1}^{\infty}(\sigma(n)-5\sigma(5n))q^n\right) \times \prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}$$

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- G. (2012): For $a$, $b$, $c \geq 3$,

$$\mathrm{spt}(5^a n + \delta_a) + 5\,\mathrm{spt}(5^{a-2} n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}},$$
$$\mathrm{spt}(7^b n + \lambda_b) + 7\,\mathrm{spt}(7^{b-2} n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2) \rfloor}},$$
$$\mathrm{spt}(13^c n + \gamma_c) - 13\,\mathrm{spt}(13^{c-2} n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}},$$

where $\delta_a$, $\lambda_b$ and $\gamma_c$ are the least nonnegative residues of the reciprocals of 24 mod $5^a$, $7^b$ and $13^c$ respectively.

- G. (2008); ONO (2011): If $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^2 n - \tfrac{1}{24}(\ell^2 - 1)) \equiv 0 \pmod{\ell},$$

for any prime $\ell \geq 5$. This follows from $(\mathcal{M}_\ell(z) - \chi_{12}(\ell)(1+\ell)\mathcal{A}(z) \equiv 0 \pmod{\ell}$.

- AHLGREN, BRINGMANN and LOVEJOY (2011) If $\left(\frac{-23-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^{2m} n + d_{\ell,2m}) \equiv 0 \pmod{\ell^m},$$

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# SPT mod 2 and 3

- FOLSOM and ONO (2008); ANDREWS, G. and LIANG (2013): $\mathrm{spt}(n)$ is odd and if and only if $24n-1 = p^{4a+1}m^2$ for some prime $p \equiv 23 \pmod{24}$ and some integers $a$, $m$, where $(p,m)=1$.
- FOLSOM and ONO (2008) If $\ell \geq 5$ is prime then

$$\mathrm{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathrm{spt}(n) + \ell\,\mathrm{spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \chi_{12}(\ell)\,(1+\ell)\,\mathrm{spt}(n) \pmod{3}.$$

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# SPT mod 4

**G. CONJECTURE (? and 2017)**

Conjecture

[1] *Suppose* $\ell > 3$ *is prime and* $\ell \not\equiv 23 \pmod{24}$. *Let* $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\ell) = 1$ *if* $\ell \equiv 1 \pmod{24}$ *and* $-1$ *otherwise. Then*

$$\mathrm{spt}(\ell n - s(\ell)) \equiv 0 \pmod{4}, \quad (\textit{where } s(\ell) = \tfrac{1}{24}(\ell^2 - 1)),$$

*when* $\left(\frac{n}{\ell}\right) = \widetilde{\varepsilon}$.

[1] This conjecture was presented in a talk, entitled *The Andrews spt-function mod 4*, at the AMS Special Session on Arithmetic Properties of Sequences from Number Theory and Combinatorics, AMS Annual Meeting, Atlanta, January 4, 2017.

# RONG CHEN

**ANDREWS, G. and LIANG (2013)** $\mathrm{spt}(n)$ is odd and if and only if $24n-1$ has the form

$$24n-1=p^{4a+1}m^2,$$

for some prime $p \equiv 23 \pmod{24}$ and some integers $a$, $m$, where $(p,m)=1$.

**RONG CHEN'S OBSERVATION** For $n > 0$ be an integer, $\mathrm{spt}(n) \equiv 2 \pmod 4$ if and only if $24n - 1$ has the form

$$24n - 1 = p_1^{4a+1} p_2^{4b+1} m^2,$$

where $p_1$ and $p_2$ are primes such that $\left(\frac{p_1}{p_2}\right) = -\varepsilon(p_2)$ for $\varepsilon(p) = -1$ if $p \equiv \pm 5 \pmod{24}$ and $\varepsilon(p) = 1$ otherwise, $(m, p_1 p_2) = 1$ and $a, b \geq 0$ are integers.

## THE UNIMODAL SEQUENCE CONJECTURES

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# BRYSON, ONO, PITMAN AND RHOADES

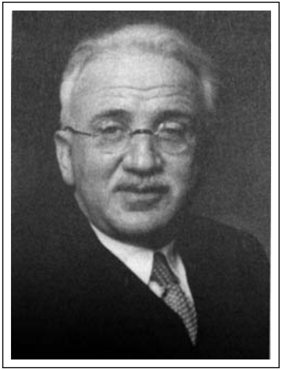
# LIM, KIM AND LOVEJOY

# ADOLF HURWITZ (1859 – 1919)

# GEORGES HUMBERT (1859 – 1921)

# RAMANUJAN (1887 – 1920)

# L.J. ROGERS (1862 – 1933) E. HECKE (1887 – 1947)

A sequence of integers $\{a_j\}_{j=1}^{s}$ is a **strongly unimodal sequence** of size $n$ if it satisfies

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0 \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n,$$

for some $k$. Let $u(n)$ be the number of such sequences.

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EXAMPLE: $n = 5$:

$$0 < 1 < 4 > 0$$
$$0 < 1 < 3 > 1 > 0$$
$$0 < 2 < 3 > 0$$
$$0 < 3 > 2 > 0$$
$$0 < 4 > 1 > 0$$
$$0 < 5 > 0$$

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$u(5) = 6$

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**GENERATING FUNCTION**:

$$\mathcal{U}(q) := \sum_{n} u(n)q^n = \sum_{n=0}^{\infty}(-q;q)_n q^{n+1}(-q;q)_n,$$
$$= q + q^2 + 3\,q^3 + 4\,q^4 + 6\,q^5 + 10\,q^6 + 15\,q^7 + 21\,q^8$$
$$+ 30\,q^9 + 43\,q^{10} + 59\,q^{11} + 82\,q^{12} + 111\,q^{13} + 148\,q^{14} + \cdots$$

where we use the usual $q$-notation

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## THE RANK OF A UNIMODAL SEQUENCE

**rank** of such a sequence as $s-2k+1$; i.e. the number terms after the maximum minus the number of terms before it. Let $u(m,n)$ be the number of strongly unimodal sequences of size $n$ and rank $m$.

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Then

$$\mathcal{U}(z;q) := \sum_{m,n} u(m,n) z^m q^n = \sum_{n=0}^{\infty} (-zq;q)_n q^{n+1} (-z^{-1}q;q)_n$$

$$= q + q^2 + \frac{z^2+z+1}{z} q^3 + \frac{z^2+2z+1}{z} q^4 + 2\,\frac{z^2+z+1}{z} q^5 + \cdots$$

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EXAMPLE: $n = 5$:

| Sequence | Rank |
|---|---|
| $0 < 1 < 4 > 0$ | 1 |
| $0 < 1 < 3 > 1 > 0$ | 0 |
| $0 < 2 < 3 > 0$ | 1 |
| $0 < 3 > 2 > 0$ | $-1$ |
| $0 < 4 > 1 > 0$ | $-1$ |
| $0 < 5 > 0$ | 0 |

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Let $u(a, b; n)$ be the number of strongly unimodal sequences of $n$ with rank congruent to $a$ mod $b$.

**BRYSON, ONO, PITMAN, RHOADES CONJECTURE (2012)** Suppose $\ell \equiv 7, 11, 13, 17 \pmod{24}$ is prime and $\left(\frac{k}{\ell}\right) = -1$. Then for all $n$ we have

$$u(\ell^2 n + kl - s(\ell)) \equiv 0 \pmod 4, \tag{4}$$

where $s(\ell) = \frac{1}{24}(\ell^2 - 1)$. Moreover, for $a \in \{0, 1, 2, 3\}$ we have

$$u(a, 4; \ell^2 n + kl - s(\ell)) \equiv 0 \pmod 2, \tag{5}$$

and

$$u(0, 4; \ell^2 n + kl - s(\ell)) \equiv u(2, 4; \ell^2 n + kl - s(\ell)) \pmod 4. \tag{6}$$

**EXAMPLE** $\ell = 7,\ k = 3,\ s(\ell) = 2,\ n = 20,$

$$\ell^2 n + k\ell - s(\ell) = 999.$$

$$u(0,4;999) = 1803774045752479268841040614319 8 \equiv 2$$

$$u(1,4,999) = u(3,4,999) = 18037740457524791096264174417626 \equiv 2$$

$$u(2,4;999) = 18037740457524789504117942692058 \equiv 2$$

$$u(999) = 72150961830099164385056697670508 \equiv 0$$

**RHOADES (2012)**

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$$\begin{aligned} u(0,4;999) &= 180377404575247926884104061431 98 \equiv 2 \\ u(1,4,999) = u(3,4,999) &= 180377404575247910962641744176 26 \equiv 2 \\ u(2,4;999) &= 180377404575247895041179426920 58 \equiv 2 \\ u(999) &= 721509618300991643850566976705 08 \equiv 0 \end{aligned}$$

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A sequence of integers $\{a_j\}_{j=1}^{s}$ is **unimodal** of size $n$ if it satisfies

$$0 < a_1 \leq a_2 \leq \cdots \leq a_{k-1} < a_k > a_{k+1} \geq \cdots \geq a_{s-1} \geq a_s > 0 \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n,$$

Such a unimodal sequence is called **odd-balanced** if the peak $a_k$ is even, even parts to the left and right of the peak are distinct and the odd parts to the left of the peak are identical with those to the right. As before the **rank** is the number to right of the peak minus the number to the left. We let $v(n)$ be the number of odd-balanced unimodal sequences of size $2n+2$ and let $v(m,n)$ be the number with rank $m$.

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## THE GENERATING FUNCTION

$$\mathcal{V}(z;q) := \sum_{m,n} v(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-zq;q)_n(-z^{-1}q;q)_n q^n}{(q;q^2)_{n+1}}$$

$$= 1 + 2\,q + \frac{z^2+3\,z+1}{z} q^2 + \frac{2\,z^2+5\,z+2}{z} q^3 + 4\,\frac{z^2+2\,z+1}{z} q^4 + \cdots$$

$$\mathcal{V}(q) := \mathcal{V}(1;q) = \sum_{n} v(n) q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_n(-q;q)_n q^n}{(q;q^2)_{n+1}}$$

$$= 1 + 2\,q + 5\,q^2 + 9\,q^3 + 16\,q^4 + 29\,q^5 + 48\,q^6 + 77\,q^7 + 123\,q^8 + 191\,q^9 + 290\,q^{10} + 436\,q^{11} + 643\,q^{12} + 936\,q^{13} + 1352\,q^{14} + \cdots$$

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**KIM, LIM and LOVEJOY'S CONJECTURE (2016)** Let $p \not\equiv \pm 1 \pmod 8$ be an odd prime, suppose $8\delta_p \equiv 1 \pmod{p^2}$ and $k, n \in \mathbb{Z}$ where $\left(\frac{k}{p}\right) = 1$. Then

$$v(p^2 n + (pk - 7)\delta_p) \equiv 0 \pmod 4.$$

## ODD BALANCED UNIMODAL SEQUENCES AND A MOCK THETA FUNCTION OF ORDER 2

$$\mathcal{V}(i;q) := \sum_{m,n} v(m,n) i^m q^n = \sum_{n} (v(0,4;n) - v(2,4;n)) q^n$$

$$= \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^n}{(q;q^2)_{n+1}} = \frac{A(q)}{q}$$

$$= q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 8\,q^5 + 11\,q^6 + 16\,q^7 + 23\,q^8 + 31\,q^9$$

$$+ 43\,q^{10} + 58\,q^{11} + 76\,q^{12} + 101\,q^{13} + 132\,q^{14} + 170\,q^{15} + \cdots$$

Let $N_A(n)$ denote the coefficient of $q^n$ in $A(q)$ so that
$N_A(n+1) = v(0,4;n) - v(2,4;n)$.

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$$N_A(p^2 n + (pk+1)\delta_p) \equiv 0 \pmod 4.$$

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HAVE YOU SEEN THIS MOD 4 BEHAVIOUR BEFORE?

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# WEIGHT 3/2 ETA-PRODUCTS

The SEARCH for similar congruences in the theory of modular forms.
We define

$$
\begin{aligned}
a(n) &= \text{the number of representations of } n \text{ as a sum of two pentagonal} \\
&\qquad \text{and three times a triangular number,} \\
b(n) &= \text{the number of representations of } n \text{ as a sum of a pentagonal} \\
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c(n) &= \text{the number of representations of } n \text{ as a sum of a pentagonal} \\
&\qquad \text{andtwo triangular numbers,}
\end{aligned}
$$

so that

$$\sum_{n=0}^{\infty} a(n)q^n = \left(\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2}\right)^2 \sum_{m=0}^{\infty} q^{3m(m+1)/2} = \frac{J_3^3 J_2^2}{J_1^2} = q^{-11/24}\frac{\eta(3\tau)^3\eta(2\tau)^2}{\eta(\tau)^2},$$

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{3m(m+1)/2}\right)^2 = \frac{J_6^3 J_2}{J_1} = q^{-19/24}\frac{\eta(6\tau)^3\eta(2\tau)}{\eta(\tau)},$$

$$\sum_{n=0}^{\infty} c(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{m(m+1)/2}\right)^2 = \frac{J_3^2 J_2^5}{J_6 J_1^3} = q^{-7/24}\frac{\eta(3\tau)^2\eta(2\tau)^5}{\eta(6\tau)\eta(\tau)^3}.$$

Here we have used the usual notation for infinite products and the Dedekind eta-function

$$J_k = \prod_{n=1}^{\infty}(1-q^{kn}), \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = \exp(2\pi i\tau)$ and $\Im(\tau) > 0$.

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**[RONG CHEN and G. (2021)]**

Let $p > 3$ be prime, suppose $24\delta_p \equiv 1 \pmod{p^2}$, and $k$, $n \in \mathbb{Z}$ where $\left(\frac{k}{p}\right) = 1$. Then

$$a(p^2 n + (pk - 11)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 11 \pmod{24},$$
$$b(p^2 n + (pk - 19)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 19 \pmod{24},$$
$$c(p^2 n + (pk - 7)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 7 \pmod{24}.$$

## CONNECTION WITH SUM OF THREE SQUARES

$$\sum_{n=0}^{\infty} A(n)q^n = \sum_{n=0}^{\infty} a(n)q^{24n+11}$$
$$= \sum_{x,y,z\in\mathbb{Z}} q^{(6x+1)^2+(6y+1)^2+9(4z+1)^2}$$
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## CONNECTION WITH THE CLASS NUMBER

**Theorem (GAUSS)**

*If $n$ is square-free, $n > 3$ and $n \equiv 3 \pmod 8$, then we have*

$$r_3(n) = 24h(-n),$$

*where $h(-n)$ is the class number of $\mathbb{Q}(\sqrt{-n})$.*

**The Hurwitz class number** $H(N)$:

(1) If $N \equiv 1, 2 \pmod 4$ then $H(N) = 0$.

(2) If $N = 0$ then $H(0) = -1/12$.

(3) If $N > 0$, $N \equiv 0, 3 \pmod 4$, then $H(N)$ is the class number of positive definite binary quadratic forms of discriminant $-N$, with those classes that contain a multiple of $x^2 + y^2$ or $x^2 + xy + y^2$ counted with weight $1/2$ or $1/3$, respectively.

| $D$ | 0 | 3 | 4 | 7 | 8 | 11 | 12 | 15 | 16 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
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where $-D$ is a fundamental discriminant, $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$, $\omega(D)$ is the number of units in the ring of integers of $\mathbb{Q}(\sqrt{D})$.

More generally,

$$H(n) = \frac{2h(D)}{\omega(D)} \sum_{d|f} \mu(d) \left(\frac{D}{d}\right) \sigma_1(f/d),$$

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## PROPERTIES OF HURWITZ CLASS NUMBER when $n \equiv 3 \pmod 4$ is square-free

- $$H(n) = h(-n) = 2^{t-1}k,$$
  where $t$ is the number of distinct prime factors of $n$ and $k$ is the number of classes in each genus of $\mathbb{Q}(\sqrt{-n})$.
- $$\left(2 - \left(\frac{n}{2}\right)\right) H(n) = \left(2 - \left(\frac{n}{2}\right)\right) h(-n) = \sum_{r=1}^{(n-1)/2} \left(\frac{r}{n}\right).$$

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**HIRZEBRUCH AND ZAGIER (1976)**

$$\mathcal{H}(\tau) := \sum_{n=0}^{\infty} H(n)q^n + \frac{1}{8\sqrt{2}\pi i}\int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau+w))^{3/2}}\,dw \in H_{3/2}(\Gamma_0(4))$$

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$$H(p^2n) + \left(\frac{-n}{p}\right)H(n) + pH(n/p^2) = (p+1)H(n),$$

for all $n \geq 0$.

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Define

$$\mathscr{H}_{a,b}(q) := \sum_{n=0}^{\infty} H(an+b)q^n$$

**HUMBERT (1907)**

$$\mathscr{H}_{8,7}(q) = \frac{1}{qJ_1^3}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}n^2q^{n(n+1)/2}}{1+q^n}.$$

where

$$J_k := (q^k;q^k)_\infty := \prod_{n=1}^{\infty}(1-q^{kn}), \quad \text{and} \quad (z;q)_\infty := \prod_{n=1}^{\infty}(1-zq^{n-1}).$$

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## ELEMENTARY CONGRUENCES

$$\frac{J_1^2}{J_2} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \equiv 1 \pmod 2,$$

$$\frac{J_2^5}{J_4^2 J_1^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \equiv 1 \pmod 2,$$

$$\text{and} \quad \frac{J_1^4}{J_2^2} \equiv 1 \pmod 4.$$

**LEMMA**

$$N_A(n) \equiv (-1)^{n+1} H(8n-1) \pmod 4.$$

**PROOF:** RAMANUJAN

$$A(q) = q\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1-q^{2n+1}}, \quad \mathscr{H}_{8,7}(q) = \frac{1}{qJ_1^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n^2 q^{n(n+1)/2}}{1+q^n}$$

$$\frac{A(-q)}{-q} = \frac{J_1}{J_2^2}\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{2n+1}}, \quad \mathscr{H}_{8,7}(q) \equiv \frac{1}{J_1^3}\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{2n+1}} \pmod 4$$

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## CONSEQUENCE OF HURWITZ MOD 4

- $N_A(n)$ is odd if and only if $8n-1$ has the form

$$8n-1 = p^{4a+1}m^2,$$

where $p$ is prime, and $m$ and $a$ are integers satisfying $(m,p)=1$ and $a \geq 0$.

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## WANG (2020)

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## EXAMPLE

$$\prod_{n=1}^{\infty}(1-q^n)^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+m} q^{(n^2-3m^2)/2+(n+m)/2}$$

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$$\left(1+\frac{1}{z}\right)\mathcal{V}(z,q)=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^{n+r}(z^r+z^{-r-1})q^{n^2+2n-r(r+1)/2}.$$

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NOW let $m = p^2 n + (pk+1)\delta_p$, where $p \not\equiv \pm 1 \pmod 8$ is an odd prime, $8\delta_p \equiv 1 \pmod{p^2}$ and $\left(\frac{k}{p}\right) = 1$

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$$D_v(q) := \sum_{n=0}^{\infty} d_v(n)q^n := \sum_{m=0}^{\infty}\sum_{r=0}^{m} q^{m^2+2m-r(r+1)/2}.$$

lemdv

**Lemma 4.2.** *If* $p \equiv 3, 5 \pmod 8$ *is prime and* $p \| 8n+7$ *then* $d_v(n) = 0$.

*Proof.* Suppose that $p \equiv 3, 5 \pmod 8$ is prime and $p \| 8n+7$. Suppose by way of contradiction that $d_v(n) \neq 0$. Then

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$$N_A(m+1) \equiv v(m) \pmod 4$$

THIS COMPLETES THE PROOF OF LIM, KIM AND LOVEJOYS MOD 4 CONJECTURES FOR ODD-BALANCE UNIMODAL SEQUENCES

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WANG (2020)

**THEOREM**

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  where $p$ is prime, and $m$ and $a$ are integers satisfying $(m,p)=1$ and $a \geq 0$.

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## RAMANUJAN'S THIRD ORDER MOCK THETA FUNCTION $\psi(q)$

$$\mathcal{U}(\pm i; q) = \psi(q) = \sum_{n=1}^{\infty} N_{\psi}(n) q^n = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$

**THEOREM [ANDREWS, G. AND LIANG (2013)]** For each $n > 0$, $N_{\psi}(n)$ is odd if and only if

$$24n - 1 = p^{4a+1} m^2,$$

for some prime $p$, and some integers $a$, $m$ satisfying $(m, p) = 1$ and $a \geq 0$.

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## MORE HECKE-ROGERS SERIES

- $$\frac{J_1^2}{J_2}\phi_-(q) = \sum_{n=1}^{\infty}\sum_{m=1-n}^{n}(-1)^{n-1}q^{n(3n-1)-2m^2+m}(1-q^{2n})$$

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**THEOREM [G. (2013)]** If $\ell \geq 5$ is prime then

$$N_\psi(\ell^2 n - s_\ell) + (-1)^{s_\ell} \left(\frac{3}{\ell}\right) \left(\frac{1-24n}{\ell}\right) N_\psi(n) + \ell\, N_\psi\left(\frac{n+s_\ell}{\ell^2}\right) \equiv (-1)^{s_\ell} \left(\frac{3}{\ell}\right) (1+\ell)\, N_\psi(n) \pmod{4}.$$

**THEOREM** For $n > 0$ be an integer, $N_\psi(n) \equiv 2 \pmod 4$ if and only if $24n-1$ has the form

$$24n-1 = p_1^{4a+1} p_2^{4b+1} m^2,$$

where $p_1$ and $p_2$ are primes such that $\left(\frac{p_1}{p_2}\right) = -\varepsilon(p_2)$ for $\varepsilon(p) = -1$ if $p \equiv \pm 5 \pmod{24}$ and $\varepsilon(p) = 1$ otherwise, $(m, p_1 p_2) = 1$ and $a, b \geq 0$ are integers.

**THEOREM** Let $p > 3$ be a prime where $p \not\equiv 23 \pmod{24}$. Suppose $24\delta_p \equiv 1 \pmod{p^2}$, $k, n \in \mathbb{Z}$ and $\left(\frac{k}{p}\right) = \varepsilon(p)$ where $\varepsilon(p) = -1$ if $p \equiv \pm 5 \pmod{24}$ and $\varepsilon(p) = 1$ otherwise. Then

$$N_\psi(p^2 n + (pk+1)\delta_p) \equiv 0 \pmod{4}, \tag{8}$$

$$\mathrm{spt}(p^2 n + (pk+1)\delta_p) \equiv 0 \pmod{4}. \tag{9}$$

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