Calderón Problem with quasilinear anisotropic conductivity

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Calderón Inverse Problem

Consider the boundary value problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) =& 0, \\ u|_{\partial\Omega} =& f. \end{aligned}$$

The measurements that one can perform on the boundary are the voltage $u|_{\partial\Omega}$ and the current $\gamma(\partial u/\partial \nu)|_{\partial\Omega}$, where ν denotes the unit outer normal to the boundary.
If $\gamma \in L^\infty(\Omega)$, for every $f \in H^{1/2}(\partial\Omega)$ we can define the
Dirichlet-to-Neumann map

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial v}\bigg|_{\partial\Omega},$$

which has values in $H^{-1/2}(\partial\Omega)$.
Calderón's inverse problem: Does Λ_γ determine γ?

Review: CGO Solutions

Theorem (Sylvester-Uhlmann, 1986, 1987 [4])

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max\left(C_0\|q\|_{L^\infty(\Omega)}, 1\right)$, and for any function $a \in H^2(\Omega)$ satisfying

$$\zeta \cdot \nabla a = 0 \quad \text{in } \Omega$$

the equation $(-\Delta + q)u = 0$ in Ω has a solution $u \in H^2(\Omega)$ of the form

$$u(x) = e^{i\zeta\cdot x}(a + r)$$

where $r \in H^2(\Omega)$ satisfies

$$\|r\|_{H^k(\Omega)} \leq C_0|\zeta|^{k-1}\|(-\Delta + q)a\|_{L^2(\Omega)}, \quad k = 0, 1, 2$$

Calderón problem with quasilinear conductivity

Consider the boundary value problem

$$\begin{cases} \nabla \cdot (\gamma(x,u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We define the associated Dirichlet-to-Neumann map by

$$\Lambda_\gamma(f) = (\gamma(x,u)\partial_\nu u)|_{\partial\Omega}$$

where ν is the unit outer normal to $\partial\Omega$.

Theorem (Sun 1996 [2])

Let $n \geq 2$. Assume $\gamma_i, \in C^{1,1}(\bar{\Omega} \times [-T, T])\ \forall T > 0, i = 1, 2$, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1(x,t) = \gamma_2(x,t)$ on $\bar{\Omega} \times \mathbb{R}$.

The linearization formula below is the key to the proof:

$$\lim_{s \to 0} \left\| \frac{1}{s}\Lambda_\gamma(t + sf) - \Lambda_{\gamma^t}(f) \right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0.$$

where $\gamma^t(x) = \gamma(x,t)$.

Calderón problem with quasilinear conductivity

In addition, we consider the quasilinear conductivity depending also on ∇u:

$$\begin{cases} \nabla \cdot (\gamma(x, u, \nabla u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The associated Dirichlet-to-Neumann map is given by

$$\Lambda_\gamma(f) = (\gamma(x, u, \nabla u)\partial_\nu u)|_{\partial\Omega}$$

where ν is the unit outer normal to $\partial\Omega$.

Theorem (Cârstea, Feizmohammadi, Kian, Krupchyk and Uhlmann, 2021[1])

Let $n \geq 3$, assume that $\gamma_1, \gamma_2 : \bar{\Omega} \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ is C^∞ in x, real-analytic in other variables and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2 \quad in \quad \bar{\Omega} \times \mathbb{C} \times \mathbb{C}^n$.

Sketch of the Proof

Let $\lambda = (\zeta, \mu) = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{C} \times \mathbb{C}^n$, by writing the Taylor series of γ

$$\gamma_j(x, \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \gamma_j^{(k)}(x, 0; \underbrace{\lambda, \ldots, \lambda}_{k \text{ times}}), \quad x \in \Omega, \quad j = 1, 2$$

We can **linearize the problem** and obtain

$$\sum_{(l_1,\ldots,l_{m+1}) \in \pi(m+1)} \sum_{j_1,\ldots,j_m=0}^{n} \int_\Omega T^{j_1 \cdots j_m}(x) \left(u_{l_1}, \nabla u_{l_1}\right)_{j_1} \ldots \left(u_{l_m}, \nabla u_{l_m}\right)_{j_m} \nabla u_{l_{m+1}} \cdot \nabla u_{m+2} dx = 0$$

for all $u_l \in C^\infty(\bar{\Omega})$ solving $\nabla \cdot (\gamma_0 \nabla u_l) = 0$ in $\Omega, l = 1, \ldots, m+2$, where

$$T^{j_1 \cdots j_m}(x) := \left(\partial_{\lambda_{j_1}} \ldots \partial_{\lambda_{j_m}} \gamma_1\right)(x, 0) - \left(\partial_{\lambda_{j_1}} \ldots \partial_{\lambda_{j_m}} \gamma_2\right)(x, 0),$$

$$\gamma_0 := \gamma_1(x, 0) = \gamma_2(x, 0)$$

and $\left(u_l, \nabla u_l\right)_j, j = 0, 1, \ldots, n$, stands for the jth component of the vector $(u_l, \partial_{x_1} u_l, \ldots, \partial_{x_n} u_l)$.

Sketch of the Proof

For $m = 1$, we have

$$0 = \sum_{(l_1, l_2) \in \pi(2)} \sum_{j=0}^{n} \int_{\Omega} T^j(x) \left(u_{l_1}, \nabla u_{l_1}\right)_j \nabla u_{l_2} \cdot \nabla u_3 dx$$

We'll use the fact that

$$\text{span} \left\{ \gamma_0 \nabla v_1 \cdot \nabla v_2 : v_j \in C^\infty(\bar{\Omega}), \nabla \cdot (\gamma_0 \nabla v_j) = 0, j = 1, 2 \right\}$$

is dense in $L^2(\Omega)$.

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For $m = 2$, we have

$$0 = \sum_{(l_1,l_2,l_3)\in\pi(3)} \sum_{j,k=0}^{n} \int_\Omega T^{jk}(x) \left(u_{l_1}, \nabla u_{l_1}\right)_j \left(u_{l_2}, \nabla u_{l_2}\right)_k \nabla u_{l_3} \cdot \nabla u_4 dx$$

Construct **CGO solutions** as in the linear problem

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Construct **CGO solutions** as in the linear problem

Set amplitudes being supported near a ray

$$\{x \in \mathbb{R}^n : x = p + t \operatorname{Re} \zeta, t \in \mathbb{R}\}.$$

Use solutions $U_{\lambda\zeta}, U_{-\lambda\zeta}, U_{\lambda\tilde{\zeta}}, U_{-\lambda\tilde{\zeta}} \in C^\infty(\bar{\Omega})$ of the form

$$U_{\pm\lambda\zeta}(x) = e^{\pm\lambda\zeta\cdot x}\gamma_0(x)^{-\frac{1}{2}} \left(a(x) + r_{\pm\lambda\zeta}(x)\right)$$

With properly chosen ζ, $\tilde{\zeta}$, a, $\tilde{a}$, show $T^{ij} = 0$ by inverse Fourier transform.

Review: Anisotropic Problem

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- There exists a natural obstruction in the unique determination in the anisotropic problem

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- There exists a natural obstruction in the unique determination in the anisotropic problem
- Let $A = (A_{ij})$ be an $n \times n$ matrix conductivity in the $C^{1,\alpha}(\bar{\Omega})$ class, $0 < \alpha < 1$, and $\Phi : \bar{\Omega} \to \bar{\Omega}$ **be a $C^{2,\alpha}$ diffeomorphism which is the identity map on $\partial\Omega$**, define

$$(H_\Phi A)(x) = \frac{(D\Phi(x))^T A(x)(D\Phi(x))}{|D\Phi|} \circ \Phi^{-1}(x)$$

where $D\Phi$ denotes the Jacobian matrix of Φ and $|D\Phi| = \det(D\Phi)$, then

$$\Lambda_{H_\Phi A} = \Lambda_A$$

In dimension 2

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Lemma (Isothermal Coordinates)

Let σ be a bounded and positive definite 2×2 matrix, there exists diffeomorphism $F : \mathbb{C} \to \mathbb{C}$ such that

$$F(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty$$

and such that

$$\left(F_* \sigma\right)(z) = \tilde{\sigma}(z) := \det\left(\sigma\left(F^{-1}(z)\right)\right)^{\frac{1}{2}}.$$

where $$F_*\sigma(y) = \frac{1}{J_F(x)} DF(x)\sigma(x)DF(x)^t \Big|_{x=F^{-1}(y)},$$

is the push-forward of the conductivity σ by F.

Anisotropoic Quasilinear Problem

Theorem (Sun and Uhlmann, 1997 [3])

Let $n = 2$, $A_1(x, u)$ and $A_2(x, u)$ be quasilinear coefficient matrices in $C^{2,\alpha}(\bar{\Omega} \times \mathbb{R})$ such that $\Lambda_{A_1} = \Lambda_{A_2}$. Then there exists a $C^{3,\alpha}$ diffeomorphism $\Phi : \bar{\Omega} \to \bar{\Omega}$ with $\Phi|_{\partial\Omega} =$ identity, such that $A_2 = H_\Phi A_1$.

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Theorem (Sun and Uhlmann, 1997 [3])

Let $n \geq 3$, $A_1(x,u)$ *and* $A_2(x,u)$ *be* **real-analytic** *quasilinear coefficient matrices such that* $\Lambda_{A_1} = \Lambda_{A_2}$. *Assume that either* A_1 *or* A_2 *extends to a real-analytic quasilinear coefficient matrix on* $\mathbb{R}^n$. *Then there exists a real-analytic diffeomorphism* $\Phi : \bar{\Omega} \to \bar{\Omega}$ *with* $\Phi|_{\partial\Omega} =$ *identity, such that* $A_2 = H_\Phi A_1$.

Sketch of the Proof

Denote $A^t(x) = A(x,t)$.
Then by first order linearization, for any $f \in C^{2,\alpha}(\partial\Omega), 0 < \alpha < 1, t \in \mathbb{R}$

$$\lim_{s \to 0} \left\| \frac{1}{s} \Lambda_A(t + sf) - \Lambda_{A^t}(f) \right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0.$$

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By result for the linear anisotropic case, for each fixed t, there exists a $C^{3,\alpha}$ diffeomorphism Φ^t when $n=2$ and a real analytic one when $n \geq 3$, and the identity at the boundary such that

$$A_2^t = H_{\Phi^t} A_1^t.$$

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We want to show that Φ^t **is independent on** t, which can be reduced to show

$$\left(\frac{\partial A_1}{\partial t} - \frac{\partial A_2}{\partial t} \right) \bigg|_{t=0} = 0$$

by differentiating the original equation with respect to t.

Sketch of the Proof

By second-order linearization, we have

$$\int_{\Omega} \nabla u_1 \cdot A_t(x,t) \nabla u_2^2 dx = 2 \int_{\partial\Omega} f_1 \frac{d}{dt} \left(t^{-1} \Lambda_A \left(t + s f_2 \right) \right) \Big|_{t=0} dx,$$

where u_i is the solution to the boundary value problem with $u_i|_{\partial\Omega} = f_i$.

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This gives

$$\int_\Omega \nabla u \cdot B(x) \nabla (u_1 u_2) \, dx = 0$$

where $B = \left(\frac{\partial A_1}{\partial t} - \frac{\partial A_2}{\partial t} \right) \Big|_{t=0}$.

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Use the density result as following:

(a)If

$$\int_\Omega h(x) \cdot \nabla (u_1 u_2)\, dx = 0$$

for solutions u_i, then $h(x)$ lies in the tangent space $T_x(\partial\Omega)$ for all $x \in \partial\Omega$.

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(b)Let A be a linear coefficient matrix in $C^{2,\alpha}(\bar{\Omega})$. Define

$$D_A = \mathrm{Span}_{L^2(\Omega)} \left\{ uv; u, v \in C^{3,\alpha}(\bar{\Omega}), \nabla \cdot A \nabla u = \nabla \cdot A \nabla v = 0 \right\}.$$

Then if $I \perp D_A$, then $I = 0$.

Quasilinear anisotropic problem in dimension 2

Consider the boundary value problem

$$\begin{cases} \nabla \cdot (A(x, u, \nabla u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Define the Dirichlet to Neumann map as follows:

$$\Lambda_\gamma(f) = (A(x, u, \nabla u)\partial_\nu u)|_{\partial\Omega}$$

where ν is the unit outer normal to $\partial\Omega$.

Theorem (Liimatainen-W, 2024)

Let $n = 2$, A_1 and A_2 be quasilinear anisotropic conductivities such that $\Lambda_{A_1}(f) = \Lambda_{A_2}(f)$, for all f in $C^{2,\alpha}(\partial\Omega)$ small, then there exists a $W^{1,2}$ diffeomorphism Φ which is the identity map on the boundary such that $A_2 = H_\Phi(A_1)$ where

$$(H_\Phi A)(x, t) = \frac{(D\Phi(x))^T A(x, t)(D\Phi(x))}{|D\Phi|} \circ \Phi^{-1}(x)$$

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Writing down the Taylor series of γ, we obtain

$$\sum_{(l_1,\dots,l_{m+1})\in\pi(m+1)} \sum_{j_1,\dots,j_m=0}^{n} \int_\Omega T^{j_1\cdots j_m}(x)\,(u_{l_1},\nabla u_{l_1})_{j_1}\dots(u_{l_m},\nabla u_{l_m})_{j_m} \nabla u_{l_{m+1}}\cdot\nabla u_{m+2}\,dx = 0$$

for all $u_l \in C^\infty(\bar{\Omega})$ solving $\nabla\cdot(\gamma_0\nabla u_l) = 0$ in $\Omega, l = 1,\dots,m+2$, where

$$T^{j_1\cdots j_m}(x) := \left(\partial_{\lambda_{j_1}}\dots\partial_{\lambda_{j_m}}\gamma_1\right)(x,0) - \left(\partial_{\lambda_{j_1}}\dots\partial_{\lambda_{j_m}}\gamma_2\right)(x,0),$$

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- Use singular solutions for boundary determination;

- Use Buhkgeim's CGO solutions and limiting Carleman weights to apply the method of stationary phase

Boundary determination

- Use **singular solutions**: let ν be an arbitrary outer pointing nontangential vector of Ω at x_0, and $z_\sigma = x_0 + \sigma\nu$ for some $\sigma > 0$. Then we have solution $u(x)$ to $\nabla \cdot (\gamma_0 \nabla u) = 0$ with singularity at z_σ:

$$u(x) = \log|x - z_\sigma| + w(x - z_\sigma)$$

where ω satisfies

$$|\omega_n(x)| + |x|\,|\nabla\omega_n(x)| \leq C|x|^\beta, \quad x \in \Omega$$

with $0 < \beta < 1$.

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- e.g. $m = 1$,

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let $u_1 = u_2 = u \implies \mathrm{T}^0(x_0) = 0$ for $x_0 \in \partial\Omega$.

- Then let $u_1 = u_2 = u_3 = u \implies \mathrm{T}^1(x_0)\partial_1 u(x_0) + T^2(x_0)\partial_2 u(x_0) = 0$ for $x_0 \in \partial\Omega$.

Bukhgeim's CGO Solutions

Bukhgeim constructed CGO solutions to the equation $(\Delta + q)u = 0$ in dimension 2, by considering the system

$$(D + Q)\mathbf{u} = 0$$

where

$$D = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -1 \\ q & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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Choose a holomorphic function ψ and let

$$\Phi = \begin{bmatrix} \psi & 0 \\ 0 & \bar{\psi} \end{bmatrix}, \quad \varphi(x) = 2\,\mathrm{Im}\,\psi$$

we seek solutions of the form

$$\mathbf{u} = e^{\Phi/h}(v + w)$$

Bukhgeim's CGO Solutions

In the algebraic computations, we would encounter the Cauchy operator $\overline{\partial}^{-1}$ defined by

$$(\overline{\partial}^{-1}u)(z) = \frac{1}{\pi}\int_{\Omega}\frac{u(w)}{z-w}\,\mathrm{d}w, \qquad \mathrm{d}w = \mathrm{d}w_1\,\mathrm{d}w_2$$

which satisfies $\bar{\partial}^{-1}\bar{\partial} = id$; similarly ∂^{-1} satisfying $\partial^{-1}\partial = id$.

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For convenience, we denote

$$\bar{\partial}_{\varphi}^{-1}f := \bar{\partial}^{-1}e^{-i\varphi/h}f, \quad \partial_{\varphi}^{-1}f := \partial^{-1}e^{i\varphi/h}f$$

Bukhgeim's CGO Solutions

In the algebraic computations, we would encounter the Cauchy operator $\overline{\partial}^{-1}$ defined by

$$(\overline{\partial}^{-1} u)(z) = \frac{1}{\pi} \int_\Omega \frac{u(w)}{z-w} \, \mathrm{d}w, \qquad \mathrm{d}w = \mathrm{d}w_1 \, \mathrm{d}w_2$$

which satisfies $\bar{\partial}^{-1}\bar{\partial} = id$; similarly ∂^{-1} satisfying $\partial^{-1}\partial = id$.
For convenience, we denote

$$\bar{\partial}_\varphi^{-1} f := \bar{\partial}^{-1} e^{-i\varphi/h} f, \quad \partial_\varphi^{-1} f := \partial^{-1} e^{i\varphi/h} f$$

Buhkhgeim found solutions to the equation $(\Delta + q)u = 0$

$$u = e^{\psi/h}(v + r_h),$$
$$r_h = (I - S)^{-1} S v = \sum_{n=1}^{\infty} S^n v$$

for small $h > 0$, where v is holomorphic, $Su = -\frac{1}{4}\bar{\partial}_\varphi^{-1}(\partial_\varphi^{-1}(qu))$.

Bukhgeim's CGO solutions

Symmetrically, we also have solutions of the form

$$u^t = e^{\bar{\psi}/h}(\bar{v} + \tilde{r}_h),$$
$$\tilde{r}_h = \sum_{n=1}^{\infty} (S^t)^n \bar{v}$$

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Using classical estimates on $\bar{\partial}^{-1}$ and ∂^{-1}, Bukhgeim showed for any $u \in L^2(\Omega)$,

$$||Su||_{L^2} \le ch^{1/3}||u||_{L^2}$$

which ensures the remainder r_h is well defined.

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which ensures the remainder r_h is well defined.
Later, Guillarmou and Tzou showed that for any $u \in W^{1,p}(\Omega)$, there is some $\epsilon > 0$ such that

$$||\bar{\partial}_{\varphi}^{-1}u||_{L^p} \leq Ch^{\frac{1}{2}+\epsilon}||u||_{W^{1,p}}$$

which may improve the estimates for the remainder r_h.

Bukhgeim's solutions solving the linear problem

Using these solutions, Bukhgeim proved for linear potentials

Theorem

If $q_j \in L^\infty(\Omega)$ *and* $C_{q_1} = C_{q_2}$*, then* $q_1 = q_2$.

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The proof used the following key lemma:

Lemma

Products of the form $u_i u_j^t$ *are dense in* L^2.

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The proof used the following key lemma:

Lemma

Products of the form $u_i u_j^t$ are dense in L^2.

To prove this, let

$$u_1 = e^{z^2/h}(1 + r_h)$$
$$u_2^t = e^{-\bar{z}^2/h}(1 + \tilde{r}_h)$$

so that for any $f \in L^2$,

$$\int f(z) u_1 u_2^t = \int e^{(z^2 - \bar{z}^2)/h} f(z)(1 + r_h + \tilde{r}_h + r_h \tilde{r}_h) = 0$$

would imply $f \equiv 0$ by the stationary phase method and remainder estimate.

Stationary phase method

We consider the oscillatory integral

$$I(h) := \int_U e^{\frac{i\varphi(x)}{h}} a(x) dx$$

- $U \subset \mathbb{R}^n$ is an open set
- $\varphi \in C^\infty(U; \mathbb{R})$, called "phase function"
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- $a \in C_c^\infty(U)$, called "amplitude"

- If **supp** a has no critical points of φ, by repeatedly integrating parts we have $I(h) = O(h^N), \forall N$.

Stationary phase method

- If x_0 is a nondegenerate critical point of φ, then $d^2\varphi(x_0)$ has k positive eigenvalues and $n-k$ negative eigenvalues for some k. We define

$$\text{sgn } d^2\varphi(y_0) := k - (n-k)$$

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Theorem

Let $\varphi \in C^\infty(U;\mathbb{R})$ have the non-degenerate critical point $x_0 \in X$ and assume that $\varphi'(x) \neq 0$ for $x \neq x_0$. Then there are differential operators $A_{2\nu}(D)$ of order $\leq 2\nu$ such that for every compact $K \subset X$ and every $N \in \mathbb{N}$, there is a constant $C = C_{K,N}$ such that for every $u \in C^\infty(X) \cap \mathcal{E}'(K)$

$$\left| \int e^{\frac{i\varphi(x)}{h}} u(x)dx - \left(\sum_0^{N-1} \left(A_{2\nu}\left(D_x\right)u\right)\left(x_0\right) h^{\nu+\frac{n}{2}} \right) e^{\frac{i\varphi(x_0)}{h}} \right|$$

$$\leq Ch^{N+\frac{n}{2}} \sum_{|x|\leq 2N+n+1} ||u||_{C^{2N+n+1}}.$$

$$\textit{Moreover } A_0 = \frac{(2\pi)^{\frac{n}{2}} \cdot e^{i\frac{\pi}{4}\operatorname{sgn}\varphi''(x_0)}}{\left|\det \varphi''\left(x_0\right)\right|^{\frac{1}{2}}}.$$

Key element of the proof

Lemma (Morse lemma)

Let $\varphi \in C^\infty(U;\mathbb{R})$ and let $x_0 \in U$ be a non-degenerate critical point. Then there are neighborhoods U of $0 \in \mathbb{R}^n$ and V of x_0 and a C^∞ diffeomorphism $\mathcal{H} : V \to U$ such that

$$\varphi \circ \mathcal{H}^{-1}(x) = \varphi(x_0) + \frac{1}{2}\left(x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_n^2\right).$$

Here $(r, n-r)$ is the signature of $\varphi''(x_0)$ (so that $r, n-r$ are respectively the number of positive and negative eigenvalues).

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Here $(r, n-r)$ is the signature of $\varphi''(x_0)$ (so that $r, n-r$ are respectively the number of positive and negative eigenvalues).

-Use the Morse lemma above to reduce the problem to the quadratic stationary phase case:

$$I(h) = \int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Qx,x\rangle} a(x)dx$$

- Q is an invertible symmetric Heal $n \times n$ matrix, $a \in C_c^\infty(\mathbb{R}^n)$.

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-Note: $\varphi(x) = \frac{1}{2}\langle Qx, x\rangle$ is a Morese function, the only critical part is $x = 0$, and

$$d_\varphi^2(0) = Q.$$

Key element of the proof

Theorem (Quadratic stationary phase)

$$\int e^{\frac{i\langle x,Qx\rangle}{h}/2}u(x)dx = \sum_{k=0}^{N-1} \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\operatorname{sgn} Q} h^{k+\frac{n}{2}}}{k!|\det Q|^{\frac{1}{2}}} \left(\frac{1}{2i}\left\langle D_x, Q^{-1}D_x\right\rangle\right)^k u(0) + S_N(u,\lambda),$$

where $$|S_N(u,h)| \leq C_{Q.\varepsilon}(N!)^{-1}h^{N+\frac{n}{2}} \left\| \left(\frac{1}{2}\left\langle D, Q^{-1}D\right\rangle\right)^N u \right\|_{H^{\frac{n}{2}+\epsilon}(\mathbf{R}^n)}$$

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-By Fourier transform,

$$\int e^{i\langle x,Qx\rangle/2h}u(x)dx = (2\pi)^{-\frac{n}{2}}h^{\frac{n}{2}}|\det Q|^{-\frac{1}{2}}e^{i\frac{\pi}{4}\operatorname{sgn} Q}\int e^{-ih\left(\xi,Q^{-1}\xi\right\rangle/2}\hat{u}(\xi)d\xi.$$

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-Use the Taylor expansion

$$e^{\frac{h}{2i}\left\langle Q^{-1}\xi,\xi\right\rangle} \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{h}{2i}\left\langle Q^{-1}\xi,\xi\right\rangle\right)^j$$

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-Inverse Fourier transform

Example

Back to the integral identity in Bukhgeim's paper for linear problem:

$$\int e^{(z^2-\bar{z}^2)/h} f(z)(1 + r_h + \tilde{r}_h + r_h \tilde{r}_h) = 0$$

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$$\int e^{(z^2-\bar{z}^2)/h} f(z) = Chf(0) + O(h^2)$$

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Therefore, we can show $f(0) = 0$, and similarly, we can show f vanishes at all the other points.

Quasilinear anisotropic problem in dimension 2: first try

It would be natural to first try solutions of the form

$$u_1 = \frac{1}{\sqrt{\gamma_0}} e^{z^2/h}(1 + r_h)$$
$$u_2^t = \frac{1}{\sqrt{\gamma_0}} e^{-\bar{z}^2/h}(1 + \tilde{r}_h)$$

for the quasilinear problem. Recall the integral identity we have in this case:

$$\sum_{(l_1,\dots,l_{m+1})\in\pi(m+1)} \sum_{j_1,\dots,j_m=0}^{n} \int_\Omega T^{j_1\cdots j_m}(x)\left(u_{l_1}, \nabla u_{l_1}\right)_{j_1} \dots \left(u_{l_m}, \nabla u_{l_m}\right)_{j_m} \nabla u_{l_{m+1}} \cdot \nabla u_{m+2} dx = 0$$

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Unfortunately, it turns out that the remainder estimate may not be good enough since we are taking derivatives of the solution. As an example case, we look at $m = 2$, where the integral identity reads

$$\sum_{(l_1,l_2,l_3)\in\pi(3)} \sum_{j,k=0}^{2} \int_\Omega T^{jk}(x)\left(u_{l_1}, \nabla u_{l_1}\right)_j \left(u_{l_2}, \nabla u_{l_2}\right)_k \nabla u_{l_3} \cdot \nabla u_4 dx = 0$$

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We mention here that by choosing $u_1 = u_2 = 1$, and then $u_1 = 0$, we would get an integral identity that already appears in the previous case $m = 1$. Therefore, assume $m = 1$ case is solved, we have $T^{00} = T^{01} = T^{02} = 0$, and obtain

$$\sum_{(l_1,l_2,l_3)\in\pi(3)} \sum_{j,k=1}^{2} \int T^{jk}(x)\partial_{x_j} u_{l_1} \partial_{x_k} u_{l_2} \nabla u_{l_3} \cdot \nabla u_4 dx = 0$$

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We focus first on the term in the expansion where the derivatives hit the phases:

$$\int \frac{1}{h^4\gamma_0^2}(T^{11} + T^{22})e^{(z^2-\bar{z}^2)/h} z^2\bar{z}^2(1 + r_h)(1 + r_h)(1 + \tilde{r}_h)(1 + \tilde{r}_h)$$

which by stationary phase would include the term $\frac{C}{h}(T^{11} + T^{22})(0)$ for some constant C.

Quasilinear anisotropic problem: first try

However, if one of the derivative hits the $\frac{1}{\sqrt{\gamma_0}}$ instead of the phase, we get

$$\int \frac{1}{h^3 \gamma_0^{3/2}} \partial \left(\frac{1}{\sqrt{\gamma_0}} \right) T e^{(z^2 - \bar{z}^2)/h} z \bar{z}^2 (1 + r_h)(1 + r_h)(1 + \tilde{r}_h)(1 + \tilde{r}_h)$$

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Therefore, we may not ensure that the above term involving the remainder is of smaller size compared to the principal term $\frac{C}{h}(T^{11} + T^{22})(0)$.

Solution

To solve the above problem, we instead choose phase functions with no critical point. Consider without loss of generality

$$u = \frac{1}{\sqrt{\gamma}} e^{(z+\frac{1}{2}z^2)/h}(1 + r_h),$$

$$r_h = \sum_{n=1}^{\infty} S^n 1$$

Integrating by parts, we have

$$\bar{\partial}^{-1} e^{\mathrm{i}\varphi/h} f = \frac{ih}{2}\left[e^{\mathrm{i}\varphi/h}\frac{f}{\bar{\partial}\varphi} + \frac{ih}{2}\bar{\partial}^{-1}\left(e^{\mathrm{i}\varphi/h}\bar{\partial}\left(\frac{f}{\bar{\partial}\varphi}\right)\right)\right],$$

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$$||\bar{\partial}^{-1} e^{\mathrm{i}\varphi/h} f||_{L^p} \leq Ch||f||_{W^{1,p}}$$

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for all $p \in (1, \infty)$. This leads to better estimate for remainders in the above solutions: $||r_h||_{L^2}, ||\partial r_h||_{L^2}, ||\bar{\partial} r_h||_{L^2} = O(h)$

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Let us check how these solutions help solve the problematic case. Again, consider the case $m = 2$, where we have the integral identity

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We mention that the idea of choosing phases without critical points has previously appeared in limiting Carleman weights. What is special in our case is that in dimension 2, we may have the explicit form for the remainder r_h using Bukhgeim's construction.

Let us check how these solutions help solve the problematic case. Again, consider the case $m = 2$, where we have the integral identity

$$\sum_{(l_1,l_2,l_3)\in\pi(3)} \sum_{j,k=1}^{2} \int T^{jk}(x)\partial_{x_j} u_{l_1} \partial_{x_k} u_{l_2} \nabla u_{l_3} \cdot \nabla u_4 dx = 0$$

Let

$$u_1 = \frac{1}{\sqrt{\gamma_0}} e^{(z+\frac{1}{2}z^2)/h}(1 + r_1),$$
$$u_2 = \frac{1}{\sqrt{\gamma_0}} e^{(-z+\frac{1}{2}z^2)/h}(1 + r_2),$$
$$u_3 = \frac{1}{\sqrt{\gamma_0}} e^{(-\bar{z}-\frac{1}{2}\bar{z}^2)/h}(1 + r_3)$$
$$u_4 = \frac{1}{\sqrt{\gamma_0}} e^{(\bar{z}-\frac{1}{2}\bar{z}^2)/h}(1 + r_4)$$

Solution

Now if the derivatives all hit the exponential component of the solutions, we will get

$$\int \frac{1}{h^4\gamma_0^2}(T^{11}+T^{22})e^{(z^2-\bar{z}^2)/h}(1+z)(-1+z)(1-\bar{z})(-1-\bar{z})(1+r_1)(1+r_2)(1+r_3)(1+r_4)$$

which includes the term

$$\int \frac{1}{h^4\gamma_0^2}(T^{11}+T^{22})e^{(z^2-\bar{z}^2)/h} = \frac{C}{h^3}(T^{11}+T^{22})(0)$$

while by remainder estimates, we can check all the terms involving the remainder term are $O(\frac{1}{h^2})$.

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Therefore, we can show $(T^{11}+T^{22})(0)=0$, and similarly we can prove $T^{11}+T^{22}$ vanishes at all the other points.

Similarly, by choosing other sets of solutions u_1, u_2, u_3, u_4 properly, we get a system of linear equatoins for T^{11}, T^{12}, T^{22}. In particular, for $m=2$, we obtain

$$\begin{aligned} T^{11}+2iT^{12}-T^{22} &= 0 \\ T^{11}+T^{22} &= 0 \\ T^{11}-2iT^{12}-T^{22} &= 0 \end{aligned}$$

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which has the unique solution $T^{11}=T^{12}=T^{22}=0$. The proof for other m is similar.

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Thank you!