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Clarifying the underlined terms:

1. "Classical" ≠ "old!" A classical mathematical concept is one whose definition does not involve computability or related aspects of mathematical logic.
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Clarifying the underlined terms:

1. "**Classical**" $\neq$ "**old!**" A **classical** mathematical concept is one whose definition does not involve computability or related aspects of mathematical logic.

2. To **effectivize** a mathematical concept is to impose algorithmic (computability or complexity) constraints on some characterization of the concept.
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2. Choose a characterization of the concept:

   $Z \subseteq \mathcal{C}$ has measure 0 if and only if
   $(\forall \epsilon > 0)(\exists A \subseteq \{0, 1\}^*)$ $Z \subseteq \bigcup_{w \in A} \mathcal{C}_w$ and $\sum_{w \in A} 2^{-|w|} \leq \epsilon$. 
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3. Discretize as needed:

$(\forall k \in \mathbb{N})(\exists A \subseteq \{0, 1\}^*)$
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How we effectivize

$Z \subseteq C$ has measure 0 iff

$(\forall k \in \mathbb{N})(\exists A \subseteq \{0, 1\}^*) [Z \subseteq \bigcup_{w \in A} C_w \text{ and } \sum_{l=0}^{\infty} 2^{-|f(l)|} \leq 2^{-k}]$
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4. Skolemize as needed

\((\exists f : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}^*) (\forall k \in \mathbb{N}) [Z \subseteq \bigcup_{i=0}^{\infty} C_{f(k,i)} \text{ and } \sum_{i=0}^{\infty} 2^{-f(k,i)} \leq 2^{-k}]\).
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5. Impose computability/complexity constraint

\[ Z \subseteq C \text{ has \underline{algorithmic measure 0} iff} \]

\[ (\exists \text{ computable } f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}^*)(\forall k \in \mathbb{N}) \]

\[ [Z \subseteq \bigcup_{l=0}^{\infty} C_{f(k,l)} \text{ and } \sum_{l=0}^{\infty} 2^{-f(k,l)} \leq 2^{-k}] \].
6. Compare and contrast effective and classical concepts
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**Observation**

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7. Have fun with your friends exploring the consequences!
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Note that several different choices were made in Martin-Löf’s effectivization of Lebesgue measure 0, i.e., there were several things to decide, including at least the following.

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3. What computability/complexity constraints do we impose? That is, **how effective** do we make our new concept?
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2. Which characterization of that concept do we effectivize?

3. What computability/complexity constraints do we impose? That is, how effective do we make our new concept?

To decide how to exercise all this freedom, we need to answer a more fundamental question.
Why We Effectivize

Or, as I have often been asked:
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Q: Why would you take a beautiful mathematical theory and pollute it with Turing machines?
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A1 (early 20th century): To restrict the theory to its legitimate content.

A2 (mid 20th century - today): To create equally beautiful theories and have fun with our friends working on them.
Or, as I have often been asked:

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Why We Effectivize

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A5 (recent - today): To reveal the unity in three great theories of information.
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A4 (recent - today): To prove new theorems in the classical (non-effectivized) theory.

A5 (recent - today): To reveal the unity in three great theories of information.

Each of these four answers (and perhaps answers to come) influences how we effectivize concepts of mathematical analysis.
Terminology used here: We effectivize at various levels.
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Here are some of these levels:

Less effective

Σ₁
Δ₀
(...)
Σ₀
Δ₁
(...)
pspace
qp
p
(...)
finite-state

More effective
We should emphasize: The choice of which characterization of an analytic concept to effectivize is important. Moreover, this importance often grows as we go down to more effective levels.
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The rest of the talk is a brief preview of what these lectures will say about the last three reasons for effectivizing analytic concepts. We proceed in reverse chronological order.
Hausdorff (1919), Shannon (1948), and Kolmogorov (1965) discovered three fundamentally different ways to quantify information.

We will see tomorrow that, when the first two of these are appropriately effectivized, the identities
\[ K(x) = \log_2 m(x) = |x| \dim(x) \]
hold, up to additive constants, for all strings \( x \in \{0, 1\}^* \).
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$$K(x) = \log \frac{1}{m(x)} = |x| \text{dim}(x)$$

hold, up to additive constants, for all strings $x \in \{0, 1\}^*$. 
Definition

(Borel 1909). A sequence \( S \in \Sigma^\omega_b \) is normal if

\[
(\forall w \in \Sigma^+_b) \lim_{n \to \infty} \text{freq}_n(w, S) = b^{-|w|}.
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Theorem

*(Schnorr and Stimm 1972). Borel normality is finite-state randomness.*

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The finite-state effectivization of Lebesgue measure 0 sets came $> 60$ years before we knew that it **was** an effectivization!
Finite-state fractal dimension

For $S \in \Sigma^\omega_b$, $\dim_{FS}(S) \in [0,1]$ is the “lower asymptotic density of finite-state information” in $S$. This is a finite-state effectivization of classical Hausdorff dimension. First definition used finite-state gamblers (Dai, Lathrop, J. Lutz, Mayordomo 2004).

Equivalent characterizations include:

- Information lossless finite-state compressors (DLLM 2004).
- Block-entropy rates (Bourke, Hitchcock, Vinodchandran 2005).
- Finite-state log-loss predictors (Hitchcock 2003).
- Automatic Kolmogorov complexity (Kozachinskiy & Shen 2021).
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The dual of $\dim_{FS}$, the finite state strong dimension $\Dim_{FS}$, is a finite-state effectivization of classical packing dimension.

Fact
(Schnorr & Stimm 1972; BHV 2005):

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**Fact**

*(Schnorr & Stimm 1972; BHV 2005):*

$$S \in \Sigma_\omega^* \text{ is normal} \iff \dim_{FS}(S) = 1.$$  

Research question: **How true** is the following? **Normality/Dimension Thesis.** Every theorem about Borel normality is the dimension-1 special case of a theorem about finite-state dimension.
At least three examples of the Normality/Dimension Thesis are known:


2. Real arithmetic (Doty, J. Lutz, & Nandakumar 2007; see also Kozachinskiy & Shen 2021).

Classical Theory # 2: Geometric measure theory

The algorithmic dimension \( \dim(x) \) and the algorithmic strong dimension \( \text{Dim}(x) \) are \( \Sigma_0^1 \) effectivizations of classical Hausdorff and packing dimensions, respectively. These are defined for all \( x \in X \), where \( X \) is any separable metric space, e.g., \( X \) may be \( C, \mathbb{R}^n \), etc. Here we focus on the case \( X = \mathbb{R}^n \).
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Theorem

(Point-to-Set Principle, J. Lutz & N. Lutz 2018). For every set \( E \subseteq \mathbb{R}^n \),

\[
\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)
\]

and

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Note: The dimensions on the left are **classical**.
The dimensions on the right are **very nonclassical**, namely,

$$\dim^A(x) = \liminf_{r \to \infty} \frac{K^A_r(x)}{r}$$

and

$$\Dim^A(x) = \limsup_{r \to \infty} \frac{K^A_r(x)}{r}.$$
The Point-to-Set Principle has been used to prove several new theorems in geometric measure theory, including the following.

- Extending fundamental theorems on intersections (N. Lutz 2021), random projections (N. Lutz and Stull 2018; Stull 2022), and products (N. Lutz 2020) of fractals from analytic sets to arbitrary sets.
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- Extending fundamental theorems on intersections (N. Lutz 2021), random projections (N. Lutz and Stull 2018; Stull 2022), and products (N. Lutz 2020) of fractals from analytic sets to arbitrary sets.
- Improved lower bounds on the dimensions of Furstenberg sets (N. Lutz and Stull 2020).
Hausdorff dimensions of $\Pi^0_1$ sets not supported by their closed subsets (Slaman 2021).

Existence of Hamel bases of $\mathbb{R}$ over $\mathbb{Q}$ of all dimensions $\alpha \in [0, 1]$ (Lutz, Qi, and Lu 2024).

Research challenges:
- More applications of the Point-to-Set Principle.
- Are Point-to-Set Principle arguments in some sense “conservative” over classical arguments?
- Analogous phenomena for analytic concepts other than fractal dimensions.
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Four analytic concepts have been effectivized to work inside of complexity classes.

- Baire category (J. Lutz 1987)
- Hausdorff dimension (J. Lutz 2003)
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Analytic Structure in Small Worlds

Four analytic concepts have been effectivized to work inside of complexity classes.

- Baire category (J. Lutz 1987)
- Hausdorff dimension (J. Lutz 2003)
- Packing dimension (Athreya, Hitchcock, J. Lutz, Mayordomo 2007)

These effectivizations all endow certain complexity classes with internal structure of the indicated type, even though the complexity classes, being countable, are classically trivial.
These effectivizations have been formulated very generally, so that the classical analytic concepts are a special case (the easiest special case!) of the resource-bounded theory.
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Most importantly, these effectivizations - especially resource-bounded measures - have interacted informatively with many complexity-theoretic concepts, including polynomial-time reducibilities, NP-completeness, circuit-size complexity, bi-immunity, complexity cores, randomized algorithms, pseudorandom generators, natural proofs, lowness, etc.
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Most importantly, these effectivizations - especially resource-bounded measures - have interacted informatively with many complexity-theoretic concepts, including polynomial-time reducibilities, NP-completeness, circuit-size complexity, bi-immunity, complexity cores, randomized algorithms, pseudorandom generators, natural proofs, lowness, etc.

To achieve this, it appears essential to use Ville’s 1939 martingale characterization of Lebesgue measure. The goal here is the effectivization that gives the strongest tool that works in the complexity classes!
Thank you!

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